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ON THE EXISTENCE OF CHARACTERISTIC FUNCTIONS
IN QUANTUM FIELD THEORY

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ABSTRACT

The formalism of topological groups is used to study renormalization transformations beyond perturbation theory. As a preamble, the Stueckelberg formulation in term of Lie groups is compared to the Gell-Mann and Low analysis. The latter is shown to be not only a consequence of the group formulation in a trivial case but also of poorer mathematical predictive power, namely concerning the existence of characteristic functions. As shown in this paper, this existence is an intrinsic property of the group structure.

The use of the theory of topological groups in the renormalization theory might appear to be a very heavy artillery for a very modest goal. Indeed, the practical uses in physics of renormalization transformations depend mainly on the trivial $U(1)$ subgroup for an overall scale transformation of the basic length scale in quantum field theories, [i.e., of the arbitrary length (or mass) introduced by the usual procedures of renormalization].

However, the theory of topological groups - specifically the group theory of continuous transformations with associated Lie algebra - is nowadays included in the elementary baggage of every student in mathematics and physics. Therefore, it would be somewhat unwise to be afraid of using it, even if at first sight it would appear that much simpler methods might be more economical. As a matter of fact, one can notice that all the results of the lengthy paper by Gell-Mann and Low¹⁾ are nothing but trivial consequences of the recognition that subsets of renormalization transformations form a group with associated Lie algebra^{1*}, a recognition already made ²⁾more than a year before their publication. For instance, the so-called Gell-Mann-Low ϕ -function is nothing but the coefficient function of the generator $\tilde{P}$ of the infinitesimal transformations in the co-ordinates space (here, the one-dimensional coupling constant space)³⁾

$$\tilde{P} \equiv h(x) \frac{\partial}{\partial x} = \psi(x) \frac{\partial}{\partial x} \quad 2^* \quad (1)$$

(1*)

In the hybrid renormalization scheme of GML, treating the electron mass parameter in a special way, this is true at least for the asymptotic limit they consider.

(2*)

The effect of a change of co-ordinates like $x \rightarrow x' = q(x)$ also follows from the general theory of topological groups:

$$h'(x') = h(x) \frac{\partial x'}{\partial x}$$

which is the kind of relation connecting ϕ to the Callan-Symanzik β ⁴⁾

$$\phi(q(\alpha)) = \frac{1}{2} \beta(\alpha) \frac{\partial q(\alpha)}{\partial \alpha}$$

where

$$h(x) = \left. \frac{\partial x'}{\partial c} \right|_{c=0}, \quad x' = x T_c = f(x, c),$$

c is the dilatation parameter of the mass scale: $\lambda \rightarrow \lambda' = e^c \lambda$, (here a one-dimensional group parameter space) so that, in conventional notations:

$$\psi(\alpha) = h(\alpha) = \lambda \frac{\partial}{\partial \lambda} [\bar{\alpha}(\alpha, \lambda'/\lambda)]_{\lambda'=\lambda} \equiv \lambda \frac{\partial}{\partial \lambda} \alpha_\lambda.$$

As mentioned above, the $U(1)$ group of scale transformations T_c , which are the kind of renormalization transformations considered in Ref. 1), is trivial but allows one self to invoke all the properties of topological groups. This ensures simplicity and the mathematical rigour which is needed if one wants to look at existence properties of characteristic functions. In addition, this also allows one to consider more complicated renormalization transformations^{4),5)}, even if their practical use in physics is questionable. At this point, one might object that there is nothing new in what has been said. It is in a sense quite true since this was discovered more than 30 years ago²⁾. But the conclusions about the existence were established in the framework of perturbation theory. As a matter of fact, the renormalization transformations are defined for the Lagrangian density or the action itself which can be chosen in such a way that it is invariant under those transformations. Therefore, the group structure transcends perturbation theory. By the time of Ref. 2), this fact was unimportant since no other methods were known to deal with quantum field theories or other quantum systems but perturbation theory. With the advent of powerful numerical and analytic methods^{6),7)}, this peculiarity is nowadays of primordial importance.

Independently of perturbation theory, the existence of characteristic functions, like RG β -functions, anomalous dimensions of operators, masses and fields, is bound to the existence of the Lie algebra structure of subsets of

renormalization transformations which a theory does exhibit or does not exhibit. These results follow from the fact that Lie groups are topological groups whose underlying spaces are analytic manifolds (i.e., manifolds of class C^∞). Therefore, one can state the following:

In a given quantum field theory with appropriate continuous renormalization transformations T_c , a sufficient condition for the existence of characteristic functions is that the transformations T_c , $c = (c_1, c_2, \dots, c_r)$, have a Lie group structure. The condition is not necessary.

(Specifically, $T_c \in G$; $c = (c_1, c_2, \dots, c_r)$; (Group space)
 $x \in C$; $x = (x_1, x_2, \dots, x_n)$; (Co-ordinates' space)
 with the property

$$x' = x T_c = f(x, c); \quad x = f(x, c_0 = 0)$$

where G and C are differentiable manifolds and f is a map of $G \times C$ into C which is differentiable. The characteristic functions $h_{ij}(x)$ are the coefficient functions of the infinitesimal generators $\tilde{P}$ of the group in C ,

$$\begin{aligned} \tilde{P}_i &= \left. \frac{\partial f_j(x, c)}{\partial c_i} \right|_{c=0} \frac{\partial}{\partial x_j} \\ &\equiv h_{ij}(x) \frac{\partial}{\partial x_j} \end{aligned} \quad (2)$$

Nothing can be said about the differentiability of f_j and h_{ij} in the co-ordinate space.

The release of some constraints in the group structure like the absence of an inverse, does not exclude existence proofs in some specific cases. However, this problem is beyond the scope of the present paper and will be treated, as discussed below, in forthcoming papers⁸⁾.

Remarks:

1) The definition of the coefficient function of $\tilde{P}_i$ in Eq. (2) provides the so-called "renormalization group equation" (RGE) for the coupling constants, masses, etc.

2) The invariance of quantities $Q(x,c)$ under T_c gives rise to the so-called RGE of the asymptotic Callan-Symanzik type, called also Lie equations^{2),5)} (Ovsiannikov). It is noteworthy that there exists a continuous set S of schemes for which the group property - and therefore the algebra - does not hold. This group structure breakdown by the set S is not specific to renormalization transformations. Indeed, one can also show that the continuous set S' of schemes violates the internal properties of the theory under consideration. (Moreover S does not necessarily coincide with S' , but $S \cap S'$ is itself a continuous set.) This is the reason why we only consider subsets $\hat{S}$ which do have the group property and one can show that $\hat{S}$ is also a continuous set of schemes with the properties

$$\hat{S} \subset S_T, \quad \hat{S} \cap S = \emptyset, \quad \hat{S} \cap S' = \emptyset$$

where S_T is the total set of possible schemes and is of course also continuous.

Examples:

$$S_{\varepsilon}\text{-schemes} \subset \hat{S}$$

$$S_{D_0}\text{-schemes} \subset \hat{S}.$$

β -functions, which of course depend on the schemes' choice, satisfy the existence theorems only if they belong to the subset $\hat{B}$ of the whole space B of β -functions. For example, β_s are shown to exist if

$$\beta_s \in \hat{B}, \text{ with } \hat{B} \subset B, \quad \hat{B} \neq B$$

$$\text{and } \hat{B} \neq \emptyset.$$

If a β_s is such that

$$\beta_s \notin \hat{B}, \quad \beta_s \in B$$

it is not excluded in any way that it may exist since $\beta_s \in \hat{B}$ is only a

sufficient condition for the existence to hold. That a groupoid structure is perhaps enough to ensure the existence is quite possible. This question, together with other releases of other group constraints (like the absence of an inverse) will be discussed in a forthcoming paper in some specific cases. The author indeed was not able to draw definite conclusions in the general case.

In summary, the formulation of renormalization transformations in terms of topological groups appears rather as existence theorems permitting to justify rigorously the existence of known non-perturbative solutions, either approximate, obtained numerically⁶⁾, or analytic^{7a-c)} by improved Borel transforms, but does not give the means for determining these solutions.

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