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THE QUASI-CLASSICAL EQUILIBRIUM WIGNER DISTRIBUTION
FUNCTION OF AN ELECTRON GAS IN A NON-UNIFORM ELECTRO-
MAGNETIC FIELD

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Quantum corrections up to order $\hbar^4$ are found for the equilibrium distribution function of an electron gas in a non-uniform electromagnetic field in the case of Boltzmann statistics. The quantum corrections are also found for the classical free energy.

§ 1. Introduction

The aim of this work is to find the quantum corrections to the classical free energy in the form of the first orders of a series in terms of $\hbar$ using density matrix of an equilibrium nonrelativistic electron gas with interaction in a non-uniform steady electromagnetic field in the Weyl representation (or the Wigner function). For the first time this method was used in the pioneer works of Wigner and Kirkwood [1,2], but in these papers the magnetic field was absent. Quantum corrections to the classical distribution function of an electron gas with interaction in a uniform magnetic field was discussed in papers [3-6]. Taking the trace from the density matrix, we can find the partition function and then the free energy, which is of practical interest (e.g. for calculating magnetic susceptibility of the gas).

The exchange effects represented by the terms of order $\hbar^3$ [7] cannot be taken into account in this problem, treated as the problem of quantum corrections to the classical value. The obtained result is true for high temperature, i.e. in the Boltzmann statistics, in which exchange effects cannot appear.

The corrections will be calculated up to order $\hbar^4$, but those of order $\hbar^2$ will be computed in three ways: 1) taking

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Weyl transform from $e^{-\beta \hat{H}}$, 2) using Bloch's equation and 3) by applying analogy between density matrix and evolutionary operator of an equilibrium system. In order to calculate the Wigner function in the third way the knowledge of the $2N$ integrals of motion of a system with N degrees of freedom is needed. The utilization of integrals of motion for calculation of some values is of interest itself [11, 12].

§ 2. Bloch's equation in the Weyl representation

2.1 Weyl transform of the Hamiltonian

Hamiltonian of an electron gas with interaction in an electromagnetic field is:

$$(2.1) \quad \hat{H} = \frac{(\hat{p}_i - \tilde{A}^i)(\hat{p}_i - \tilde{A}^i)}{2m} + U(\vec{q}) \quad i = 1, 2, \dots, 3N \quad \tilde{A} = \frac{e}{c} \bar{A}$$

where $\hat{\vec{p}}$ stands for the operator $-i\hbar \frac{\partial}{\partial \vec{q}}$, $\tilde{A}$ as well as $\bar{A}$ (the tilde will be omitted lower, and return to the old designation will be stipulated for) is a $3N$ -dimensional vector $\bar{A} = (\bar{A}_1, \dots, \bar{A}_N)$ ($\bar{A}_i$ is a three-dimensional vector-potential of a point of the i -th particle position). We use $3N$ -dimensional position and momentum vectors such as $\vec{q}, \vec{p}$, etc... Here and lower the summation in the case of repeating index is implied. Lower we shall discuss the Hamiltonian of more general character.

$$(2.2) \quad \hat{H} = \frac{M_{ik}}{2} (\hat{p}_i - A^i)(\hat{p}_k - A^k) + U(\vec{q})$$

which corresponds to the case of an electron gas in a solid state (M_{ik} - symmetric matrix). Usually one defines the Weyl transform of a quantum operator $\hat{A}$ as follows [1,8]:

$$(2.3) \quad a(\bar{p}, \bar{q}) = \int d\bar{v} e^{(i/\hbar) \bar{p} \cdot \bar{v}} \langle \bar{q} - \frac{\bar{v}}{2} | \hat{A} | \bar{q} + \frac{\bar{v}}{2} \rangle$$

Then, the Weyl transform of the Hamiltonian of concern here will be:

$$(2.4) \quad h_c = \frac{M_{ik}}{2} (p_i - A^i) (p_k - A^k) + U(\bar{q})$$

i.e., the classical Hamiltonian

2.2 Bloch's equation in the Weyl representation

The Bloch's equation as an operator equality is:

$$(2.5) \quad -\frac{\partial \hat{\rho}}{\partial \beta} = \hat{H} \hat{\rho}, \quad \text{where } \hat{\rho} \text{ is the density mat-}$$

rix. Weyl transform of the product of two commuting operators $\hat{A}$ and $\hat{B}$ ($\hat{A}\hat{B} = \hat{B}\hat{A}$) is not in general the product of their Weyl transforms $a(\bar{p}, \bar{q})$ and $b(\bar{p}, \bar{q})$ and may be written as [8]:

$$(2.6) \quad \hat{A}\hat{B} \Rightarrow a(\bar{p}, \bar{q}) b(\bar{p}, \bar{q}) \cos \left\{ \frac{\hbar}{2} \left(\frac{\partial^a}{\partial q^i} \cdot \frac{\partial^b}{\partial p_i} - \frac{\partial^a}{\partial p_i} \cdot \frac{\partial^b}{\partial q^i} \right) \right\}$$

The symbol $\Rightarrow$ signifies the correspondence between operator and its Weyl transform. The cosine is the symbol of the corresponding operator series, in which the superscript "a" of an operator indicates that it acts on $a(\bar{p}, \bar{q})$ only.

Thus, the Bloch's equation in the Weyl representation (note that $\hat{A}\hat{\rho}=\hat{\rho}\hat{A}$) may be written as [9]:

$$(2.7) \quad -\frac{\partial \rho}{\partial \beta} = h_{cl}(\bar{p}, \bar{q}) \rho(\bar{p}, \bar{q}) \cos \left\{ \frac{\hbar}{2} \left(\frac{\partial^h}{\partial q^i} \cdot \frac{\partial^p}{\partial p_i} - \frac{\partial^h}{\partial p_i} \cdot \frac{\partial^p}{\partial q^i} \right) \right\}$$

where ρ is the Wigner function. Since the Hamiltonian h_{cl} depends on $\bar{p} - \bar{A}$, it is convenient to make the substitution:

$$(2.8) \quad \bar{\pi} = \bar{p} - \bar{A}$$

where $\bar{\pi}$ is the "physical" momentum, and to find ρ as a function of $\bar{\pi}$ and $\bar{q}$. Formula (2.7) can be rewritten as:

$$-\frac{\partial \rho}{\partial \beta} = h_{cl}(\bar{\pi}, \bar{q}) \rho(\bar{\pi}, \bar{q}).$$

$$(2.9) \quad \cos \left\{ \frac{\hbar}{2} \left(\frac{\partial^h}{\partial q^i} \cdot \frac{\partial^p}{\partial \pi_i} - \frac{\partial^h}{\partial \pi_i} \cdot \frac{\partial^p}{\partial q^i} + \frac{\partial^h}{\partial \pi_i} \left[\frac{\partial}{\partial \pi_k} A_i^k \right] \rho - \left[A_k^i \frac{\partial}{\partial \pi_i} \right] \cdot \frac{\partial^h}{\partial \pi_k} \right) \right\}$$

Here and lower indexes of scalar, vector and tensor values, depending on the coordinates only, will be upper ones, and the following notation will also be used:

$$\frac{\partial C \dots (\bar{\pi}, \bar{q})}{\partial q^i} \equiv C \dots i ; \quad \frac{\partial C \dots (\bar{\pi}, \bar{q})}{\partial \pi_i} \equiv C \dots i$$

From (2.7) one can see, that $\hbar$ enters the Bloch's equation of even powers only, and hence the Wigner function is real, as well as any Weyl transformed hermitian operator.

2.3 Gauge transform of density matrix

It is necessary to discuss the question of gauge transforms of density matrix. In general the density matrix is not gauge-invariant, which becomes obvious after consideration of a pure state:

$$(2.10) \quad \hat{\rho}(x, x') = |\psi(x)\rangle \langle \psi(x')|$$

After gauge transform $|\psi(x)\rangle$ goes over into:

$$(2.11) \quad |\psi^f(x)\rangle = |\psi(x)\rangle e^{(ie/\hbar c)f(x)}$$

and $\hat{\rho}$ into:

$$(2.12) \quad \hat{\rho}^f(x, x') = \hat{\rho}(x, x') e^{(ie/\hbar c)(f(x) - f(x'))}$$

The gauge invariance of density matrix cannot follow from (2.9) either. The conclusion about gauge invariance of Wigner function, made by the authors of paper [5], is erroneous, and is a consequence of wrongly made substitution (2.8). The results of calculations, which will be carried out lower, will show explicitly the gauge non-invariance of the density matrix. But, despite the fact above, the moments of the Wigner function must be gauge invariants.

§ 3 Quantum corrections to the classical distribution function of an electron gas in an electromagnetic field.

An approximate expression for the Wigner function as the first terms of a series in $\hbar$ one can obtain in many ways. The essence of the methods considered here can be described as follows: it is necessary to expand the exponent $e^{-\beta H}$ in β to take Weyl transform of the obtained series and then to sum-

marize it over β . One may summarize it directly, what is made in app. 1.2, but one may differentiate it with respect to some parameter and, writing a differential equation for the sum of the series, solve it and obtain the expression sought. If such a parameter is β , we get the Bloch's equation. The utilization of integrals of motion realizes the analogous method, in which the parameters are $\bar{\pi}$ and $\bar{q}$. The Bloch's equation will be used lower, and the two other methods will be developed in the app. 1.

In the limit $\hbar=0$, the solution of (2.9) (with the initial condition $\rho=1$ at $\beta=0$) is the classical equilibrium distribution.

$$(3.1) \quad \rho_0 = e^{-\beta h_{cl}(\bar{\pi}, \bar{q})}$$

It is natural to look for a solution of a form:

$$(3.2) \quad \rho = e^{-\beta h_{cl}} \cdot x(\bar{\pi}, \bar{q})$$

where x is a series in $\hbar^2$:

$$(3.3) \quad x = 1 + \hbar^2 x_2(\bar{\pi}, \bar{q}) + \hbar^4 x_4(\bar{\pi}, \bar{q})$$

Expanding the cosine in (2.9) up to $\hbar^2$ we get the equation

$$(3.4) \quad -\frac{\partial \rho}{\partial \beta} = h_{cl} \rho - \frac{\hbar^2}{8} [U_{ie} \rho^{ie} + M_{ie} \rho^{ie} - M_{ik} A_{em}^i \pi_k \rho^{em} - \\ - M_{ik} A_{ik}^e \rho^e + M_{ik} F^{ei} \rho_k^e + M_{ik} F^{ie} F^{km} \rho^{em}]$$

where $F^{ei} = A_e^i - A_i^e$. Substituting (3.2) for ρ in the equation (3.4), separating the equation into powers of $\hbar$ we get, integrating over β :

$$(3.5) \quad \begin{aligned} \mathcal{Z}_2 = & -\frac{\beta^2}{8} M_{ie} U_{ie} + \frac{\beta^3}{24} M_{ie} M_{km} U_{ik} \pi_e \pi_m + \frac{\beta^3}{24} M_{ik} U_{ii} U_{kk} + \\ & + \frac{\beta^3}{12} M_{ik} M_{em} F^{ei} U_{kk} \pi_m - \frac{\beta^2}{16} M_{ik} M_{em} F^{ie} F^{km} + \\ & + \frac{\beta^3}{24} M_{ik} M_{ep} M_{mq} F^{ie} F^{km} \pi_p \pi_q + \frac{\beta^2}{8} M_{em} M_{ik} A_{em}^i \pi_k - \\ & - \frac{\beta^3}{24} M_{ik} M_{ep} M_{mq} A_{em}^i \pi_p \pi_q \pi_k \end{aligned}$$

The calculation of $\mathcal{Z}_4$ is similar. In case of the mass matrix $M_{ik} = \frac{1}{m} \delta_{ik}$ we get the expression for $\mathcal{Z}_2$:

$$(3.6) \quad \begin{aligned} \mathcal{Z}_2 = & -\frac{\beta^2}{8m} U_{ii} + \frac{\beta^3}{24m^2} U_{ik} \pi_i \pi_k + \frac{\beta^3}{24m} U_{ii} U_{ii} + \\ & + \frac{\beta^3}{12m^2} F^{ei} U_{ii} \pi_e - \frac{\beta^2}{16m^2} F^{ie} F^{ie} + \frac{\beta^3}{24m^3} F^{ie} F^{im} \pi_e \pi_m + \\ & + \frac{\beta^2}{8m^2} A_{ee}^i \pi_i - \frac{\beta^3}{24m^3} A_{em}^i \pi_i \pi_e \pi_m \end{aligned}$$

The explicit form of $\mathcal{Z}_4$ in the case of the diagonal mass matrix $M_{ik} = \frac{1}{m} \delta_{ik}$ is presented in the appendix 2.

§ 4. Free energy of a gas

One defines the partition function as a trace of a density matrix, which is evaluated in the Weyl representation:

$$(4.1) \quad Z = \text{Tr} \hat{\rho} = \int \rho(\bar{p}, \bar{q}) \frac{d\bar{p} d\bar{q}}{(2\pi\hbar)^{3N}} = \int \rho(\bar{\pi}, \bar{q}) \frac{d\bar{\pi} d\bar{q}}{(2\pi\hbar)^{3N}}$$

In the last equality the substitution of variables (2.8) with unitary Jacobian was made. One may extract in the expression (4.1) a classical partition function:

$$(4.2) \quad Z = \int e^{-\beta h_{cl}} \mathcal{X}(\bar{\pi}, \bar{q}) \frac{d\bar{\pi} d\bar{q}}{(2\pi\hbar)^{3N}} = Z_{cl} \cdot \{\mathcal{X}\}$$

where $\{\mathcal{X}\}$ is defined as follows

$$(4.3) \quad \{\mathcal{X}\} = \frac{\int e^{-\beta h_{cl}} \mathcal{X} \frac{d\bar{\pi} d\bar{q}}{(2\pi\hbar)^{3N}}}{\int e^{-\beta h_{cl}} \frac{d\bar{\pi} d\bar{q}}{(2\pi\hbar)^{3N}}}$$

$\{\mathcal{X}\}$ is a series in terms of $\hbar^2$:

$$\{\mathcal{X}\} = 1 + \hbar^2 \{\mathcal{X}_2\} + \hbar^4 \{\mathcal{X}_4\} + \dots$$

Substituting (3.5) for $\mathcal{X}$ in (4.3), one can see, that partial integration of $\{\mathcal{X}_2\}$ is possible. Using expressions

$$(4.4) \quad \begin{aligned} \int e^{-\frac{\beta}{2} M_{ik} \pi_i \pi_k} d\bar{\pi} &= \left(\frac{2\pi}{\beta}\right)^{3N/2} (\det M)^{-1/2} \\ \int e^{-\frac{\beta}{2} M_{ik} \pi_i \pi_k} \pi_e d\bar{\pi} &= \int e^{-\frac{\beta}{2} M_{ik} \pi_i \pi_k} \pi_e \pi_m \pi_n = 0 \\ \int e^{-\frac{\beta}{2} M_{ik} \pi_i \pi_k} \pi_e \pi_m &= \left(\frac{2\pi}{\beta}\right)^{3N/2} (\det M)^{-1/2} \frac{M_{em}^{-1}}{\beta} \end{aligned}$$

and introducing the symbol,

$$(4.5) \langle x \rangle = \frac{\int e^{-\beta U} x d\bar{q}}{\int e^{-\beta U} d\bar{q}}$$

we find:

$$(4.6) \begin{aligned} \langle x_2 \rangle = & -\frac{\beta^2}{42} M_{ie} \langle U_{ie} \rangle + \frac{\beta^3}{24} M_{ik} \langle U_{ik} \rangle - \\ & - \frac{\beta^2}{48} M_{ik} M_{em} \langle F^{ie} F^{km} \rangle \end{aligned}$$

Integrating by parts

$$(4.7) \int U_{ie} e^{-\beta U} d\bar{q} = \beta \int U_{ik} U_{ie} e^{-\beta U} d\bar{q}$$

where one neglects the surface integral as a small value in comparison with the volume integral, we reduce $\langle x_2 \rangle$ to the form:

$$(4.8) \langle x_2 \rangle = -\frac{\beta^2}{24} M_{ie} \langle U_{ie} \rangle - \frac{\beta^2}{48} M_{ik} M_{em} \langle F^{ie} F^{km} \rangle$$

For $M_{ik} = \frac{1}{m} \delta_{ik}$:

$$(4.9) \langle x_2 \rangle = -\frac{\beta^2}{24m} \langle U_{ii} \rangle - \frac{\beta^2}{48m^2} \langle F^{ie} F^{ie} \rangle$$

For the partition function we write up to $\hbar^2$:

$$(4.10) Z = Z_0 \left(1 - \frac{\hbar^2 \beta^2}{24} M_{ie} \langle U_{ie} \rangle - \frac{\hbar^2 \beta^2}{48} M_{ik} M_{em} \langle F^{ie} F^{km} \rangle \right)$$

and for the diagonal mass matrix

$$(4.11) \quad Z = Z_{cl} \left(1 - \frac{\hbar^2 \beta^2}{24m} \langle U_{ii} \rangle - \frac{\hbar^2 \beta^2}{48m^2} \langle F^{ie} F^{ie} \rangle + \dots \right)$$

From these formulae it is easy to get the expression of the free

energy $\tilde{F} = -\frac{1}{\beta} \ln Z$

$$(4.12) \quad \tilde{F} = \tilde{F}_{cl} + \frac{\hbar^2 \beta}{24} M_{ie} \langle U_{ie} \rangle + \frac{\hbar^2 \beta}{48} M_{ik} M_{em} \langle F^{ie} F^{km} \rangle + \dots$$

and for $M_{ik} = \frac{1}{m} \delta_{ik}$

$$\begin{aligned} \tilde{F} = \tilde{F}_{cl} + \frac{\hbar^2}{24} \left[\frac{\beta}{m} \langle U_{ii} \rangle + \frac{\beta}{2m^2} \langle F^{ie} F^{ie} \rangle \right] + \\ + \frac{\hbar^4}{8 \cdot 6!} \left[5 \frac{\beta^3}{m^2} \langle U_{ii} \rangle^2 - 5 \frac{\beta^3}{m^2} \langle U_{ii} U_{ee} \rangle - 2 \frac{\beta^3}{m^2} \langle U_{ke} U_{ke} \rangle + \right. \\ (4.13) \quad + 5 \frac{\beta^2}{m^2} \langle U_{iie} \rangle + 5 \frac{\beta^3}{m^3} \langle U_{ii} \rangle \langle F^{ke} F^{ke} \rangle - \\ - 5 \frac{\beta^3}{m^3} \langle U_{ii} F^{ke} F^{ke} \rangle + \frac{5}{4} \frac{\beta^3}{m^4} \langle F^{ik} F^{ik} \rangle^2 - \\ - \frac{5}{4} \frac{\beta^3}{m^4} \langle F^{ik} F^{ik} F^{em} F^{em} \rangle - \frac{\beta^3}{m^4} \langle F^{ik} F^{ie} F^{mk} F^{me} \rangle + \\ \left. + 6 \frac{\beta^2}{m^3} \langle F^{ke} F^{ke} \rangle + 4 \frac{\beta^3}{m^3} \langle F^{ie} F^{ke} U_{ii} \rangle - 4 \frac{\beta^3}{m^3} \langle F^{ki} F^{ke} U_{ie} \rangle \right] + \dots \end{aligned}$$

where the corrections of $\frac{\hbar^4}{m^4}$, computed in app. 2, are written for information. Finally we rewrite the expressions (4.12,

4.13) taking into account the inverse substitution $\bar{A} \rightarrow \frac{e}{c} \bar{A}$

$$(4.14) \quad \tilde{F} = \tilde{F}_{cl} + \frac{\hbar^2}{24} \left[\beta M_{ie} \langle U_{ie} \rangle + \frac{\beta e^2}{2c^2} M_{ik} M_{em} \langle F^{ie} F^{km} \rangle \right] + \dots$$

and for $M_{ik} = \frac{1}{m} \delta_{ik}$

$$\begin{aligned}
 \mathcal{F} = & \mathcal{F}_0 + \frac{\hbar^2}{24} \left[\frac{\beta}{m} \langle U_{ii} \rangle + \frac{\beta e^2}{2m^2 c^2} \langle F_{ie} F_{ie} \rangle \right] + \\
 & + \frac{\hbar^4}{8 \cdot 6!} \left[5 \frac{\beta^3}{m^2} \langle U_{ii} \rangle^2 - 5 \frac{\beta^3}{m^2} \langle U_{iilee} \rangle - 2 \frac{\beta^3}{m^2} \langle U_{kell} \rangle + \right. \\
 & (4.15) \quad + 5 \frac{\beta^2}{m^2} \langle U_{iilee} \rangle + 5 \frac{\beta^3 e^2}{m^3 c^2} \langle U_{ii} \rangle \langle F_{ke} F_{ke} \rangle - \\
 & - 5 \frac{\beta^3 e^2}{m^3 c^2} \langle U_{ii} F_{ke} F_{ke} \rangle + \frac{5 \beta^3 e^4}{4 m^4 c^4} \langle F_{ik} F_{ik} \rangle^2 - \\
 & - \frac{5 \beta^3 e^4}{4 m^4 c^4} \langle F_{ik} F_{ik} F_{em} F_{em} \rangle - \frac{\beta^3 e^4}{m^4 c^4} \langle F_{ik} F_{ie} F_{mk} F_{me} \rangle + \\
 & + 6 \frac{\beta^2 e^2}{m^3 c^2} \langle F_{ke} F_{ke} \rangle + 4 \frac{\beta^3 e^2}{m^3 c^2} \langle F_{ke} F_{ke} U_{ii} \rangle - 4 \frac{\beta^3 e^2}{m^3 c^2} \langle F_{ik} F_{ke} U_{ie} \rangle \Big]
 \end{aligned}$$

Note, that the expressions obtained are equivalent: to the results of [10] in the special case $\bar{B} = 0$, to the results of [4,6] in case of uniform magnetic field. The expansion obtained coincides with the expansion of the exact Wigner function of three-dimensional oscillator in uniform electric and magnetic fields, computed in [11] on the basis of formulae for the Wigner function of arbitrary many-dimensional quadratic system, obtained in paper [12]. For comparison we write one of these formulae - the Wigner function of a particle of mass m and charge e , moving in plane q_1, q_2 , perpendicular to the uniform magnetic field $\bar{B}$ (vector-potential $\frac{1}{2}[\bar{B} \times \bar{q}]$), in the presence of the uniform electric field $\bar{E} = (E_1, E_2)$ and the isotropic potential $\frac{1}{2} m g \bar{q}^2$:

$$\begin{aligned}
 \rho(\bar{q}, \bar{p}, \beta) = & 2 \mathcal{D}_+^{-1}(\beta) \exp \left\{ - \frac{\text{sh}(\hbar \Omega \beta)}{\mathcal{D}_+(\beta)} \cdot \frac{\bar{V}^2}{m \hbar \Omega} - \right. \\
 (4.16) \quad & - \frac{m \Omega}{\hbar} \cdot \frac{\mathcal{D}_-(\beta)}{\text{sh}(\hbar \Omega \beta)} \cdot \bar{x}^2 + \frac{\beta (e \bar{E})^2}{2 m g} \Big\} \\
 \bar{x} = \bar{q} - \frac{e}{m g} \bar{E} \quad & \omega = \frac{e B}{m c} \quad \Omega = \left(\frac{\omega^2}{4} + g \right)^{1/2}
 \end{aligned}$$

$$D_{\pm}(\beta) = ch(\hbar\Omega\beta) \pm ch(\frac{1}{2}\hbar\omega\beta)$$

$$v_1 = p_1 + \frac{m\Omega sh(\frac{1}{2}\hbar\omega\beta)}{sh(\hbar\Omega\beta)} x_2 + \frac{e\omega}{2g} E_2$$

$$v_2 = p_2 - \frac{m\Omega sh(\frac{1}{2}\hbar\omega\beta)}{sh(\hbar\Omega\beta)} x_1 - \frac{e\omega}{2g} E_1$$

In a uniform electric field $g=0$ from (4.16) we get:

$$\rho(\bar{q}, \bar{p}, \beta) = [ch(\frac{1}{2}\hbar\omega\beta)]^{-1} \exp\left\{-th(\frac{1}{2}\hbar\omega\beta) \frac{\bar{v}^2}{m\hbar\omega} + \right.$$

$$(4.17) \left. + \beta e \bar{E} \bar{q} + \frac{e}{2m} \left(\frac{e\bar{E}}{\omega}\right)^2 \left[\frac{1}{2}\hbar\omega\beta cth(\frac{1}{2}\hbar\omega\beta) - 1\right]\right\}$$

$$\bar{v} = \bar{p} - \frac{e}{2c} [\bar{B} \times \bar{q}] + \frac{e^2}{mc\omega^2} \left[\frac{1}{2}\hbar\omega\beta cth(\frac{1}{2}\hbar\omega\beta) - 1\right] [\bar{E} \times \bar{B}]$$

Appendix 1

1.1 For calculating accurate propagators and equilibrium density matrix and Wigner function of a system with quadratic Hamiltonian the method of integrals of motion is effective. This method can also be applied for calculations of quantum corrections to the distribution function.

Let the explicit view of operator be known:

$$(A.1.1) \quad \hat{X}(\beta) = e^{-\beta \hat{H}} \hat{x} e^{\beta \hat{H}}$$

Then the operator $\hat{\rho}(\beta)$ can be found from equation

$$(A.1.2) \quad \hat{X}(\beta) \hat{\rho}(\beta) = \hat{\rho}(\beta) \hat{x}$$

Upon substitution of $\beta = \frac{i\tau}{\hbar}$, operator $\hat{S}(\beta)$ becomes the evolution operator of a system, and the $\hat{X}(\beta)$ becomes the integral of motion, which coincides with $\hat{x}$ at $t=0$. Thus the equation (A.1.2) means that the evolution operator can be found from a solution of an eigenvalue problem for operators - integrals of motion).

Operator $\hat{X}(\beta)$ can be represented as a series in $\hbar$ on the basis of the Baker-Campbell-Hausdorff formula:

$$\hat{X}(\beta) = \sum_{n=0}^{\infty} \hbar^n \hat{X}^{(n)} \quad (A.1.3)$$

$$\hat{X}^{(n)} = -\frac{\beta}{\hbar} [\hat{H}, \hat{X}^{(n-1)}], \quad \hat{X}^{(0)} = \hat{x}$$

For $\hat{x}$ the first terms of expansion are as follows (we omit the symbol " $\wedge$ ")

$$(A1.4) \quad X_{\alpha}^{(1)} = i\beta M_{\alpha\gamma} \pi_{\gamma}$$

$$(A.1.5) \quad X_{\alpha}^{(2)} = \frac{1}{2} \beta^2 M_{\alpha\gamma} \left\{ V_{\gamma} + \frac{1}{2} M_{\delta\delta} (F^{\epsilon\gamma} \pi_{\delta} + \pi_{\delta} F^{\epsilon\gamma}) \right\}$$

$$X_{\alpha}^{(3)} = \frac{i\beta^3}{6} M_{\alpha\gamma} \left\{ \frac{1}{2} M_{\rho\nu} (V_{\rho} \pi_{\nu} + \pi_{\nu} V_{\rho}) + \frac{1}{4} M_{\rho\nu} M_{\delta\delta} \cdot \right.$$

$$(A1.6) \cdot \left[(F_{\rho}^{\epsilon\gamma} \pi_{\nu} + \pi_{\nu} F_{\rho}^{\epsilon\gamma}) \pi_{\delta} + \pi_{\delta} (F_{\rho}^{\epsilon\gamma} \pi_{\nu} + \pi_{\nu} F_{\rho}^{\epsilon\gamma}) - \right.$$

$$\left. - F^{\epsilon\gamma} (F_{\rho}^{\delta} \pi_{\nu} + \pi_{\nu} F_{\rho}^{\delta}) - (F_{\rho}^{\delta} \pi_{\nu} + \pi_{\nu} F_{\rho}^{\delta}) F^{\epsilon\gamma} \right] M_{\epsilon\delta} F^{\epsilon\gamma} V_{\delta} \}$$

Further it is necessary to take into account that in the Weyl representation the operators $\hat{x}_{\alpha}$ and $\hat{\pi}_{\alpha}$ depend on the Plank's constant:

$$\hat{x}_\alpha = q_\alpha + \frac{i\hbar}{2} \frac{\partial}{\partial p_\alpha} = q_\alpha + \frac{i\hbar}{2} \frac{\partial}{\partial \pi_\alpha}$$

$$(A.1.7) \quad \hat{\pi}_\alpha = p_\alpha - \frac{i\hbar}{2} \frac{\partial}{\partial q_\alpha} - A^\alpha(\bar{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \bar{p}}) = p_\alpha - A^\alpha(\bar{q}) -$$

$$(A.1.8) \quad - \frac{i\hbar}{2} \left(\frac{\partial}{\partial q_\alpha} + A^\alpha_\lambda \frac{\partial}{\partial p_\lambda} \right) + \frac{\hbar^2}{8} A^\alpha_{\mu\lambda} \frac{\partial^2}{\partial p_\mu \partial p_\lambda} + \dots =$$

$$= \pi_\alpha - \frac{i\hbar}{2} \left(\frac{\partial}{\partial q_\alpha} + F^{\lambda\alpha} \frac{\partial}{\partial \pi_\lambda} \right) + \frac{\hbar^2}{8} A^\alpha_{\mu\lambda} \frac{\partial^2}{\partial \pi_\mu \partial \pi_\lambda}$$

(Here the left operators are implied). The difference between the last equality and the preceding one is that in the first case the Wigner function depends on $\bar{q}$ and $\bar{p}$, but in the other case - on $\bar{q}$ and $\bar{\pi}$. Using (A.1.7) and (A.1.8) it is to expand operators $\hat{X}^{(n)}_\alpha$ in $\hbar$ and to assemble the same powers of $\hbar$ in the series $\sum \hbar^n \tilde{X}^{(n)}_\alpha$. As a result we obtain the new operators $\tilde{X}^{(n)}_\alpha$. Then in the first order in $\hbar$ the equation (A.1.2) becomes (it is necessary to take into account the following form of the right operator $\hat{x}_\alpha = q_\alpha - \frac{i\hbar}{2} \frac{\partial}{\partial \pi_\alpha}$):

$$i\hbar \frac{\partial \mathcal{Z}}{\partial \pi_\alpha} + i\beta\hbar M_{\alpha\gamma} \pi_\gamma \rho = 0$$

from which we again get the zero term of the expansion $\rho_0(\beta) = e^{\beta h_\alpha}$ (precisely, the part which depends on $\bar{\pi}$). The absence of terms of odd orders in $\hbar$ follows from the reality of Wigner function and from the fact, that $\tilde{X}^{(n)}$ contains i^n as a multiplier. Assembling in (A.1.2) the terms of the third order in $\hbar$, we get the equation for $\mathcal{Z}_2$:

$$(A.1.9) \quad i \frac{\partial \mathcal{Z}_2}{\partial \pi_\alpha} = e^{\beta h_\alpha(\bar{\pi}, \bar{q})} \hat{X}^{(3)}_\alpha e^{-\beta h_\alpha(\bar{\pi}, \bar{q})}$$

From (A.1.9) we can find the part of $\mathcal{L}_2$ which depends on π .

The remainder can be calculated analogously, using operators

$\hat{\Pi}_2(\beta)$ which one can find using (A.1.1) for $\hat{\pi}_\alpha$. It is not necessary to calculate these operators since we can write using (A.1.4):

$$(A.1.10) \quad \hat{X}_\alpha^{(n)} = \frac{i\beta}{n} M_{\alpha\gamma} \hat{\Pi}_\gamma^{(n-1)}$$

The additive components, depending on the coordinates only, are found from equation

$$(A.1.11) \quad i \frac{\partial \mathcal{L}_2}{\partial q^\alpha} + i \Gamma^\alpha_{\gamma\delta} \frac{\partial \mathcal{L}_2}{\partial \pi_\gamma} = e^{\beta h_\alpha} \hat{\Pi}_\alpha^{(3)} e^{-\beta h_\alpha}$$

As a result the expression (3.5) is obtained. But from this example one sees that for calculation of next terms of expansion in $\hbar$ of the Wigner function the method of integrals of motion is not effective. The reason is that it is necessary to re-expand the operators $\hat{X}^{(n)}$ in $\hbar$ and again to assemble the terms of equal powers of $\hbar$, which at high powers demands very much computation in comparison with others methods. However mind that the difficulties arise only if $\Gamma^\alpha_{\gamma\delta} \neq 0$. When the magnetic field is absent the integrals of motion become far more simple, and their utilization for computation of quantum corrections causes no technical difficulties.

1.2. Now we get the expression (3.5) by direct Weyl transform of $\hat{\rho} = e^{-\beta \hat{H}}$. It is necessary to rewrite the density matrix as a series:

$$(A.1.12) \quad e^{-\beta \hat{H}} = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \hat{H}^n$$

From formula for the Weyl transform of the product of two commuting operators $\hat{A}$ and $\hat{B}$ ($\hat{A}\hat{B}=\hat{B}\hat{A}$) [8]

$$(A.1.13) \quad \hat{A}\hat{B} = a(\bar{p}, \bar{q}) b(\bar{p}, \bar{q}) \cos \left\{ \frac{\hbar}{2} \left[\frac{\partial^2}{\partial q^i \partial p_i} - \frac{\partial^2}{\partial p_i \partial q^i} \right] \right\}$$

it is easy to obtain its generalization for the case of n commuting operators $\hat{A}_i \hat{A}_j = \hat{A}_j \hat{A}_i$:

$$(A.1.14) \quad \hat{A}_1 \dots \hat{A}_n = a_1 \dots a_n \cdot \cos \left\{ \frac{\hbar}{2} \sum_{i,j} \left[\frac{\partial^i}{\partial q^k} \frac{\partial^j}{\partial p_k} - \frac{\partial^i}{\partial p_k} \frac{\partial^j}{\partial q^k} \right] \right\}$$

Taking Weyl transform from (A.1.12) and performing the substitution (2.8) we get:

$$(A1.15) \quad \rho = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \hbar^1 \dots \hbar^n$$

$$\cdot \cos \left\{ \frac{\hbar}{2} \sum_{i,j} \left[\frac{\partial^i}{\partial q^k} \frac{\partial^j}{\partial p_k} - \frac{\partial^i}{\partial p_k} \frac{\partial^j}{\partial q^k} + \frac{\partial^i}{\partial p_k} \frac{\partial^j}{\partial p_m} A_e^{m,j} - \frac{\partial^i}{\partial p_m} \frac{\partial^j}{\partial p_k} A_e^{m,j} \right] \right\}$$

where $\hbar^i$ is the classical Hamiltonian (2.4), the superscript

" i " is necessary only to indicate the operator, which acts on this Hamiltonian. The zero approximation of ρ is obvious $\rho_0 = e^{-\beta \hbar^0}$. For brevity we introduce the symbol

$$(A.1.16) \quad \mathcal{D}^{ij} = \frac{\partial^i}{\partial q^k} \frac{\partial^j}{\partial p_k} - \frac{\partial^i}{\partial p_k} \frac{\partial^j}{\partial q^k} + \frac{\partial^i}{\partial p_k} \frac{\partial^j}{\partial p_m} A_e^{m,j} - \frac{\partial^i}{\partial p_m} \frac{\partial^j}{\partial p_k} A_e^{m,j}$$

The operator D_{ij} is antisymmetric with respect to permutation $i \leftrightarrow j$. Evaluating the expression (A.1.15) one can see that it decays into the sum of terms of different order in $\hbar$. The terms of order $\hbar^2$ obtained by evaluation of n -th term of series in β is represented by the sum, in which the operator $\sum_{i < j} D_{ij}$ acts twice:

$$(A.1.17) - \frac{\hbar^2 (-\beta)^n}{8 \cdot n!} \sum_{i < j} \sum_{k < l}^n D_{ij} D_{kl} \cdot h_{ce}^1 \dots h_{ce}^n$$

We designate the action of D_{ij} on two multipliers of the product $h_{ce}^1 \dots h_{ce}^n$ by the symbol $\boxed{j}$. In our case the numbers of terms, on which the operators act, is not essential (they are identical, and under action of D such terms become C -numbers, for which the order of multiplication is not essential), as well as the order of action of differential operators D . We consider one of these terms of order $\hbar^2$ (we omit the next terms of a series). This term can be represented as $\boxed{j} \boxed{k}$ where $i \neq j; k \neq l$. If there are no equal indexes, such term is equal to 0, as a result of action of the antisymmetric operator D_{ij} on the symmetric with respect to permutation pair of multipliers $\dots h_{ce}^i \dots h_{ce}^j \dots$. The same is true for the odd number of catching operators (the operators are named catching if, in the product there is at least one multiplier h_{ce} on which more than one operator acts). The non-zero result is given by pairs of operators, catching at least by one end. One can segregate 4 different modifications of catching of two operators:

(A.1.18)

a)

b)

c)

d)

where $l \neq j; l \neq k; j \neq k$. It is easy to see that (this follows from the antisymmetry of $\mathcal{D}$)

(A.1.19) = = -

Since the multipliers in which the operators act is identical, the sum (A.1.17) can be calculated.

(A.1.20)
$$-\frac{(-\beta)^{n+2}}{8n!} h \left\{ \mathcal{L}_n \left\{ \begin{array}{c} \text{Diagram 1} \end{array} \right\} h_{cl}^{n-2} + 2! \mathcal{L}_n \left\{ \begin{array}{c} \text{Diagram 2} \end{array} \right\} h_{cl}^{n-3} \right\}$$

The computation gives:

(A.1.21)
$$\begin{aligned} \left\{ \begin{array}{c} \text{Diagram 1} \end{array} \right\} &= 2M_{iellie} + M_{em} M_{ik} (F_{ie}^i F_{ik}^{km} - 2A_{em}^i \pi_k) \\ \left\{ \begin{array}{c} \text{Diagram 2} \end{array} \right\} &= M_{ik} \mathcal{L}_{illk} + M_{em} M_{ik} (2F_{ei}^i \mathcal{L}_{ik} \pi_m + \mathcal{L}_{ei} \pi_m \pi_k) + \\ &+ M_{ik} M_{ep} M_{mq} (F_{ie}^i F_{ik}^{km} \pi_p \pi_q - A_{em}^i \pi_k \pi_p \pi_q) \end{aligned}$$

Now one can write the explicit form of the approximate expression of Wigner function:

(A.1.22)
$$\begin{aligned} \rho &= \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} h_{cl}^n + \sum_{n=2}^{\infty} \frac{(-\beta)^{n-2}}{(n-2)!} h_{cl}^{n-2} \left(-\frac{3\hbar^2}{16} \right) \left\{ \begin{array}{c} \text{Diagram 1} \end{array} \right\} + \\ &+ \sum_{n=3}^{\infty} \frac{(-\beta)^{n-3}}{(n-3)!} h_{cl}^{n-3} \frac{\hbar^2 \beta^3}{2n!} \left\{ \begin{array}{c} \text{Diagram 2} \end{array} \right\} \end{aligned}$$

The obtained result is nothing but the expansion of exponent $e^{-\beta \hbar \alpha}$ with multiplier $1 + \hbar^2 \alpha_2$ and the explicit form of α_2 is equal to the obtained earlier. Note that the realized method, besides reducing calculations in comparison with the method developed in the main text, gives the understanding of structure of α . The advantage of Bloch's equation method is the possibility of performing calculations automatically, without time-consuming search of scheme of catching four $\mathcal{D}$ operators, in case of α_4 .

Appendix 2

We write for information the corrections of order $\hbar^4$ to the classical distribution function in case of $M_{ik} = \frac{1}{m} \delta_{ik}$. They can be computed in two ways, either by the method realized in the main text, or in the Appendix 1.2

$$\begin{aligned} \alpha_4 = & \frac{1}{8 \cdot 6!} \left[\frac{\beta^4}{m^2} U_{iellkm} (45 \delta_{ie} \delta_{km} + 30 \delta_{ik} \delta_{em} - \right. \\ & \left. - \frac{\beta}{m} [30 \delta_{ie} \pi_k \pi_m + 24 \delta_{ik} \pi_e \pi_m] + 5 \frac{\beta^2}{m^2} \pi_i \pi_e \pi_k \pi_m \right) + \\ & (A.2.1) + 5 \frac{\beta^6}{m^2} U_{ielllell} + \frac{\beta^5}{m^2} U_{iellklm} \cdot \\ & \cdot (-30 \delta_{ie} \delta_{km} - 24 \delta_{ik} \delta_{em} + 10 \frac{\beta}{m} \delta_{km} \pi_i \pi_e) + \\ & + \left(\frac{\beta^4}{m^2} U_{iellke} + \frac{\beta^4}{m^3} F_{im}^{\text{im}} U_{mke} \pi_i + \frac{\beta^4}{m^3} F_{ke}^{\text{im}} U_m \pi_i \right) \cdot \\ & \cdot (60 \delta_{ke} - 12 \frac{\beta}{m} \pi_k \pi_e) + \left(\frac{\beta^3}{m^3} A_{empq}^k - \frac{\beta^3}{m^2} U_{empq} \right) \cdot \\ & \cdot (45 \delta_{em} \delta_{pq} - 30 \frac{\beta}{m} \delta_{pq} \pi_e \pi_m + 3 \frac{\beta^2}{m^2} \pi_e \pi_m \pi_p \pi_q) + \end{aligned}$$

$$\begin{aligned}
 & + \frac{\beta^5}{m^3} \mathcal{U} e m \Gamma^{kr} \mathcal{U}_q (-60 \delta e m \delta r q \pi_k - 48 \delta r q \delta k e \pi_m - \\
 & - 48 \delta r e d m q \pi_k + 20 \frac{\beta}{m} \delta r q \pi e \pi m \pi_k) + \frac{\beta^4}{m^3} F_i^{ke} \cdot \\
 & \cdot (-24 \beta \mathcal{U} e \mathcal{U}_i \pi_k + 30 \mathcal{U} e i \pi_k) + 20 \frac{\beta^6}{m^3} F^{ki} \mathcal{U}_i \mathcal{U} e \mathcal{U} e \pi_k + \\
 & + \frac{\beta^4}{m^3} \mathcal{U} i e A m^k (-90 \delta i e d m r \pi_k - 60 \delta e k d m r \pi_i - \\
 & - 30 \delta i m d e r \pi_k - 30 \delta i k d e m \pi_r + \frac{\beta}{m} [30 \delta i e \pi_k \pi m \pi_r + \\
 & + 30 d m r \pi_k \pi i e + 12 \delta e k \pi i \pi m \pi_r + 48 d e r \pi i \pi k \pi m] - \\
 & - 10 \frac{\beta^2}{m^2} \pi i \pi e \pi k \pi m \pi_r) + \frac{\beta^5}{m^3} \mathcal{U} m \mathcal{U} m A i^k (30 \delta i e \pi_k - \\
 & - 10 \frac{\beta}{m} \pi i \pi e \pi_k) + \frac{\beta^4}{m^3} \mathcal{U} i e \Gamma^{mk} \Gamma^{qr} (45 d m q d i e \delta k r + \\
 & + 60 d m q d i k d e r - \frac{\beta}{m} [30 d i e d m q \pi_k \pi_r + 15 d m q d k r \pi i e + \\
 & + 48 d m q d e r \pi i \pi_k + 24 d i m d e q \pi_k \pi_r] + 10 \frac{\beta^2}{m^2} d m q \pi i \pi e \pi_k \pi_r) + \\
 & + \frac{\beta^5}{m^3} \mathcal{U} e \mathcal{U} m \Gamma^{ik} \Gamma^{rq} (-15 d e m d k q d i r - 24 d i r d e k d m q + \\
 & + \frac{\beta}{m} [10 d e m d i r \pi_k \pi_q + 20 d e k d m q \pi i \pi_r]) + \\
 & + \frac{\beta^4}{m^3} F_i^{ke} \Gamma^{km} \mathcal{U}_r (60 d i r d e m + \frac{\beta}{m} [24 d e r \pi i \pi_m - \\
 & - 48 d i r \pi e \pi_m]) + \frac{\beta^3}{m^3} F_i^{kr} \Gamma^{km} (-30 d i r d e m + \\
 & + 60 \frac{\beta}{m} d i r \pi e \pi_m - 12 \frac{\beta^2}{m^2} \pi i \pi_r \pi e \pi_m) + (\frac{\beta^3}{m^3} F_i^{ke} \Gamma_i^{km} - \\
 & - 4 \frac{\beta^3}{m^3} \Gamma^{ki} A i^k e m) (-30 d e m + 15 \frac{\beta}{m} \pi e \pi_m) +
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\beta^4}{m^3} F_{\mu i}^{\nu j} U_i A_{\mu p}^K (-60 \delta_{\mu k} \delta_{\nu p} + \frac{\beta}{m} [60 \delta_{\mu p} \pi_k \pi_e + \\
 & + 12 \delta_{\mu k} \pi_{\mu} \pi_p + 48 \delta_{\mu p} \pi_k \pi_{\mu}] - 20 \frac{\beta^2}{m^2} \pi_k \pi_e \pi_{\mu} \pi_p) + \\
 & + \frac{\beta^3}{m^3} A_{\mu e}^K A_{\mu q}^M (-45 \delta_{\mu k} \delta_{\nu e} \delta_{\mu q} - 30 \delta_{\mu k} \delta_{\nu p} \delta_{e q} + \\
 & + \frac{\beta}{m} [45 \delta_{\nu e} \delta_{\mu q} \pi_k \pi_{\mu} + 60 \delta_{\mu e} \delta_{\mu q} \pi_k \pi_i + 30 \delta_{\mu k} \delta_{\mu q} \pi_i \pi_e + \\
 & + 45 \delta_{\mu k} \delta_{\nu p} \pi_e \pi_q + 30 \delta_{\mu p} \delta_{\nu q} \pi_e \pi_{\mu} + 15 \delta_{\nu p} \delta_{\mu q} \pi_k \pi_{\mu}] - \\
 & - \frac{\beta^2}{m^2} [30 \delta_{\mu q} \pi_k \pi_i \pi_e \pi_{\mu} + 24 \delta_{\mu q} \pi_i \pi_k \pi_{\mu} \pi_p + \\
 & + 12 \delta_{\mu q} \pi_i \pi_e \pi_{\mu} \pi_p + 9 \delta_{\mu k} \pi_i \pi_e \pi_p \pi_q] + \\
 & + 5 \frac{\beta^3}{m^3} \pi_k \pi_i \pi_e \pi_{\mu} \pi_p \pi_q) + \frac{\beta^5}{m^4} F_{\mu i}^{\nu j} F_{\mu k}^{\nu l} U_i U_k (-30 \delta_{\mu k} \pi_p - \\
 & - 48 \delta_{\mu p} \pi_i + 20 \frac{\beta}{m} \pi_i \pi_{\mu} \pi_p) + \frac{\beta^4}{m^4} F_{\mu e}^{\nu j} F_{\mu k}^{\nu l} U_i U_k \cdot \\
 & \cdot (30 \delta_{\mu p} \pi_k + 60 \delta_{\mu k} \pi_p - 24 \frac{\beta}{m} \pi_k \pi_e \pi_p) + \frac{\beta^4}{m^4} F_{\mu k}^{\nu j} F_{\mu e}^{\nu l} A_{\mu q}^M \cdot \\
 & \cdot (-45 \delta_{\mu k} \delta_{\nu e} \delta_{\mu q} \pi_{\mu} - 60 \delta_{\mu e} \delta_{\nu q} \pi_k - 30 \delta_{\mu p} \delta_{\nu e} \pi_{\mu} - \\
 & - 30 \delta_{\mu k} \delta_{\nu p} \pi_q + \frac{\beta}{m} [15 \delta_{\mu k} \pi_{\mu} \pi_p \pi_q + 30 \delta_{\mu q} \pi_k \pi_e \pi_{\mu} + \\
 & + 48 \delta_{\mu p} \pi_k \pi_{\mu} \pi_q + 12 \delta_{\mu e} \pi_k \pi_p \pi_q] - 10 \frac{\beta^2}{m^2} \pi_k \pi_e \pi_{\mu} \pi_p \pi_q) + \\
 & + \frac{\beta^4}{m^4} F_{\mu k}^{\nu j} F_{\mu e}^{\nu l} F_{\mu p}^{\nu m} F_{\mu q}^{\nu n} (45 \delta_{\mu k} \delta_{\nu p} + 15 \delta_{\mu p} \delta_{\nu k} - \\
 & - \frac{\beta}{m} [15 \delta_{\mu k} \pi_p \pi_q + 24 \delta_{\nu p} \pi_k \pi_q] + 5 \frac{\beta^2}{m^2} \pi_k \pi_i \pi_p \pi_q)
 \end{aligned}$$

The average $\langle \mathcal{Z}_4 \rangle$ can be expressed in a variety of ways by integrations by parts such as (4.7). We write one of possible forms:

$$\begin{aligned} \langle \mathcal{Z}_4 \rangle = & \frac{1}{8 \cdot 6!} \left(-5 \frac{\beta^3}{m^2} \langle U_{iie} \rangle + 5 \frac{\beta^4}{m^2} \langle U_{iille} \rangle + 2 \frac{\beta^4}{m^2} \langle U_{ke} U_{ke} \rangle + \right. \\ (A.2.2) & + 5 \frac{\beta^4}{m^3} \langle F_{ki} F_{ki} U_{ee} \rangle - 6 \frac{\beta^3}{m^3} \langle F_{ii}^k F_{ke} \rangle + 4 \frac{\beta^3}{m^3} \langle F_{ki} F_{ke} U_{ie} \rangle - \\ & - 4 \frac{\beta^4}{m^3} \langle F_{ke} F_{ke} U_{ii} \rangle + \frac{5 \beta^4}{4 m^4} \langle F_{ik} F_{ik} F_{im} F_{im} \rangle + \frac{\beta^4}{m^4} \langle F_{ik} F_{ie} F_{im} F_{im} \rangle \Big) \end{aligned}$$

Utilizing the obtained result it is easy to get the approximate expression for the free energy up to order $\hbar^4$, which is written in the main text.

Appendix 3

The properties of the Weyl transform

Let us extract the known quantum mechanics relations (the material of this appendix is based on the book [8]):

a) The relations of completeness:

$$(A.3.1) \quad \int d\bar{p} |\bar{p}\rangle \langle \bar{p}| = I$$

$$(A.3.2) \quad \int d\bar{q} |\bar{q}\rangle \langle \bar{q}| = I \quad \text{where } I \text{ is unitary operator}$$

b) The scalar products

$$(A.3.3) \quad \langle \bar{p} | \bar{p}' \rangle = \delta(\bar{p} - \bar{p}') ; \langle \bar{q} | \bar{q}' \rangle = \delta(\bar{q} - \bar{q}')$$

$$(A.3.4) \quad \langle \bar{q} | \bar{p} \rangle = \frac{1}{(2\pi\hbar)^{N/2}} e^{(i/\hbar) \bar{p} \cdot \bar{q}}$$

where N is the number of degrees of freedom or the dimension

sionality of $\bar{P}$ and $\bar{Q}$, where $|\bar{p}\rangle$ and $|\bar{q}\rangle$ are eigenvectors of operators of momentum and coordinate, i.e. these vectors satisfy equations:

$$(A.3.5) \quad \begin{aligned} \hat{P}|\bar{p}\rangle &= \bar{p}|\bar{p}\rangle \\ \hat{Q}|\bar{q}\rangle &= \bar{q}|\bar{q}\rangle \end{aligned}$$

Using (A.3.1, A.3.2) we can write the following identity for an arbitrary operator $\hat{A}$:

$$(A.3.6) \quad \hat{A} = \int d\bar{p}' d\bar{p}'' d\bar{q}' d\bar{q}'' |\bar{p}'\rangle \langle \bar{p}''| \hat{A} |\bar{q}''\rangle \langle \bar{q}'|$$

Introducing the new variables:

$$\begin{aligned} \bar{p}' &= \bar{p} + \frac{1}{2}\bar{u} & \bar{q}' &= \bar{q} + \frac{1}{2}\bar{v} \\ \bar{p}'' &= \bar{p} - \frac{1}{2}\bar{u} & \bar{q}'' &= \bar{q} - \frac{1}{2}\bar{v} \end{aligned}$$

with unitary Jacobian and using (A.3.4), we get from (A.3.6):

$$(A.3.7) \quad \hat{A} = \frac{1}{(2\pi\hbar)^n} \int d\bar{p} d\bar{q} a(\bar{p}, \bar{q}) \Delta(\bar{p}, \bar{q})$$

where

$$(A.3.8) \quad a(\bar{p}, \bar{q}) = \int d\bar{v} e^{(i/\hbar)\bar{p}\cdot\bar{v}} \langle \bar{q} - \frac{\bar{v}}{2} | \hat{A} | \bar{q} + \frac{\bar{v}}{2} \rangle$$

$$(A.3.9) \quad \Delta(\bar{p}, \bar{q}) = \int d\bar{u} e^{(i/\hbar)\bar{q}\cdot\bar{u}} |\bar{p} - \frac{\bar{u}}{2}\rangle \langle \bar{p} + \frac{\bar{u}}{2}|$$

Rearranging the unitary operators in the identity (A.3.6)

$$\hat{A} = \int d\bar{p}' d\bar{p}'' d\bar{q}' d\bar{q}'' |\bar{p}'\rangle \langle \bar{p}''| \hat{A} |\bar{q}''\rangle \langle \bar{q}'|$$

and making the above substitution of variables, we get the other form for $\Delta(\bar{p}, \bar{q})$ and $a(\bar{p}, \bar{q})$

$$(A.3.10) \quad a(\bar{p}, \bar{q}) = \int d\bar{u} e^{(i/\hbar)\bar{q} \cdot \bar{u}} \langle \bar{p} + \frac{\bar{u}}{2} | \hat{A} | \bar{p} - \frac{\bar{u}}{2} \rangle$$

$$(A.3.11) \quad \Delta(\bar{p}, \bar{q}) = \int d\bar{v} e^{(i/\hbar)\bar{p} \cdot \bar{v}} \langle \bar{q} + \frac{\bar{v}}{2} | \langle \bar{q} - \frac{\bar{v}}{2} |$$

One can show that $a(\bar{p}, \bar{q})$ and $\Delta(\bar{p}, \bar{q})$ defined by the formulae (A.3.8-A.3.11) are identical respectively [8].

Using the known identity

$$(A.3.12) \quad \text{Tr}(\hat{A} | \bar{q} \rangle \langle \bar{q}' |) = \langle \bar{q}' | \hat{A} | \bar{q} \rangle$$

and the definitions (A.3.8), (A.3.11) we get for a $a(\bar{p}, \bar{q})$:

$$\begin{aligned} \text{Tr}(\hat{A} \Delta(\bar{p}, \bar{q})) &= \text{Tr}(\hat{A} \int d\bar{v} e^{(i/\hbar)\bar{p} \cdot \bar{v}} | \bar{q} + \frac{\bar{v}}{2} \rangle \langle \bar{q} - \frac{\bar{v}}{2} |) = \\ &= \int d\bar{v} e^{(i/\hbar)\bar{p} \cdot \bar{v}} \langle \bar{q} - \frac{\bar{v}}{2} | \hat{A} | \bar{q} + \frac{\bar{v}}{2} \rangle = a(\bar{p}, \bar{q}) \end{aligned}$$

or briefly

$$(A.3.13) \quad a(\bar{p}, \bar{q}) = \text{Tr}(\hat{A} \Delta(\bar{p}, \bar{q}))$$

From (A.3.12) and (A.3.2) it is easy to get for $\hat{A}$:

$$(A.3.14) \quad \text{Tr} \hat{A} = \int d\bar{q} \langle \bar{q} | \hat{A} | \bar{q} \rangle$$

From here we have for $\Delta(\bar{p}, \bar{q})$ (using (A.3.11) and (A.3.3)):

$$\text{Tr} \Delta(\bar{p}, \bar{q}) = \int d\bar{q}' d\bar{v} \delta(\bar{q}' - \bar{q} - \frac{\bar{v}}{2}) \delta(\bar{q} - \frac{\bar{v}}{2} - \bar{q}') e^{(i/\hbar)\bar{p} \cdot \bar{v}} = 1$$

Hence, utilizing (A.3.7), follows the relation:

$$(A.3.15) \quad \text{Tr } \hat{A} = \frac{1}{(2\pi\hbar)^N} \int d\bar{p} d\bar{q} a(\bar{p}, \bar{q})$$

Let us find the Weyl transform of $\Delta(\bar{p}, \bar{q})$. In order to get it, it is necessary to substitute $\Delta(\bar{p}, \bar{q})$ defined by (A.3.11) for $\hat{A}$ in (A.3.10):

$$\begin{aligned} \Delta(\bar{p}, \bar{q}) &\Rightarrow \delta_w(\bar{p}, \bar{q}) = \\ &= \frac{1}{(2\pi\hbar)^N} \int d\bar{u} d\bar{v} e^{(i/\hbar)\bar{u} \cdot (\bar{q} - \bar{q}') - (i/\hbar)\bar{v} \cdot (\bar{p} - \bar{p}')} \end{aligned}$$

From here we obtain the Weyl transform of $\Delta(\bar{p}, \bar{q})$

$$(A.3.16) \quad \delta_w(\bar{p}, \bar{q}) = (2\pi\hbar)^N \delta(\bar{p} - \bar{p}') \delta(\bar{q} - \bar{q}')$$

Then from (A.3.13) we have:

$$\text{Tr}(\Delta(\bar{p}, \bar{q}) \Delta(\bar{p}', \bar{q}')) = (2\pi\hbar)^N \delta(\bar{p} - \bar{p}') \delta(\bar{q} - \bar{q}')$$

Using the obtained relation it is easy to get the formula for the trace of the product of two operators:

$$(A.3.17) \quad \text{Tr } \hat{A}\hat{B} = \text{Tr } \hat{B}\hat{A} = \frac{1}{(2\pi\hbar)^N} \int d\bar{p} d\bar{q} a(\bar{p}, \bar{q}) b(\bar{p}, \bar{q})$$

Here after we shall feel the need of trace of three Δ -operators. From (A.3.14), using (A.3.2) and (A.3.11) we find:

$$\begin{aligned}
 & \text{Tr}(\Delta(\bar{p}, \bar{q}) \Delta(\bar{p}', \bar{q}') \Delta(\bar{p}'', \bar{q}'')) = \\
 & = \int d\bar{q}_1 d\bar{q}_2 d\bar{q}_3 \langle q_1 | \Delta | q_2 \rangle \langle q_2 | \Delta' | q_3 \rangle \langle q_3 | \Delta'' | q_1 \rangle = \\
 & = \int d\bar{q}_1 d\bar{q}_2 d\bar{q}_3 \delta(\bar{q} - \frac{\bar{q}_1 + \bar{q}_2}{2}) \cdot \\
 & \cdot \delta(\bar{q}' - \frac{\bar{q}_2 + \bar{q}_3}{2}) \delta(\bar{q}'' - \frac{\bar{q}_3 + \bar{q}_1}{2}) \cdot \\
 & \exp\left\{\frac{i}{\hbar}(\bar{p} \cdot (\bar{q}_1 - \bar{q}_2) + \bar{p}' \cdot (\bar{q}_2 - \bar{q}_3) + \bar{p}'' \cdot (\bar{q}_3 - \bar{q}_1))\right\}
 \end{aligned}$$

Changing the variables

$$\frac{\bar{q}_1 + \bar{q}_2}{2} = \bar{k}_1, \quad \frac{\bar{q}_2 + \bar{q}_3}{2} = \bar{k}_2, \quad \frac{\bar{q}_3 + \bar{q}_1}{2} = \bar{k}_3$$

and integrating we get finally

$$\begin{aligned}
 & \text{Tr}(\Delta(\bar{p}, \bar{q}) \Delta(\bar{p}', \bar{q}') \Delta(\bar{p}'', \bar{q}'')) = \\
 & = 2^{2N} \exp\left\{\frac{2i}{\hbar}(\bar{p} \cdot (\bar{q}'' - \bar{q}') + \bar{p}' \cdot (\bar{q} - \bar{q}'') + \bar{p}'' \cdot (\bar{q}' - \bar{q}))\right\}
 \end{aligned}$$

Proceeding from the (A.3.13) and using the obtained formula, for Weyl transform of two operators, which is not generally the product

of Weyl transforms of these operators we get:

$$\begin{aligned}
 \hat{A}\hat{B} &= \left(\frac{2}{2\pi\hbar}\right)^{2N} \int d\bar{p}' d\bar{q}' d\bar{p}'' d\bar{q}'' a(\bar{p}', \bar{q}') b(\bar{p}'', \bar{q}'') \cdot \\
 & \cdot \exp\left\{\frac{2i}{\hbar}((\bar{q}' - \bar{q}) \cdot (\bar{p}'' - \bar{p}) - (\bar{p}' - \bar{p}) \wedge (\bar{q}'' - \bar{q}))\right\}
 \end{aligned}$$

Introducing the new variables $\hat{p} = \bar{p}'' - \bar{p}$ and $\hat{q} = \bar{q}'' - \bar{q}$ and expanding $b(\bar{p} + \hat{p}, \bar{q} + \hat{q})$ by its Taylor series expansion at $b(\bar{p}, \bar{q})$ we obtain:

$$\hat{A}\hat{B} \Rightarrow \left(\frac{2}{2\pi\hbar}\right)^{2N} \int d\bar{p}' d\bar{q}' d\hat{p} d\hat{q} a(\bar{p}, \bar{q}) \cdot \exp\left[\frac{2i}{\hbar}(\bar{q}' - \bar{q})\hat{p} - (\bar{p}' - \bar{p})\hat{q}\right] \exp\left[\hat{p} \cdot \frac{\partial}{\partial \bar{p}} + \hat{q} \cdot \frac{\partial}{\partial \bar{q}}\right] b(\bar{p}, \bar{q})$$

In the last exponent one may substitute $\hat{p}$ by differential operator $-(\hbar/2i)\frac{\partial}{\partial \bar{q}}$, and $\hat{q}$ by $(\hbar/2i)\frac{\partial}{\partial \bar{p}}$, (the arrowed lines show that the operators act on the left). Then, integrating for $\hat{p}$ and $\hat{q}$, we get the δ -functions, the integration for $\bar{p}'$ and $\bar{q}'$ becomes possible. We obtain:

$$(A.3.18) \hat{A}\hat{B} \Rightarrow a(\bar{p}, \bar{q}) \exp\left[\frac{\hbar}{2i}\left(\frac{\partial^a}{\partial \bar{q}} \cdot \frac{\partial^b}{\partial \bar{p}} - \frac{\partial^a}{\partial \bar{p}} \cdot \frac{\partial^b}{\partial \bar{q}}\right)\right] b(\bar{p}, \bar{q})$$

where $\frac{\partial^a}{\partial \bar{p}}$ and $\frac{\partial^b}{\partial \bar{q}}$ act on $a(\bar{p}, \bar{q})$. The consequence of (A.3.18) is the formulae of Weyl transform of commutator and anticommutator of operators $\hat{A}$ and $\hat{B}$:

$$(A.3.19) [\hat{A}, \hat{B}] \Rightarrow 2sh\left[\frac{\hbar}{2}\left(\frac{\partial^a}{\partial \bar{q}} \cdot \frac{\partial^b}{\partial \bar{p}} - \frac{\partial^a}{\partial \bar{p}} \cdot \frac{\partial^b}{\partial \bar{q}}\right)\right] a(\bar{p}, \bar{q}) b(\bar{p}, \bar{q})$$

$$(A.3.19) [\hat{A}, \hat{B}]_+ \Rightarrow 2ch\left[\frac{\hbar}{2}\left(\frac{\partial^a}{\partial \bar{q}} \cdot \frac{\partial^b}{\partial \bar{p}} - \frac{\partial^a}{\partial \bar{p}} \cdot \frac{\partial^b}{\partial \bar{q}}\right)\right] a(\bar{p}, \bar{q}) b(\bar{p}, \bar{q})$$

From (A.3.18), as a natural generalization, the formula of Weyl transform of the product of N operators follows

$$\hat{A}_1 \dots \hat{A}_n \rightleftharpoons a_1(\bar{p}, \bar{q}) \dots a_n(\bar{p}, \bar{q}).$$

$$(A.3.21) \cdot \exp\left\{\frac{i\hbar}{2} \sum_{i,j}^n \left(\frac{\partial^i}{\partial \bar{q}} \cdot \frac{\partial^j}{\partial \bar{p}} - \frac{\partial^i}{\partial \bar{p}} \cdot \frac{\partial^j}{\partial \bar{q}} \right)\right\}$$

where $\frac{\partial^i}{\partial \bar{q}}$ and $\frac{\partial^i}{\partial \bar{p}}$ act on $a_i(\bar{p}, \bar{q})$

In conclusion we get Weyl transform of Hamiltonian

$$\hat{H} = \frac{M_{ik}}{2} (\hat{p}_i - A^i)(\hat{p}_k - A^k) + \mathcal{U}(\hat{q})$$

where $A^i = A^i(\hat{q})$

$$\hat{H} = \frac{M_{ik}}{2} (\hat{p}_i \hat{p}_k + A^i A^k - [\hat{p}_i, A^k]_+) + \mathcal{U}(\hat{q})$$

Here the symmetry of M_{ik} is used. From this form of $\hat{H}$ using formulae (A.3.18) and (A.3.20), it is easy to find:

$$\hat{H} \rightleftharpoons h_a(\bar{p}, \bar{q}) = \frac{M_{ik}}{2} (p_i p_k + A^i A^k - 2p_i A^k) + \mathcal{U}(\bar{q})$$

and transforming back, we get:

$$(A.3.22) \quad h_a(\bar{p}, \bar{q}) = \frac{M_{ik}}{2} (p_i - A^i)(p_k - A^k) + \mathcal{U}(\bar{q})$$

Here we use the following correspondence:

$$f(\hat{p}) \rightleftharpoons f(\bar{p}) ; f(\hat{q}) \rightleftharpoons f(\bar{q})$$

It is easy to get these relations, using the formulae (A.3.10) and (A.3.8) respectively.

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