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HIGHER DERIVATIVE QUANTUM GRAVITY IN TWO DIMENSIONS

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ABSTRACT

It is proposed to study geometrodynamics in two space-time dimensions described by a higher derivative action. The Hamilton-Jacobi equation in the superspace approach is exactly solved. Based on the WKB approximation, we argue that the system is stable. It is suggested that within a canonical framework for the exact quantization, there is a built-in mechanism by which the physical cosmological constant always vanishes, irrespective of its bare value.

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In view of the divergence difficulty in constructing the quantized gravity theory, it seems not unreasonable to suppose that the Einstein theory of gravity is a low-energy effective theory. Then, one of the attractive candidates for the microscopic action of quantum gravity is the conformally invariant Weyl action. Although the Weyl action contains higher derivatives, there has been a hope [1] that this defines a renormalizable and asymptotically free quantum theory of gravity. The usual scenario presupposed in this approach is that the breaking of scale invariance, necessarily associated with quantization, would result in the Einstein theory in the low-energy region.

Clearly the realization (or disproof) of this program requires advanced understanding of the non-perturbative properties of the higher derivative theories with general coordinate (and wider) invariance. The situation is somewhat reminiscent of the confinement problem in quantum chromodynamics. The ideas -- such as renormalization group, lattice regularization, instanton gas, various analogies with lower dimensional models, and so on -- which have been useful for unveiling the non-perturbative structure of Yang-Mills theories, may presumably be fruitful for investigating different aspects of the non-perturbative quantum geometrodynamics.

The purpose of the present note is to study a higher derivative quantum gravity in two dimensions. Of course, in two dimensions there is no conformal (or scale) invariance in higher derivative theories. Worse than this, it is not clear what, if any, should be the two-dimensional analogue of the Einstein theory. It is, however, perhaps possible that the common key feature -- the coexistence of higher derivatives and general coordinate invariance -- makes the two-dimensional system share some of the important characteristics of the higher dimensional systems. After all,

the scale invariance in four dimensions is broken and the theory becomes infrared unstable. In a broad sense, the two-dimensional higher derivative theory is also infrared unstable, just as the two-dimensional gauge theories are in general. It should be noted also that the lowest order scalar potentials, if they are meaningful at all, would be "confining" both in four and, more strongly, in two dimensions. The higher derivative gravity in two dimensions thus seems to be not an unworthy subject to study.

For simplicity, we consider the model which contains the derivatives up to fourth order. Then the most general (apart from a total derivative) local action for pure gravity is^{*)}

$$A = \int \left(-\frac{1}{2\kappa^2} R^2 + \alpha R - \Lambda \right) \sqrt{-g} d^2x \quad (1)$$

The usual covariant perturbational approach is totally inapplicable to (1) owing to the infrared divergences. We therefore study this system in the framework of the so-called "superspace" (not to be confused with the usage in SUSY theories) quantization [2]. Incidentally, it will turn out that the system is also interesting as a possible non-trivial testing ground of this important, if yet rather unfamiliar, approach to the quantization of gravity.

We have first to explore the Hamiltonian structure of the system. By introducing an auxiliary scalar field χ , (1) can equivalently be rewritten as

^{*)} Here, the sign of the R^2 term is chosen so that it has the correct sign when continued to the positive metric Riemannian space. Classically, however, the sign is irrelevant. See below.

$$A^x = \int \left(\frac{\kappa^2}{2} \chi^2 + (\chi + \gamma) R - \Lambda \right) \sqrt{-g} d^2x \quad (2)$$

The metric tensor is decomposed in the standard form

$$g_{\mu\nu} = e^\Phi \begin{pmatrix} (\eta^1)^2 - (\eta^\perp)^2 & \eta^1 \\ \eta^1 & 1 \end{pmatrix} \quad (3)$$

using the lapse, $e^{\Phi/2} \eta^\perp$, and shift, η^1 , functions. In order to eliminate the second derivatives in (2), the R term must be replaced by $V^\alpha \partial_\alpha (\chi + \gamma)$ by adding a surface term, where $R/\sqrt{-g} = -\partial_\alpha V^\alpha$, with $V^0 = -(\dot{\phi} - \eta^1 \phi' - 2\eta^{1'})/\eta^\perp$ and $V^1 = \eta^1 V^0 + 2e^{-\Phi/2} (e^{\Phi/2} \eta^\perp)'$. Thus the constant γ plays no role in the Hamiltonian formulation. A^X now takes the canonical form

$$A^X = \int d^2x [\dot{\chi} P + \dot{\phi} \pi - \eta^\perp \mathcal{H}_\perp - \eta^1 \mathcal{H}_1], \quad (4)$$

$$\mathcal{H}_\perp = -\pi P + 2\chi'' - \phi' \pi' - \frac{1}{2} \kappa^2 e^\Phi \chi^2 + \Lambda e^\Phi \quad (5a)$$

$$\mathcal{H}_1 = \chi' P + \phi' \pi - 2\pi' \quad (5b)$$

where π and P are the momenta conjugate to ϕ and χ , respectively

$$([\phi(x), \pi(y)]_{PB} = [\chi(x), P(y)]_{PB} = \delta(x, y)):$$

$$\pi = -(\eta^\perp)^{-1} (\dot{\chi} - \eta^1 \chi') \quad (6a)$$

$$P = -(\eta^\perp)^{-1} (\dot{\phi} - \eta^1 \phi' - 2\eta^{1'}) \quad (6b)$$

Formulae (5) satisfy the closed algebra [3] without central charge:

$$[\mathcal{H}_\perp(x), \mathcal{H}_\perp(y)]_{PB} = \partial_x \delta(x, y) (\mathcal{H}_\perp(x) + \mathcal{H}_\perp(y)) \quad (7a)$$

$$[\mathcal{H}_\perp(x), \mathcal{H}_\perp(y)]_{PB} = \partial_x \delta(x, y) (\mathcal{H}_\perp(x) + \mathcal{H}_\perp(y)) \quad (7b)$$

$$[\mathcal{H}_\perp(x), \mathcal{H}_\perp(y)]_{PB} = \partial_x \delta(x, y) (\mathcal{H}_\perp(x) + \mathcal{H}_\perp(y)) \quad (7c)$$

These are the crucial relations ensuring the general covariance of the Hamiltonian formulation. The equations of motion demand that $H_\perp = H_1 = 0$, corresponding to the existence of the Lagrangian multiplier fields η^μ .

Before proceeding to quantization, let us here examine two relevant questions. Since this system, with two pairs of canonical field variables, has two first-class constraints at each point, it might seem that there is no dynamics unless matter fields are included. This is of course correct in the sense that the system does not possess gravitational wave. However, it does not necessarily mean that the system has no dynamical content, since there is evidence for the existence of an infinite number of solutions which are not diffeomorphic to each other. Let us see why this can happen. We first note that χ is a world scalar directly proportional to the curvature scalar. A property which is related to this is that the equation of motion describing the time evolution of χ is, in contrast with the one for φ , homogeneous with respect to the canonical variables: $\dot{\chi} = -\eta^\perp \pi + \eta^1 \chi'$. Suppose that there are solutions for which π and χ' are not everywhere zero. Then, if each such solution was diffeomorphic to another, or in other words there was only one space-time, an infinitesimal change of the permissible initial data for χ at a Cauchy surface (= curve)

should have been equivalently described by an infinitesimal gauge transformation, which amounts to an infinitesimal time development from one Cauchy surface to a different Cauchy surface for a suitable choice of the lapse and shift functions. This is, however, impossible when there are regions (or points) where π and χ' vanish simultaneously. That π and χ' can be simultaneously zero at some regions, but not be zero everywhere, is easy to see by inspecting the constraint equations $H_{\perp}(x) = H_1(x) = 0$. This indicates^{*)} that the choices of the initial data for χ and π with different distributions of the simultaneous zeros of χ' and π yield, in general, different space-times.

The next question is the definition of the energy of this system. This can easily be answered by following the general analysis of Regge and Teitelboim [4]. If the Cauchy surface is not a closed space, the Hamiltonian $H = \int (\eta^{\perp} H_{\perp} + \eta^1 H_1) dx^1$ is not, in general, functionally differentiable (and hence cannot generate the equations of motion) when η^{μ} does not vanish at infinity. For example, if $\eta^{\perp} \neq 0$ and $\eta^1 = 0$ at infinity, H picks up the surface term $2\eta^{\perp} \delta\chi' - \delta\psi\chi'\eta^{\perp} - \eta^{\perp}\psi'\delta\chi - 2\eta^{\perp}\delta\chi$ under the variation of ψ and χ . The functional differentiability is restored if this surface term can be consistently subtracted. Then the energy is equal to the subtraction term, because of the constraint $H_{\perp} = 0$. This is only possible when the surface term is also a total variation. It is easy to see that the unique choice for $\eta^{\perp}$ to meet this requirement is $\eta^{\perp} = e^{-\psi/2}$ at infinity, apart from an arbitrary proportional constant. The energy is then given by

*) This argument is certainly not rigorous since the possibility of finite gauge transformation is not considered. Taking this into account would require much more sophisticated mathematical analysis.

$$E = -2 \left(e^{-\frac{\phi(\infty)}{2}} \chi'(\infty) - e^{-\frac{\phi(-\infty)}{2}} \chi'(-\infty) \right) \quad (8)$$

Formula (8) already shows that asymptotically flat ($\Lambda = 0$) solutions [or more generally, any solutions which have constant curvature ($\Lambda \neq 0$) at infinity] always have zero energy. Formula (8) is also valid when arbitrary matter fields are included, provided that the Hamiltonian formulation is possible. This result is the two-dimensional analogue of the classical zero energy theorem [5] recently discovered for the four-dimensional (scale invariant) higher derivative gravity, and shows that the system is classically stable in contrast with the ordinary higher derivative field theories without general covariance.

Let us now proceed to quantization. We assume that the classical constraints should be interpreted as conditions (called the Wheeler-DeWitt equation) on permissible state vectors in quantum theory, in analogy with the Klein-Gordon equation for the case of a relativistic point particle. We first give a step toward the WKB approximation, without bothering about the precise nature of the representation of the canonical commutation relations (CCR). Namely, we take the formal representation in which ϕ and χ are diagonal and $\pi(x) = (\hbar/i) \delta/\delta\phi(x)$, $P(x) = (\hbar/i) \delta/\delta\chi(x)$. At this level, there is no problem of the operator ordering, and (7) is still valid without central charge. Then, by assuming the WKB form $\Psi = A \exp iW/\hbar$ for the wave functional, we find

$$-\frac{\delta W}{\delta\phi(x)} \frac{\delta W}{\delta\chi(x)} + 2\chi''(x) - \phi'(x)\chi'(x) - \left(\frac{\kappa^2}{2}\chi(x)^2 - \Lambda\right)e^{\phi(x)} = 0 \quad (9a)$$

$$\chi'(x) \frac{\delta W}{\delta\chi(x)} + \phi'(x) \frac{\delta W}{\delta\phi(x)} - 2\left(\frac{\delta W}{\delta\phi(x)}\right)' = 0 \quad (9b)$$

$$\frac{\delta}{\delta \phi(x)} \left(A^2 \frac{\delta W}{\delta \chi(x)} \right) + \frac{\delta}{\delta \chi(x)} \left(A^2 \frac{\delta W}{\delta \phi(x)} \right) = 0 \quad (10a)$$

$$\chi'(x) \frac{\delta A}{\delta \chi(x)} + \phi'(x) \frac{\delta A}{\delta \phi(x)} - 2 \left(\frac{\delta A}{\delta \phi(x)} \right)' = 0 \quad (10b)$$

Formulae (9b) and (10b) merely express the reparametrization invariance on a given Cauchy curve, and the real dynamical content is expressed by (9a) and (10a). Remarkably, the Hamilton-Jacobi equation (9a) with (9b) is exactly soluble. The general solution is given by

$$W = \int dx w_{(x)}[\phi, \chi] + \text{const.} \quad (11)$$

$$\begin{aligned} \pm w_{(x)}[\phi, \chi] = & 2 \sqrt{\chi'^2 + e^\phi (c - \frac{1}{6} k^2 \chi^3 + \Lambda \chi)} \\ & + \chi' \log \left| \frac{\sqrt{\chi'^2 + e^\phi (c - \frac{1}{6} k^2 \chi^3 + \Lambda \chi)} - \chi'}{\sqrt{\chi'^2 + e^\phi (c - \frac{1}{6} k^2 \chi^3 + \Lambda \chi)} + \chi'} \right| \end{aligned}$$

where c is an arbitrary integration constant. Classically, c can be fixed by imposing boundary conditions at infinity.

In this note we do not elaborate on determining the amplitude A ; it requires a more precise definition of the operator products at coincident points, such as the one we shall propose later. Assuming that it can be done, we can suppose, on the basis of the "continuity equation" (10a), that A has a power-behaved asymptotic form with respect to the arguments χ and $\exp \phi$, since the momenta $\delta W / \delta \chi$ and $\delta W / \delta \phi$ are power-behaved. Now, when W is real and sufficiently large, Ψ will take the form

$$\Psi \sim \int dx \left\{ A_+[\phi, x, c] e^{\frac{i}{\hbar} \int w_x[\phi, x, c] dx} + A_-[\phi, x, c] e^{-\frac{i}{\hbar} \int w_x[\phi, x, c] dx} \right\} \quad (12a)$$

and, when ϕ and x lie deep in the classically forbidden region and hence W is imaginarily large,

$$\Psi \sim \int dx A[\phi, x, c] e^{-\frac{1}{\hbar} \left| \int w_x[\phi, x, c] dx \right|} \quad (12b)$$

Using these forms we now argue that the wave functional only consists of the states with non-negative total energy. By spatial reparametrization invariance we can always assume that $\phi(\pm\infty) = 1$. For a state having a definite energy, $\chi'(\pm\infty)$ must then be well defined. Let us therefore assume that $\chi(x) \rightarrow \tilde{c}_\pm x$ for $x \rightarrow \pm\infty$. Negative energy can occur only when either i) $\tilde{c}_+ > 0$ or ii) $\tilde{c}_+ < 0$, $\tilde{c}_- < 0$. Evidently, we then have $\chi(+\infty) \rightarrow +\infty$ for (i) and $\chi(-\infty) \rightarrow +\infty$ for (ii). Provided that $\kappa^2 > 0$, this and (11) show that the WKB phase always becomes imaginarily infinite for the negative energy states, and hence^{*)} that the wave functional never contains the negative energy states. This should be valid beyond the WKB approximation, since it becomes accurate when the WKB phase is large. In fact, we may further infer that the physically realizable states would be only of zero energy. This is because $\tilde{c}_+$ and $\tilde{c}_-$ cannot be simultaneously zero for finite energy states. Then the momentum $\delta W / \delta \phi(x) = \pm [\chi'^2 + e^\psi (c - (1/6) \kappa^2 \chi^3 + \Lambda \chi)]^{1/2}$ diverges at spatial infinity. If the momentum diverges, the continuity equation (10a) dictates that the "density" A^2 must vanish, relative to the zero-energy states for which $\delta W / \delta \phi(x)$ is finite, even if W is not imaginarily. Thus, energy seems to be absolutely "confined" or rather

*) Here we assumed that the physical wave functional should not be exponentially divergent.

"screened"^{*)}. Perhaps one of the key problems in the conformal quantum gravity will be how to evade energy confinement or screening. We have constructed the locally supersymmetric version of the model (1) and have confirmed that not only the energy but also the global supercharge is screened semi-classically in that case.

However, to make these arguments well founded beyond the WKB approximation, we would need more precise definitions for the representation space of CCR and for the local operator products. In the remaining part of this note, we shall suggest a possible approach to this problem.

It is known that the formal representation diagonalizing the fields is not well defined. At the best we could only approach such a representation by some limiting procedure. A traditional way of proceeding is to define the operator fields as expansions over some reasonably complete set of independent functions. Although this is perhaps not essential, let us for the moment assume that the cosmological constant is zero and that space-times are asymptotically flat. Then it seems natural to try the Fourier expansion^{**)}.

$$\begin{aligned}
 \phi(x) &= \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{4\pi\omega_1(k)}} (e^{ikx} a(k) + e^{-ikx} a^+(k)) , \\
 \pi(x) &= -i \int_{-\infty}^{\infty} dk \sqrt{\frac{\omega_1(k)}{4\pi}} (e^{ikx} a(k) - e^{-ikx} a^+(k)) , \\
 \chi(x) &= \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{4\pi\omega_2(k)}} (e^{ikx} b(k) + e^{-ikx} b^+(k)) , \\
 P(x) &= -i \int_{-\infty}^{\infty} dk \sqrt{\frac{\omega_2(k)}{4\pi}} (e^{ikx} b(k) - e^{-ikx} b^+(k)) .
 \end{aligned}
 \tag{13}$$

*) For a related result in four dimensions based on a strong coupling expansion, see a recent work by Kaku [6].

***) If the Cauchy curve is closed, the Fourier integral is replaced by the Fourier series, which does not change the following discussions.

The CCRs are reproduced by demanding that $[a(k), a^\dagger(k')] = [b(k), b^\dagger(k')] = \hbar\delta(k-k')$. (All other commutators are zero.) We define the CCR representation space by introducing the "Fock vacuum" $|0\rangle$, $[a(k)|0\rangle = b(k)|0\rangle = 0]$. The property of $\omega_i(k)$ [$= \omega_i(-k)$] ($i = 1, 2$) will further specify the nature of the representation. On this basis, the local operator products are naturally defined by the normal products with respect to $|0\rangle$.

Now we require that i) the operators $:H_\perp:$ and $:H_\parallel:$ satisfy the algebra (7); ii) there exist solutions to the constraint equations. For the first condition we try to check whether the directly normal-ordered forms of (5) meet our demand. By inspection, it becomes clear that the only dangerous part of the algebra after the normal ordering is (7b). Specifically the operator $:e^{\Phi_X^2}: in :H_\perp:$ may not transform correctly under the spatial reparametrization,

$$[:\mathcal{H}_1(x):, :x(y)^2:] = -i\hbar\delta(x,y)\partial_y :x(y)^2: - \hbar \sum_1(x,y)\delta(x,y) \quad (14a)$$

$$\begin{aligned} [: \mathcal{H}_1(x):, :e^{P\Phi(y)}:] &= -i\hbar\delta(x,y)\partial_y :e^{P\Phi(y)}: + 2i\hbar p :e^{P\Phi(y)}: \partial_x \delta(x,y) \\ &\quad - \frac{1}{2} p^2 \hbar \sum_2(x,y) :e^{P\Phi(y)}: \delta(x,y) \end{aligned} \quad (14b)$$

$$\sum_2(x,y) = \frac{\hbar}{4\pi} \int_{-\infty}^{\infty} dk \frac{k}{\omega_2(k)} e^{ik(x-y)} \quad (15)$$

The last terms in (14), which are absent for classical Poisson bracket relations, may prevent the closing of the algebra. Within the present framework there are only two possibilities of resolving this difficulty. One is obviously that Γ_i 's have no short-distance singularity and hence

vanish like $x-y$ as $x \rightarrow y$. The other is that Γ_i 's do not vanish and behave like $i\pi\alpha_i^2/2\pi(x-y)$ at short distances. In the latter case the anomalous terms in (14) are both proportional to $\partial_x \delta(x,y)^*$. Then the classical transformation property of $e^{\Phi_X^2}$ is restored and the whole algebra closes without central charge as given in (7), if $e^{\Phi_X^2}$ is replaced by

$$: e^{p\phi} (x^2 + a\phi) : \quad (16)$$

provided that p and a are determined by $p(1 - \alpha_2^2 \hbar p/8\pi) = 1$ and $a = \alpha_1^2 \hbar p/4\pi(2-p)$. The second term in the parenthesis compensates the anomaly term of (14a). The first possibility corresponds to the special case $\alpha_1^2 = 0$.

On dimensional grounds, the second possibility is much more natural than the first one, since the latter necessarily implies that the operator products $\pi(x)\pi(y)$ or $P(x)P(y)$ exhibit stronger singularities than $(x-y)^{-3^{**}}$, at least, for the matrix elements between any normalizable states.

There still remains the second major question whether the Wheeler-DeWitt equation admits acceptable, at least mathematically, solutions within our quantization scheme, to which we have no clear answer. Certainly, the closed algebra (7) is necessary for the existence of solutions. We do not, however, know whether it is sufficient or not. Most likely it is not, and further conditions for restricting the CCR representation [7] must perhaps be found. It should be noted here that the criterion for the "acceptable" solution is, however, by no means clear. In the first place, the possibility of an infinite renormalization of the undetermined constants (κ^2, Λ) should still not be excluded. Secondly, the

*) For a related computation, see also [8].

***) Note that in this case $\omega(k) \sim O(|k|^{2+\epsilon})$ ($\epsilon > 0$) for large $|k|$. The second possibility corresponds to $\omega_1(k) \sim (1/\alpha_1^2) |k|$.

normalizability of the wave functional should not be required; otherwise CCR would be incompatible with the constraints. Remember that the solutions of the Klein-Gordon equation with Minkowski metric are not normalizable, either, in four dimensions; they become normalizable in spatial dimensions after the time variable is identified. Even if the wave functional is not normalizable in the ordinary sense, it may be possible, provided that $|\Psi|^2$ has no non-integrable singularity, to assign a reasonable physical interpretation to the wave functional based on the notion of "relative" or "conditional" probability, referring only to the correlations of the events. This is, however, an open question. The problem of unitarity could only be properly formulated when the interpretation of the wave functional is settled. In connection with this, we note that negative metric has not been required at all in our discussions.

With these warnings in mind, we now point out a surprising consequence of the quantum modification (16) when $a \neq 0$. Suppose we add the cosmological term $\Lambda :e^{p\phi}:$ to $:H_{\perp}:$, which does not violate the algebra with p being the same as for (16). We then observe that a redefinition of the field ϕ by $\phi \rightarrow \phi - (2\Lambda/a\kappa^2)$ eliminates the cosmological term. This means that the theory with two (dimensional) constants κ^2 and Λ is equivalent to the theory in which κ^2 is replaced by $\kappa^2 \exp - 2p\Lambda/a\kappa^2$ but the cosmological constant vanishes. The extra factor in the metric tensor, resulting from the redefinition, is cancelled if the unit for the coordinate is uniformly dilated by the factor $\exp 2p\Lambda/a\kappa^2$. This phenomenon is somewhat reminiscent of the role played by the axion field in the θ vacuum in gauge theories. If we maintain that the Fourier expansion is only natural in the asymptotically flat case, the existence of this mechanism seems to ensure the self-consistency of our scheme when $a \neq 0$. It should also be remarked that this is an inherently non-perturbative phenomenon [$a = 0(\hbar)$]. Whether

similar mechanisms exist in higher dimensions and in other theories is an important future question.

Finally, we note that when matter fields are included, they induce a non-vanishing central charge upon quantization. It is, however, not difficult to find a suitable modification of the pure gravitational operators $H_{\perp}$ and H_1 such that the total central charge is always zero. It turns out that the above disappearing mechanism of the cosmological constant still works. Related matters will be discussed in a future publication.

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