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UNIVERSAL BOSE-FERMI MASS-RELATIONS IN KALUZA-KLEIN SUPERGRAVITY
AND HARMONIC ANALYSIS ON COSET MANIFOLDS WITH KILLING SPINORS

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A B S T R A C T

Universal mass-relations between Bose and Fermi modes are obtained via general relations existing among the spectra of invariant operators on coset manifolds with Killing spinors. In particular zero-modes for pseudoscalars, spinors and Lichnerowicz scalars are always related to the isometry and cohomology properties of the manifold. The structure of the four-dimensional supersymmetry rules on physical modes is given. Furthermore, the harmonics of the possible massless multiplets, namely the gravitational multiplet, the Killing gauge multiplet, and the Betti gauge multiplet, are exhibited for general manifolds. Comments on matter and quasi-massless multiplets conclude the paper.

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TABLE OF CONTENTS

1. - Introduction	1
2. - Harmonic analysis on seven-dimensional coset manifolds with Killing spinors	2
3. - Mass-formulae for the "physical modes" on Freund-Rubin solutions of $D = 11$ supergravity	8
4. - Bose-Fermi mass relations from the supersymmetry transformation rules of bosons	18
5. - Massless, and quasi-massless and matter multiplets	40
6. - Conclusions	51
Tables	52
References	61

1. - INTRODUCTION

In this paper we consider those Freund-Rubin¹⁾ solutions of $D = 11$ supergravity²⁾ whose internal coset manifold / admits Killing spinors. These correspond to possible "supersymmetric ground states", the number of supersymmetries in $d = 4$ being equal to the number of Killing spinors in $d = 7$. The structure of such coset spaces has been analyzed in Ref. 3) and their list was found in Refs. 4) and 5): they have a very specific structure which we shall recall in Section 2. In Ref. 3), they were called "Englert spaces" because they also admit Englert solutions.

The goal of this paper is to analyze the general features of the mass-spectrum of $D = 11$ supergravity compactified on this type of spaces. Starting from the mass-formulae for the bosonic and fermionic "physical modes" already obtained in Refs. 3) and 6) and also worked out by other authors dealing with the specific solutions on the round S^7 and squashed S^7 ⁷⁾⁻¹⁰⁾, we remark that supersymmetry implies a relation among the spectra of the bosonic and fermionic mass operators. These latter are linear and quadratic functions of the standard invariant operators on irreducible SO^7 harmonics (Laplacian, Lichnerowicz, Dirac, Rarita-Schwinger, Hodge de Rham operators).

Hence on coset manifolds with Killing spinors, there must be relations among the spectra of the Laplace-Beltrami operators of the Fermi type with those of the Bose type. In this paper we find all these relations via a constructive procedure. We show how, given the eigenfunctions of the Fermi type operators and the Killing spinors, we can construct the eigenfunctions of the Bose-type operators and we work out the relations among the two sets of eigenvalues. These relations immediately reflect into relations among the masses of the physical modes sitting in the same $Osp(4/N)$ multiplet which we are able to give in full generality.

In a way, what we obtain is the mirror image, in the seven-dimensional space, of the supersymmetry algebra in four dimensions, the fields being replaced by their corresponding harmonics and the supersymmetry parameters by the Killing spinors.

Indeed, we are guided, in constructing our bosonic harmonics out of the fermionic ones, by the linearized supersymmetry transformations of the $D = 11$ theory, of which we give a complete analysis.

As an outcome of this construction, we obtain interesting relations among the occurrence of critical eigenvalues for massless modes in sectors where their geometric interpretation is clear and in sectors where it was not so far. In particular, we obtain a relation between the occurrence of zero-modes of Rarita-Schwinger and Lichnerowicz and $\star d$ operator on three-forms with the second cohomology of the manifold (B_2 Betti number).

In the last section, relying on these general relations, we construct the explicit form of the harmonics for the possible massless multiplets associated to Killing spinors (the gravitational multiplet) to Killing vectors (the Killing gauge multiplet) and to Betti harmonic forms (the Betti gauge multiplets).

We also discuss the occurrence, which seems to take place on all manifolds, of quasi-massless multiplets where a scalar is massless and all the other partners are massive. We formulate the question whether these quasi-massless multiplets, in the case of $N = 1$ and $N = 2$ solutions, can be matter multiplets stopping at spin- $\frac{1}{2}$. If this is the case they might be retained in the effective four-dimensional theory giving possibly rise to the superHiggs phenomenon. In the $N = 2$ theory from $M^{p,q,r}$ -spaces this is not the case but it might be true for the squashed sphere. We point out, however, that the vanishing mass of the 0^+ scalar has probably no special significance.

There might be matter multiplets whose elements are all massive which can be equally well used. The essential point is that they should stop at spin- $\frac{1}{2}$ and should not contain spin-1 fields and higher. Given the information provided by our general techniques for the harmonic analysis on the internal space and the results of Freedman and Nicolai for the structure of $Osp(4/N)$ multiplets we think that this interesting problem can now be attacked.

2. - HARMONIC ANALYSIS ON SEVEN-DIMENSIONAL COSET MANIFOLDS WITH KILLING SPINORS

In this paper we consider the properties of $D = 11$ supergravity when we take as ground state a Freund-Rubin solution¹⁾ possessing N -supersymmetries. Such solutions appear when the internal compactified seven-dimensions correspond to a very specific class of coset manifolds endowed with Killing spinors¹¹⁾ whose structure can be analyzed in complex generality. This was

done in Section 5 of Ref. 3) where these special cosets were classified under the name of Englert spaces since they lead to Englert solutions. The same manifolds also lead to Freund-Rubin solutions with supersymmetry and it is in this respect that they are relevant to us here. Let us then introduce a seven-dimensional Einstein coset manifold $\mathcal{S}/\mathcal{R}$ whose geometry is described by the following structural equations (Greek indices run from 1 to 7):

$$\mathcal{R}^\alpha = d\mathcal{B}^\alpha - \mathcal{B}^{\alpha\beta} \wedge \mathcal{B}_\beta = 0 \quad (2.1a)$$

$$\mathcal{R}^{\alpha\beta} = d\mathcal{B}^{\alpha\beta} - \mathcal{B}^{\alpha\gamma} \wedge \mathcal{B}_\gamma{}^\beta = \mathcal{R}_{\mu\nu}^{\alpha\beta} \mathcal{B}^\mu \wedge \mathcal{B}^\nu \quad (2.1b)$$

$$\mathcal{R}_{\beta\mu}^{\alpha\mu} = 12e^2 \delta_\beta^\alpha \quad (2.1c)$$

In Eq. (2.1c), e is a constant which in our applications to supergravity will play the role of vacuum expectation value of the compactifying field F_{mnr} . It will be the only dimensionful parameter of the entire theory and all the masses we shall discuss later will be measured in units of e .

We assume that our manifold $\mathcal{S}/\mathcal{R}$ admits N Killing spinors, namely N -solutions of the following equation:

$$\mathcal{D}\eta^M \equiv \left(d - \frac{1}{4} \mathcal{B}^{\alpha\beta} \tau_{\alpha\beta}\right) \eta^M = \mathcal{B}^\alpha \tau_\alpha \eta^M \quad (2.2)$$

where $\tau_{\alpha\beta}$ and τ_α are gamma-matrices in seven-dimensions [conventions are the same as in Refs. 3), 12), 4) and 5)] and the capital Latin indices from the middle of the alphabet (M, N, P, Q) run on N -values and label the Killing spinors. These latter are Majorana

$$\zeta_7 (\bar{\eta}_M)^T = \eta_M \quad (2.3)$$

and are orthonormal among themselves:

$$\bar{\eta}^M \eta^N = \delta^{MN} \quad (2.4)$$

Recalling the general results of Section 5 of Ref. 3) we know that:

a) The isometry group $\mathcal{G}$ has the following structure:

$$\mathcal{G} = \mathcal{G}' \otimes SO(N) \quad (2.5)$$

where N is the number of Killing spinors and $\mathcal{G}'$ is some suitable group;

b) The holonomy algebra $\mathcal{H}$ of $\mathcal{G}/R$ is:

$$\mathcal{H} = \begin{cases} \mathfrak{G}_2 & \text{for } N=1 \\ SU_3 & \text{for } N=2 \\ SU_2 & \text{for } N=3 \\ 1 & \text{for } N=8 \end{cases} \quad (2.6)$$

c) There is a metrical equivalence

$$\mathcal{G}/R \simeq \mathcal{G}^*/R^* \quad (2.7a)$$

where

$$\mathcal{G}^* = \mathcal{G}' \otimes SO(N-1) ; R^* = \mathcal{G}^* \cap R \quad (2.7b)$$

d) The Killing spinors η^M belong to the vector representation of $SO(N)$ and the identity representation of $\mathcal{G}'$. This means that they are invariant under the action of the $\mathcal{G}'$ Killing vectors, namely they satisfy:

$$\mathcal{L}_{k'} \eta^M \equiv \left(l_{k'} - \frac{1}{4} W_{k'}^{\alpha\beta} \tau_{\alpha\beta} \right) \eta^M = 0 \quad (2.8)$$

where $\mathcal{L}_{k'}$ is the covariant Lie derivative, thoroughly discussed in Ref. 3).

Introducing moreover the Weyl tensor

$$C^{\alpha\beta\mu\nu} = R^{\alpha\beta}_{\mu\nu} - 4e^2 \delta^{\alpha\beta}_{\mu\nu} \quad (2.9a)$$

one has that the 21 objects $\hat{C}_{\alpha\beta} = C_{\alpha\beta\mu\nu} \tau_{\mu\nu}$ are linear dependent, span the holonomy algebra (2.6) and annihilate the Killing spinors:

$$\hat{C}_{\alpha\beta} \eta^M \equiv C_{\alpha\beta\mu\nu} \tau^{\mu\nu} \eta^M = 0 \quad (2.9b)$$

On our coset manifold $\mathcal{G}/\mathcal{R}$ we introduce a harmonic analysis according to the methods discussed in 3). Indeed, we need an appropriate harmonic expansion for each irreducible SO^7 representation which appears in our field theory and for each of them we have an invariant operator (commuting with the Lie derivative along Killing vectors) whose eigenvalues define the masses of the corresponding x-space fields. The SO^7 representations relevant to the analysis of compactified $D = 11$ supergravity are the following ones (for each of them we write the associated irreducible harmonic with its transversality properties):

$$i) \quad \underline{1} = [0, 0, 0] \Rightarrow \text{scalar harmonic } Y(z) \quad (2.10a)$$

$$ii) \quad \underline{7} = [1, 0, 0] \Rightarrow \text{transverse vector harmonic } Y_\alpha(z); \quad (2.10b) \\ \mathcal{D}^\alpha Y_\alpha = 0$$

$$iii) \quad \underline{21} = [1, 1, 0] \Rightarrow \text{transverse 2-form } Y_{[\alpha\beta]}(z); \quad (2.10c) \\ \mathcal{D}^\alpha Y_{[\alpha\beta]} = 0$$

$$iv) \quad \underline{35} = [1, 1, 1] \Rightarrow \text{transverse 3-form } Y_{[\alpha\beta\gamma]}(z); \quad (2.10d) \\ \mathcal{D}^\alpha Y_{[\alpha\beta\gamma]} = 0$$

$$v) \quad \underline{27} = [2, 0, 0] \Rightarrow \text{symmetric transverse traceless tensor:} \quad (2.10e) \\ Y_{(\alpha\beta)}; \quad Y_{(\alpha\alpha)} = 0; \quad \mathcal{D}^\alpha Y_{(\alpha\beta)} = 0$$

$$vi) \quad \underline{8} = [1/2, 1/2, 1/2] \Rightarrow \text{spinor } \underline{\psi} \quad (2.10f)$$

$$vii) \quad \underline{48} = [3/2, 1/2, 1/2] \Rightarrow \text{irreducible transverse vector spinor } \underline{\psi}_\alpha \quad (2.10g) \\ \tau^\alpha \underline{\psi}_\alpha = \mathcal{D}^\alpha \underline{\psi}_\alpha = 0$$

The above irreducible harmonics are, by definition, eigenstates of the invariant operators listed hereunder which naturally emerge in the analysis of $D = 11$ field equations^{*)}:

*) From this point on we use units where $e = 1$.

i) Irrep[0,0,0] → Laplacian operator

$$\boxtimes_{(0)^3} Y \equiv \square Y \equiv \mathcal{D}^\mu \mathcal{D}_\mu Y = M_{(0)^3} Y \quad (2.11a)$$

ii) Irrep[1,0,0] → Hodge de Rham operator on one-forms

$$\boxtimes_{(1)(0)^2} Y_\alpha \equiv 2 \mathcal{D}^\mu \mathcal{D}_{[\mu} Y_{\alpha]} = (\square + 24) Y_\alpha = M_{(1)(0)^2} Y_\alpha \quad (2.11b)$$

iii) Irrep[1,1,0] → Hodge de Rham on two-forms

$$\begin{aligned} \boxtimes_{(1)^2(0)} Y_{[\alpha\beta]} &\equiv 3 \mathcal{D}^\mu \mathcal{D}_{[\mu} Y_{\alpha\beta]} = \\ &[(\square + 40) \mathcal{D}_{[\lambda\mu]}^{\alpha\beta} - 2 \mathcal{G}^{\alpha\beta\lambda\mu}] Y_{\lambda\mu} = M_{(1)^2(0)} Y_{[\alpha\beta]} \end{aligned} \quad (2.11c)$$

iv) Irrep[1,1,1] → *d operator = "square root of the Hodge de Rham operator on three-forms"

$$\boxtimes_{(1)^3} Y_{[\alpha\beta\gamma]} \equiv \frac{1}{24} \varepsilon_{\alpha\beta\gamma\lambda\mu\nu} \mathcal{D}_\lambda Y_{\mu\nu} = M_{(1)^3} Y_{[\alpha\beta\gamma]} \quad (2.11d)$$

v) Irrep[2,0,0] → Lichnerowicz-like operator on symmetric tensors

$$\boxtimes_{(2)(0)^2} Y_{(\alpha\beta)} = [(\square + 40) \mathcal{D}_{(\lambda\mu)}^{(\alpha\beta)} - 4 \mathcal{G}^{\alpha\lambda\beta\mu}] Y_{\lambda\mu} = M_{(2)(0)^2} Y_{(\alpha\beta)} \quad (2.11e)$$

vi) Irrep[$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$] → Dirac operator with respect to the de-Sitter covariant derivative

$$\begin{aligned} \boxtimes_{(1/2)^3} \varpi &\equiv \not{D} \varpi \equiv \tau^\mu (\mathcal{D}_\mu - \tau_\mu) \varpi \equiv (\not{D} - 7) \varpi = \\ &= M_{(1/2)^3} \varpi \end{aligned} \quad (2.11f)$$

vii) Irrep[$3/2, \frac{1}{2}, \frac{1}{2}$] → Irreducible Rarita-Schwinger operator with respect to the de Sitter derivative

$$\begin{aligned} \boxtimes_{(3/2)(1/2)^2} \varpi_\alpha &\equiv \tau^{\alpha\mu\nu} \nabla_\mu \varpi_\nu - \frac{5}{7} \tau^\alpha \tau^{\mu\nu} \nabla_\mu \varpi_\nu \equiv \\ &\equiv (\not{D} - 5) \varpi_\alpha = M_{(3/2)(1/2)^2} \varpi_\alpha \end{aligned} \quad (2.11g)$$

As we shall see in the next section, the masses of the "physical modes" around a Freund-Rubin solution which has $\mathcal{S}/\mathcal{R}$ as internal manifold are given by simple expressions in terms of the eigenvalues $M_{[\lambda_1\lambda_2\lambda_3]}$ defined in Eqs. (2.11). It seems therefore that in order to work out the particle

spectrum of supergravity we have to construct the spectrum of $\mathcal{F} = \mathcal{F}' \otimes \text{SO}(N)$ representations contributing to each of the above $[\lambda_1 \lambda_2 \lambda_3]$ fields and for each of them compute the eigenvalues of the corresponding $\boxtimes_{[\lambda_1 \lambda_2 \lambda_3]}$. This gigantic work, however, is not unavoidable because supersymmetry implies that the spectrum of bosons and fermions are related so that knowing one sector one can retrieve the other. Supersymmetry corresponds, for the internal manifold, to the existence of Killing spinors. This means that on a coset $\mathcal{F}/\mathcal{R}$ with Killing spinors, the spectra of the operators $\boxtimes_{[\lambda_1 \lambda_2 \lambda_3]}$ must be related to each other. Indeed this is the case because having an harmonic which is an eigenstate of a bosonic $\boxtimes_{[\lambda_1 \lambda_2 \lambda_3]}$ (integer weights $\lambda_1 \lambda_2 \lambda_3$) by multiplication with a Killing spinor η we can convert it into an eigenstate of a fermionic $\boxtimes_{[\lambda_1 \lambda_2 \lambda_3]}$ (odd-integer weights $\lambda_1 \lambda_2 \lambda_3$) and vice versa.

Moreover the eigenvalues of the two harmonics are in a simple relation. In Table 1, these relations, which will be derived in Section 4, are tabulated.

The reader will notice that the study of the fermionic spectrum is sufficient to reconstruct the entire bosonic spectrum: the eigenvalues of the bosonic operators are unambiguously defined by those of the fermionic ones and the possible representations are those contained in the fermionic spectrum multiplied by the N-dimensional representation of $\text{SO}(N)$ (representation of the Killing spinors). In particular it follows that all possible representations of the gauge group $\mathcal{F}'$ are already given by the fermions, which, however, are composed of two sectors, the longitudinal $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ and the transverse $[3/2, \frac{1}{2}, \frac{1}{2}]$ one.

It is interesting to observe that the spectrum of both types of fermions and, as a consequence, the entire spectrum of the theory is already obtainable from the spectrum of the three-form $Y_{[\alpha\beta\gamma]}$. The reason is that the bosonic operator acting on $Y_{[\alpha\beta\gamma]}$ is a first-order one and therefore is equivalent to the fermionic operators. Hence an investigation of the $Y_{[\alpha\beta\gamma]}$ spectrum gives the complete answer to one's questions.

It is perhaps important to remark that on the round seven-sphere it is one of the low-lying $Y_{[\alpha\beta\gamma]}$ modes which is responsible for the extension of SO^8 to the hidden SU^8 symmetry¹³⁾ and that in every compactification it is also $Y_{[\alpha\beta\gamma]}$ which is responsible for the Englert solution. Hence the three-form $Y_{[\alpha\beta\gamma]}$ appears to contain all the information one might be looking for.

We give a striking example: suppose there is a three-form with eigenvalue $M_{(1)3} = 1$. This implies that there is a transverse vector spinor with eigenvalue $M_{(3/2)(\frac{1}{2})} = -8$ and this in turn implies that there is a two-form with eigenvalue $M_{(1)2(0)} = 0$. This latter is a closed and coclosed two-form, hence the Betti number B_2 of $\mathcal{S}/\mathcal{R}$ is larger than zero. Hence the spectrum of $Y_{[\alpha\beta\gamma]}$ yields the cohomology of the manifold!

3. - MASS FORMULAE FOR "THE PHYSICAL MODES" ON FREUND-RUBIN SOLUTIONS OF D = 11 SUPERGRAVITY

The general form of the mass operators for the small fluctuations around a general Freund-Rubin solution¹⁾ of D = 11 supergravity²⁾ has been given in Ref. 3) for the fermion fields and in Ref. 6) for the Bose fields. Similar formulae have been derived by other authors in the specific case of the round and squashed S^7 solutions⁷⁾⁻¹⁰⁾.

In Ref. 6), the bosonic mass formulae are given in conventions and notations which do not fit with those of Ref. 3) where the fermionic mass formulae were derived in the context of the geometric formulation¹⁴⁾ of D = 11 supergravity.

In view of this and in order to make this paper self-contained, we summarize the derivation of the mass formulae we shall be using throughout this paper, using the geometric set-up of Refs. 14), 3), 11) and 15). Given the definition of the 11-dimensional curvatures

$$R^{\hat{a}\hat{b}} = d\omega^{\hat{a}\hat{b}} - \omega^{\hat{a}\hat{c}} \wedge \omega_{\hat{c}}^{\hat{b}} \quad (3.1a)$$

$$R^{\hat{a}} = dV^{\hat{a}} - \omega^{\hat{a}\hat{b}} \wedge V_{\hat{b}} - \frac{i}{2} \bar{\psi} \wedge \Gamma^{\hat{a}} \psi \quad (3.1b)$$

$$g = d\psi - \frac{1}{4} \omega^{\hat{a}\hat{b}} \wedge \Gamma_{\hat{a}\hat{b}} \psi \quad (3.1c)$$

$$R^\square = dA - \frac{1}{2} \bar{\Psi} \wedge \Gamma^{\hat{a}\hat{b}} \Psi \wedge V_{\hat{a}} \wedge V_{\hat{b}} \quad (3.1d)$$

and the field equations of supergravity:

$$R^{\hat{a}} = 0 \quad (3.2a)$$

$$R^\square = F_{\hat{a}_1 \dots \hat{a}_4} V^{\hat{a}_1} \wedge \dots \wedge V^{\hat{a}_4} \quad (3.2b)$$

$$\mathcal{G} = \mathcal{G}_{\hat{a}\hat{b}} V^{\hat{a}} \wedge V^{\hat{b}} - \frac{i}{3} \left(\Gamma^{\hat{a}_1 \dots \hat{a}_3} \Psi \wedge V^{\hat{a}_4} + \frac{1}{8} \Gamma^{\hat{a}_1 \dots \hat{a}_4 \hat{m}} \Psi \wedge V_{\hat{m}} \right) F_{\hat{a}_1 \dots \hat{a}_4} \quad (3.2c)$$

$$R^{\hat{a}\hat{b}} = R_{\hat{m}\hat{n}}^{\hat{a}\hat{b}} V^{\hat{m}} \wedge V^{\hat{n}} + i \bar{\mathcal{G}}_{\hat{m}\hat{n}} \left(\frac{1}{2} \Gamma^{\hat{a}\hat{b}\hat{m}\hat{n}\hat{c}} - \frac{2}{9} \Gamma^{\hat{m}\hat{n}} [\hat{a}\hat{b}]^{\hat{c}} + 2 \Gamma^{\hat{a}\hat{b}} [\hat{m}\hat{n}]^{\hat{c}} \right) \Psi \wedge V_{\hat{c}} + \bar{\Psi} \wedge \Gamma_{\hat{m}\hat{n}} \Psi F^{\hat{a}\hat{b}\hat{m}\hat{n}} + \frac{1}{24} \bar{\Psi} \wedge \Gamma^{\hat{a}\hat{b}\hat{c}\dots\hat{c}_4} \Psi F_{\hat{c}_1 \dots \hat{c}_4} \quad (3.2d)$$

$$\Gamma^{\hat{a}\hat{b}\hat{c}} \mathcal{G}_{\hat{b}\hat{c}} = 0 \quad (3.2e)$$

$$\mathcal{D}_{\hat{m}} F^{\hat{m}\hat{c}_1 \dots \hat{c}_3} + \frac{1}{96} \varepsilon^{\hat{c}_1 \dots \hat{c}_3 \hat{a}_1 \dots \hat{a}_8} F_{\hat{a}_1 \dots \hat{a}_4} F_{\hat{a}_5 \dots \hat{a}_8} = 0 \quad (3.2f)$$

$$R_{\hat{b}\hat{m}}^{\hat{a}\hat{m}} = 6 F^{\hat{a}\hat{c}_1 \dots \hat{c}_3} F_{\hat{b}\hat{c}_1 \dots \hat{c}_3} - \frac{1}{2} \delta_{\hat{b}}^{\hat{a}} F^{\hat{c}_1 \dots \hat{c}_4} F_{\hat{c}_1 \dots \hat{c}_4} \quad (3.2g)$$

and, using the $\mathcal{G}/\mathcal{R}$ coset manifold with Killing spinors described in the previous section, one can construct a solution of the field equations (3.2) with the following procedure. We consider the supercoset space

$$\mathcal{R}^{4/4N} = \text{Osp}(4/N) / \text{SO}(1,3) \otimes \text{SO}(N) \quad (3.3)$$

which has four bosonic and $4N$ fermionic co-ordinates. On this coset, in a completely analogous way to bosonic cosets, we have a set of 1-forms satisfying $\text{Osp}(4/N)$ Maurer-Cartan equations (Latin indices without a hat run from 1 to 4):

$$d\overset{\circ}{V}^a - \overset{\circ}{\omega}^{ab} \wedge \overset{\circ}{V}_b - \frac{i}{2} \overset{\circ}{\bar{\Psi}}_M \wedge \gamma^a \Psi_M = 0 \quad (3.4a)$$

$$d\overset{\circ}{\omega}^{ab} - \overset{\circ}{\omega}^{ac} \wedge \overset{\circ}{\omega}^{cb} + 16e^2 \overset{\circ}{V}^a \wedge \overset{\circ}{V}^b - 2i \overset{\circ}{\bar{\Psi}}_M \wedge \gamma_5 \gamma^{ab} \Psi_M = 0 \quad (3.4b)$$

$$d\overset{\circ}{A}^{MN} + e \overset{\circ}{A}^{MR} \wedge \overset{\circ}{A}^{RN} - 4i \overset{\circ}{\bar{\Psi}}_M \wedge \gamma_5 \Psi_N = 0 \quad (3.4c)$$

$$d\overset{\circ}{\Psi}_M - \frac{1}{4} \overset{\circ}{\omega}^{ab} \gamma_{ab} \overset{\circ}{\Psi}_M - 2e \gamma_5 \gamma_m \overset{\circ}{V}^m \wedge \overset{\circ}{\Psi}_M - e \overset{\circ}{A}_{MN} \wedge \overset{\circ}{\Psi}_N = 0 \quad (3.4d)$$

where e is the same parameter introduced in the treatment of the coset space S/R . In Ref. 11), it was shown that under the above conditions the following is a solution of the superspace field equations (3.2):

$$\overset{\circ}{V}^a = \overset{\circ}{V}^a \quad (3.5a)$$

$$\overset{\circ}{V}^\alpha = \mathcal{B}^\alpha + \frac{1}{8} \bar{\eta}_M \tau^\alpha \eta_N \overset{\circ}{A}_{MN} \quad (3.5b)$$

$$\overset{\circ}{\omega}^{ab} = \overset{\circ}{\omega}^{ab} \quad (3.5c)$$

$$\overset{\circ}{\omega}^{\alpha\beta} = \mathcal{B}^{\alpha\beta} - \frac{e}{4} \bar{\eta}_M \tau^{\alpha\beta} \eta_N \overset{\circ}{A}_{MN} \quad (3.5d)$$

$$\eta = \eta_M \overset{\circ}{\Psi}_M \quad (3.5e)$$

$$A = \overset{\circ}{A} \quad (3.5f)$$

where $\overset{\circ}{A}$ satisfies

$$d\hat{A} - \frac{1}{2} \hat{\bar{\Psi}}_M \wedge \gamma^{ab} \hat{\Psi}_N \wedge \hat{V}_a \wedge \hat{V}_b - \frac{1}{2} \hat{\bar{\Psi}}_M \wedge \hat{\Psi}_N \bar{\gamma}_M \tau^\alpha \gamma_N \hat{V}_\alpha \wedge \hat{V}_\beta \\ + \hat{\bar{\Psi}}_M \wedge \gamma_5 \gamma_a \hat{\Psi}_N \wedge \hat{V}^a \wedge \bar{\gamma}_M \tau^\alpha \gamma_N \hat{V}_\alpha = e \varepsilon_{abcd} \hat{V}_a \wedge \hat{V}_b \wedge \hat{V}_c \wedge \hat{V}_d \quad (3.5g)$$

This solution lives in the superspace:

$$R^{4+7/4N} \simeq \frac{osp(4/N) \otimes \mathcal{S}'}{so(1,3) \otimes \mathcal{R}} \quad (3.6)$$

which is locally the direct product of the N-extended de Sitter superspace with the coset $\mathcal{S}/\mathcal{R}$. It is manifestly evident that this ground states admits the following isometry supergroup

$$G = osp(4/N) \otimes \mathcal{S}' \quad (3.7)$$

Therefore the particle states given by the small fluctuations around this solution will fall into representations of G.

Let us begin by defining the fluctuation fields. We set:

$$V^{\hat{a}} = \hat{V}^{\hat{a}} + \mathcal{H}^{\hat{a}\hat{b}} \hat{V}_{\hat{b}} \quad (3.8a)$$

$$\Psi = \hat{\Psi} + \psi_{\hat{m}} \hat{V}^{\hat{m}} \quad (3.8b)$$

$$A = \hat{A} + A_{\hat{m}\hat{n}\hat{z}} \hat{V}^{\hat{m}} \wedge \hat{V}^{\hat{n}} \wedge \hat{V}^{\hat{z}} \quad (3.8c)$$

where $\hat{V}^{\hat{a}}, \hat{\Psi}, \hat{A}$ are the vacuum fields defined in Eq. (3.5) and $\mathcal{H}^{\hat{a}\hat{b}}, \psi_{\hat{m}}, A_{\hat{m}\hat{n}\hat{z}}$ are treated as small fluctuations. Each of them must be expanded in spherical harmonics of $\mathcal{S}/\mathcal{R}$ and inserted in the linearization of the inner field equations (3.2). The general form of these latter is easily obtained. First of all one notes that the fluctuation of the spin connection is given by

$$\Delta \omega_{\hat{z}}^{\hat{m}\hat{n}} = 2 \mathcal{D}^{[\hat{m}} \mathcal{H}^{\hat{n}]} \hat{z} \quad (3.9)$$

where, by exploiting the $SO(1,0)$ invariance, we have assumed that $\mathcal{H}^{\hat{m}\hat{n}}$ is symmetrical. Then one concludes that the fluctuation of the Ricci tensor has the following form:

$$\begin{aligned} \Delta R_{\hat{b}}^{\hat{a}} = & \frac{1}{2} (\square_x + \square_z) \mathcal{H}_{\hat{a}\hat{b}} - \mathcal{D}_{\hat{a}} \mathcal{D}^{\hat{m}} \mathcal{H}_{\hat{b}\hat{m}} \\ & + \frac{1}{2} \mathcal{D}_{\hat{a}} \mathcal{D}_{\hat{b}} \mathcal{H}_{\hat{m}\hat{m}} - 2 \overset{\circ}{R}_{\hat{a}\hat{s}}^{\hat{b}\hat{m}} \mathcal{H}_{\hat{m}\hat{s}} \end{aligned} \quad (3.10)$$

where

$$\square_x = \mathcal{D}^m \mathcal{D}_m \quad ; \quad \square_z = \mathcal{D}^\mu \mathcal{D}_\mu \quad (3.11)$$

are the Laplacians in x and $\mathcal{S}/\mathcal{R}$ space and $\overset{\circ}{R}_{\hat{r}\hat{s}}^{\hat{m}\hat{n}}$ is the vacuum value of the curvature:

$$\overset{\circ}{R}_{\hat{r}\hat{s}}^{\hat{m}\hat{n}} = \left\{ \overset{\circ}{R}_{rs}^{mn} = -16e^2 \delta_{rs}^{mn} ; R_{\sigma\sigma}^{\mu\nu} = \mathcal{R}_{\sigma\sigma}^{\mu\nu} ; \text{zero otherwise} \right\} \quad (3.12)$$

On the other hand the fluctuation of the F -field strength is:

$$\Delta F_{\hat{m}\hat{n}\hat{r}\hat{s}} = \mathcal{D}_{[\hat{m}} A_{\hat{n}\hat{r}\hat{s}]} - 4 \overset{\circ}{F}_{[\hat{m}\hat{n}\hat{r}} \hat{\mathcal{E}}_{\hat{s}]} \quad (3.13)$$

where $\overset{\circ}{F}$ is the vacuum field strength:

$$\overset{\circ}{F}_{\hat{m}\hat{n}\hat{r}\hat{s}} = \left\{ F_{mnrs} = e \varepsilon_{mnrs} ; \text{zero otherwise} \right\} \quad (3.14)$$

out of which we obtain the fluctuation of the energy momentum tensor

$$\Delta T_{\hat{b}}^{\hat{a}} = 6 \overset{\circ}{F}^{\hat{a}\hat{p}\hat{q}\hat{r}} \Delta F_{\hat{b}\hat{p}\hat{q}\hat{r}} + (\hat{a} \leftrightarrow \hat{b}) - \overset{\circ}{S}_{\hat{b}}^{\hat{a}} \overset{\circ}{F}^{\hat{p}\hat{q}\hat{r}\hat{s}} \Delta F_{\hat{p}\hat{q}\hat{r}\hat{s}} \quad (3.15)$$

This enables us to write the linearization of Eq. (3.2g):

$$\Delta R_{\hat{b}}^{\hat{a}} = \Delta T_{\hat{b}}^{\hat{a}} \quad (3.16)$$

The linearization of the Maxwell equation (3.2e) on the other hand becomes:

$$\mathcal{D}_{\hat{m}} \Delta F^{\hat{m}\hat{p}\hat{q}\hat{r}} + 4 \Delta \omega_{\hat{m}}^{\hat{t}} [\hat{m}^{\hat{o}} \cdot \hat{p}\hat{q}\hat{r}] + \frac{1}{48} \varepsilon^{\hat{p}\hat{q}\hat{r}\hat{s}\hat{t}\hat{u}\hat{v}\hat{w}\hat{x}} F_{\hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4} \Delta F_{\hat{a}_5 \hat{a}_6 \hat{a}_7 \hat{a}_8} = 0 \quad (3.17)$$

while the linearization of the fermionic equation (3.2e) is

$$\Gamma^{\hat{a}\hat{b}\hat{c}} D_{\hat{b}} \psi_{\hat{c}} = 0 \quad (3.18)$$

$D_{\hat{b}}$ being the de Sitter derivative

$$D_{\hat{b}} = \begin{cases} \rightarrow \mathcal{D}_m = \mathcal{D}_m - 2e\gamma_5 \gamma_m \\ \rightarrow \mathcal{D}_\mu = \nabla_\mu = \mathcal{D}_\mu - e\tau_\mu \end{cases} \quad (3.19)$$

In all the above equations, the derivatives $\mathcal{D}_m$ and $\mathcal{D}_\mu$ are covariant with respect to the vacuum connection and vielbein. Now, in order to find the spectrum of our field theory we just have to expand the fluctuations $\mathcal{H}_{\hat{m}\hat{n}}$, $\psi_{\hat{m}}$ and $A_{\hat{p}\hat{q}\hat{r}}$ in spherical harmonics and insert them in Eqs. (3.16), (3.17) and (3.18). Before doing that we must, however, remember that the $D = 11$ theory has a large set of local invariances (co-ordinate, photon-gauge and supersymmetry) which must be fixed in order to obtain the physical modes. In choosing the gauge fixing terms we want to break all the symmetries except those which will correspond to the visible symmetries of the effective four-dimensional theory, namely those associated to $Osp(4/N) \times \mathcal{G}'$.

A convenient choice is the following one.

$$i) \quad \tau^\alpha \psi_\alpha = 0 \quad (\text{fixes all supersymmetries except those generated by a Killing spinor } \nabla_\alpha \eta = 0) \quad (3.20a)$$

$$ii) \quad \mathcal{D}^\alpha \left(\mathcal{H}_{\alpha\beta} - \frac{15}{7} \mathcal{H}_{\alpha\beta} \right) = 0 \quad (\text{fixes seven-dimensional general co-ordinate transformations whose parameter is transverse } (\mathcal{D}^\mu \xi_\mu = 0) \text{ except those generated by a Killing vector } \mathcal{D}_{\alpha k\beta} = -\mathcal{D}_{\beta k\alpha}) \quad (3.20b)$$

$$iii) \quad \mathcal{D}^\alpha \mathcal{H}_{\alpha b} = 0 \quad (\text{fixes those seven-dimensional transformations whose parameter is longitudinal } \xi_\mu = \mathcal{D}_\mu \zeta \text{ by requiring that } \xi \propto \zeta \text{ where } \zeta \text{ is the parameter of the longitudinal four-dimensional co-ordinate transformations } \xi_m = \mathcal{D}_m \zeta) \quad (3.20c)$$

$$\text{iv)} \quad 5 \square_z \mathcal{D}^a \mathcal{H}_{ab} = (216 - \square_z + 36 \sqrt{\square_z + 36}) \mathcal{D}_b \mathcal{H}_{mm}$$

(fixes the residual symmetry $\xi_m = \mathcal{D}_m \zeta$)

(3.20d)

This last condition is imposed uniformly on all modes except the first one for which $\square_z = 0$. This leaves the freedom of standard diffeomorphisms in the four-dimensional manifold.

$$\text{v)} \quad \mathcal{D}^\alpha A_{\alpha mn} = \mathcal{D}^\alpha A_{\alpha \beta m} = \mathcal{D}^\alpha A_{\alpha \beta \gamma} = 0$$

(3.20e)

[These conditions fix the remaining photon gauges except for one possibility. If there is a closed but not exact two-form on the manifold $\mathcal{S}/R$ $\Omega_{\alpha\beta}$, this parameter is not fixed by Eq. (3.20e).]

Once all these conditions are imposed the most general parametrization of $\mathcal{H}_{ab}$, ψ_m , A_{mn} , in terms of the irreducible and transverse harmonics discussed in the previous section is the following one:

$$\begin{aligned} \mathcal{H}_{ab} = & \left\{ h_{ab}^I - \frac{3}{M_{(0)^3} + 32} \mathcal{D}_a \mathcal{D}_b \left[(2 + \sqrt{M_{(0)^3} + 36}) S^I + (2 - \sqrt{M_{(0)^3} + 36}) \bar{Z}^I \right] \right. \\ & \left. + \frac{5}{4} \delta_{ab} \left[(6 - \sqrt{M_{(0)^3} + 36}) S^I + (6 + \sqrt{M_{(0)^3} + 36}) \bar{Z}^I \right] \right\} Y^I \end{aligned} \quad (3.21a)$$

$$\mathcal{H}_{a\beta} = \left\{ (\sqrt{M_{(4)(0)^2} + 16} - 4) A_a^I + (\sqrt{M_{(4)(0)^2} + 16} + 4) W_a^I \right\} Y_\beta^I \quad (3.21b)$$

$$\mathcal{H}_{\alpha\beta} = \varphi^I Y_{\alpha\beta}^I - \delta_{\alpha\beta} \left\{ (6 - \sqrt{M_{(0)^3} + 36}) S^I + (6 + \sqrt{M_{(0)^3} + 36}) \bar{Z}^I \right\} Y^I \quad (3.21c)$$

$$A_{abc} = 2 \varepsilon_{abcm} \mathcal{D}_m (S^I + \bar{Z}^I) \quad (3.21d)$$

$$A_{ab\gamma} = \left\{ \frac{2}{3} \varepsilon_{abts} (\partial_t A_s^I + \partial_t W_s^I) \right\} Y_\gamma^I \quad (3.21e)$$

$$A_{a\beta\gamma} = Z_a^I Y_{[\beta\gamma]}^I(z) \quad (3.21f)$$

$$A_{\alpha\beta\gamma} = \pi^I Y_{[\alpha\beta\gamma]}^I(z) \quad (3.21g)$$

$$\Psi_a = \left\{ \chi_a + \frac{\frac{4}{7} M_{(1/2)^3} + 8}{M_{(1/2)^3} + 8} [D_a \lambda_L^I]^{3/2} - (6 + \frac{3}{7} M_{(1/2)^3}) \gamma_5 \gamma_a \lambda_L^I \right\} \square_a^I(z) \quad (3.21h)$$

$$\Psi_\alpha = \lambda_T^I \square_\alpha^I + \lambda_L^I [\nabla_\alpha \square]^{3/2} \quad (3.21i)$$

where h_{ab}^I , S^I , Σ^I , π^I , ϕ^I , A_a^I , W_a^I , Z_a^I , χ_a^I , λ_L^I , λ_T^I are all space-time fields carrying the index of the representation of to which their corresponding harmonics $Y_{\dots}^I$ or E^I belong to. h_{ab}^I being symmetric, traceless and transverse ($\partial^a h_{ab}^I = 0$) describes a spin-two particle, similarly χ_a^I ($\gamma^a \chi_a = 0$; $\partial^a \chi_a = 0$) describes a spin- $3/2$ object. Σ , S , π and ϕ are scalars, W_a , A_a and Z_a vectors, while λ_L and λ_T are spin- $1/2$ fields. As the reader can see, once the fields have been introduced in one place they appear in the other places where they show up with complicated normalizations due to the equations of motion and the gauge conditions (3.20). These latter, however, have been chosen in such a way that the x-space fields satisfy canonical field equations with a mass term dictated by the eigenvalues of the invariant operators (2.11). Indeed, we have:

$$\left. \begin{aligned} (\square_x - 32) \bar{Z}^I &= -m_{\bar{Z}}^2 \bar{Z}^I \\ (\square_x - 32) \bar{S}^I &= -m_S^2 \bar{S}^I \\ (\square_x - 32) \bar{\varphi}^I &= -m_\varphi^2 \bar{\varphi}^I \end{aligned} \right\} 0^+ \quad (3.22a)$$

$$(\square_x - 32) \pi^I = -m_\pi^2 \pi^I \} 0^- \quad (3.22b)$$

$$\left. \begin{aligned} \not{D}_x \lambda_L^I &= -m_{\lambda_L} \gamma_5 \lambda_L^I \\ \not{D}_x \lambda_T^I &= -m_{\lambda_T} \gamma_5 \lambda_T^I \end{aligned} \right\} 1/2 \quad (3.22c)$$

$$\left. \begin{aligned} 2 \not{D}^m \not{D}_m W_m^I &= -m_W^2 W_m^I \\ 2 \not{D}^m \not{D}_m A_m^I &= -m_A^2 A_m^I \end{aligned} \right\} 1^+ \quad (3.22d)$$

$$2 \not{D}^m \not{D}_m Z_m^I = -m_Z^2 Z_m^I \} 1^- \quad (3.22e)$$

$$(\not{D}_x - 4\gamma_5) \chi_a = -m_\chi \gamma_5 \chi_a \} 3/2 \quad (3.22f)$$

$$(\square_x - 32) h_{ab} = -m_h^2 h_{ab} \} 2 \quad (3.22g)$$

where

$$\left. \begin{aligned} m_S^2 &= M_{(0)^3} + 176 - 24 \sqrt{M_{(0)^3} + 36} \\ m_Z^2 &= M_{(0)^3} + 176 + 24 \sqrt{M_{(0)^3} + 36} \end{aligned} \right\} \quad (3.23a)$$

$$m_\varphi^2 = -M_{(2)(0)^2} \quad (3.23b)$$

$$m_\pi^2 = 16 (M_{(1)^3} - 2) (M_{(1)^3} - 1) \quad (3.23c)$$

$$m_{\lambda_L} = - (M_{(1/2)^3} + 16) \quad (3.23d)$$

$$m_{\lambda_T} = M_{(3/2)(1/2)^2} + 8 \quad (3.23e)$$

$$\left. \begin{aligned} m_A^2 &= M_{(1)(0)^2} + 48 - 12 \sqrt{M_{(1)(0)^2} + 16} \\ m_W^2 &= M_{(1)(0)^2} + 48 + 12 \sqrt{M_{(1)(0)^2} + 16} \end{aligned} \right\} \quad (3.23f)$$

$$m_Z^2 = M_{(1)^2(0)} \quad (3.23g)$$

$$m_\chi = M_{(1/2)^3} \quad (3.23h)$$

$$m_h^2 = M_{(0)^3} \quad (3.23i)$$

It is evident at this point that the relations quoted at the end of the previous section reflect into relations among the masses (3.23). It is the goal of the next section to derive them. In Table 2, we discuss the possible occurrence of zero-modes.

Looking at Table 2, we see that the existence of spin-1 or spin- $3/2$ zero-modes is linked to either the symmetries or the cohomology of the internal manifold. A general theorem states that if a manifold has Killing vectors, it cannot have harmonic one-forms ($B_1 = 0$). However, the second Betti number B_2 can be different from zero also in the presence of Killing vectors. This means that the number of gauge fields which one obtains in the effective four-dimensional theory can be larger than that predicted by the isometry group. Indeed, for the $N = 2$ $SU(3) \times SU(2) \times U(1)$ compactification¹²⁾, we find $B_2 = 1$ and we have an extra vector multiplet. This will be discussed later.

Here we remark that the meaning of the critical values for the scalar, pseudoscalar and spin- $1/2$ zero-modes is more mysterious. However, when Killing spinors are present, the spectra of all the listed operators can be related to each other and the zero-modes required by the theory of supersymmetry representations in four dimensions are guaranteed to exist. In the next section, we shall show how. As a final remark, let us point out that the scalar fields Σ and the vectors W can never have zero-modes. They are always massive.

4. - BOSE-FERMI MASS RELATIONS FROM THE SUPERSYMMETRY TRANSFORMATION RULES OF THE BOSONS

When the vacuum has a certain symmetry the small fluctuations around the ground state fall into representations of its symmetry group. In the case we are dealing with, the vacuum is supersymmetric and has a super-isometry group:

$$G = OSp(4/N) \otimes \mathcal{G}' \quad (4.1)$$

Hence, all the "physical modes" we have classified in the previous section must be arranged into supermultiplets¹⁶⁾ of $Osp(4/N)$, each one belonging to a specific representation of the gauge group $\mathcal{G}'$.

In general, within each multiplet will coexist states of different masses [see Ref. 16) for a general study of $Osp(4/N)$ multiplets]. This implies that, in the presence of Killing spinors, the spectra of the various invariant operators $\boxtimes_z [\lambda_1 \lambda_2 \lambda_3]$ must all be related to each other. This is indeed the case as it will result from our study of the supersymmetry algebra. Actually the relation between the masses of supermultiplet members are universal and do not depend either on N or $\mathcal{G}'$. This enables us to give the general structure of the supermultiplets for an arbitrary Freund-Rubin compactification. As a by-product of our analysis, certain theorems connecting objects of differential geometric interest as the Betti numbers and the zeros of the Lichnerowicz and Rarita-Schwinger operators will be obtained.

It is an open question whether these theorems are specific to seven-dimensions or can be extended to other dimensions. In this section we shall derive the supersymmetry transformation rules of all the "physical modes" and for each pair of modes which are related to each other by supersymmetry we shall obtain the relation among their masses. This is possible because the eigenvalues of "harmonics" differing by a Killing spinor are in a universal calculable relation.

In the next section, we shall consider the "massless" multiplets of physical relevance which are associated to specific objects of differential geometric relevance (the Killing multiplet, the Betti multiplet, etc.).

Let us now turn to the derivation of our results. We start from the 11-dimensional supersymmetry transformation rules, which are obtained from the Lie derivative in a fermionic direction upon use of the explicit form (2.2) of the on-shell curvatures.

They read:

$$\delta_{\varepsilon} V^{\hat{a}} = i \bar{\varepsilon} \Gamma^{\hat{a}} \Psi \quad (4.2a)$$

$$\delta_{\varepsilon} A = \bar{\varepsilon} \Gamma_{\hat{a}\hat{b}} \Psi \wedge V^{\hat{a}} \wedge V^{\hat{b}} \quad (4.2b)$$

$$\delta_{\varepsilon} \Psi = \mathcal{D}\varepsilon - \frac{i}{3} \left(\Gamma^{\hat{a}_1\hat{a}_2\hat{a}_3} \varepsilon V^{\hat{a}_4} + \right. \\ \left. + \frac{1}{8} \Gamma^{\hat{a}_1\ldots\hat{a}_4\hat{m}} \varepsilon V_{\hat{m}} \right) F_{\hat{a}_1\ldots\hat{a}_4} \quad (4.2c)$$

We consider now a global $\text{Osp}(4/N)$ supersymmetry transformation, namely a parameter $\varepsilon(x,z)$ of the following type:

$$\varepsilon(x,z) = \varepsilon^M(x) \eta^M(z) \quad (4.3)$$

where $\varepsilon^M(x)$ is a four-dimensional, Majorana Killing spinor:

$$\mathcal{D}_m \varepsilon^M(x) = 2 \gamma_5 \gamma_m \varepsilon^M(x) \quad (4.4a)$$

$$\mathcal{D}_m \bar{\varepsilon}^M(x) = 2 \bar{\varepsilon}^M \gamma_5 \gamma_m \quad (4.4b)$$

and η^M is a seven-dimensional Killing spinor, namely a solution of Eq. (2.2). Inserting (4.3) into (4.2) and linearizing the transformation, we obtain the following rules for the fluctuations:

$$\delta_{\varepsilon} h_{\hat{a}\hat{b}} = i \bar{\varepsilon} \Gamma_{\hat{a}} \Psi_{\hat{b}} \quad (4.5a)$$

$$\delta_{\varepsilon} A_{\hat{a}\hat{b}\hat{c}} = \bar{\varepsilon} \Gamma_{[\hat{a}\hat{b}} \Psi_{\hat{c}]} \quad (4.5b)$$

$$\delta_\varepsilon \psi_{\hat{a}} = -\frac{1}{4} \Gamma_{\hat{m}\hat{n}} \varepsilon \Delta \omega_{\hat{a}}^{\hat{m}\hat{n}} - \frac{i}{3} \Gamma_{\hat{m}\hat{n}\hat{t}} \varepsilon \Delta F^{\hat{m}\hat{n}\hat{t}\hat{a}} - \frac{i}{24} \Gamma_{\hat{m}\hat{n}\hat{t}\hat{s}} \varepsilon \Delta F^{\hat{m}\hat{n}\hat{t}\hat{s}} \quad (4.5c)$$

where the fluctuations of the spin connection $\Delta \omega_{\hat{a}}^{\hat{m}\hat{n}}$ and of the field strength $\Delta F_{\hat{m}\hat{n}\hat{t}\hat{s}}$ have been defined in Eqs. (3.9) and (3.13). The derivative term in the gravitino transformation rule (4.5c) is absent because of the specific choice of the ε parameter [Eq. (4.3)] which implies:

$$D_{\hat{m}}^{SO(2,3) \otimes SO(8)} \varepsilon = 0 \quad (4.6)$$

where $D_{\hat{m}}$ is the de Sitter derivative in the product space $M_4 \times M_7$:

$$D = V^{\hat{m}} D_{\hat{m}} = d - \frac{1}{4} \omega^{\hat{m}\hat{n}} \gamma_{\hat{m}\hat{n}} - 2e\gamma_{50m} V^{\hat{m}} - \frac{1}{4} \tau_{\alpha\beta} B^{\alpha\beta} - \tau_{\alpha} B^{\alpha} \quad (4.7)$$

Inserting now the harmonic expansion (3.21) in Eqs. (4.5) we should be able to extract the transformation rules of all the physical modes. We have, however, to watch out for the following subtle point. The physical modes have been defined by a previous "gauge fixing" (3.20) of all the massive symmetries (both bosonic and fermionic): after the massless global $Osp(4/N)$ transformation (4.5) has been performed, it is no longer guaranteed that the new fluctuations:

$$h'_{\hat{a}\hat{b}} = h_{\hat{a}\hat{b}} + \delta_\varepsilon h_{\hat{a}\hat{b}} \quad (4.8a)$$

$$\psi'_{\hat{a}} = \psi_{\hat{a}} + \delta_\varepsilon \psi_{\hat{a}} \quad (4.8b)$$

$$A'_{\hat{a}\hat{b}\hat{c}} = A_{\hat{a}\hat{b}\hat{c}} + \delta_\varepsilon A_{\hat{a}\hat{b}\hat{c}} \quad (4.8c)$$

still satisfy the gauge conditions (3.20). Before reading off the $Osp(4/N)$ supersymmetry laws of the physical modes we must, therefore, perform those massive symmetry transformations (with field-dependent parameters) which, at the linear level, restore the gauge conditions (3.20) broken by the transformation (4.5).

For this purpose, we introduce massive general co-ordinate, photon-gauge and supersymmetry parameters:

$$\xi_{\hat{a}} = \bar{\varepsilon} M_{\hat{a}}^{\cdot \hat{b}} \psi_{\hat{b}} \quad (4.9a)$$

$$\Lambda_{\hat{a}\hat{b}} = \bar{\varepsilon} N_{\hat{a}\hat{b}}^{\cdot \hat{c}} \psi_{\hat{c}} \quad (4.9b)$$

$$\varepsilon' = P(\mathcal{H}, A) \varepsilon \quad (4.9c)$$

which are linear functions of the fluctuation fields $\phi, \mathcal{H}, A$ and of the massless supersymmetry parameter (4.3).

In Eq. (4.9), $M_{\hat{a}}^{\cdot \hat{b}}$ and $N_{\hat{a}\hat{b}}^{\cdot \hat{c}}$ and P are matrices in the 32-dimensional space of the 11-dimensional spinors: and are purely numerical, while is a linear function of the bosonic fluctuation fields. Then we define a new transformation δ_{ε} under the combined effect of (4.5) and (4.6) where we consistently keep only the linear terms in $\mathcal{H}, A, \phi$, discarding all the rest:

$$\hat{\delta}_{\varepsilon} \mathcal{H}_{\hat{a}\hat{b}} = \delta_{\varepsilon} \mathcal{H}_{\hat{a}\hat{b}} + \mathcal{D}_{\{\hat{a}} \xi_{\hat{b}\}} \quad (4.10a)$$

$$\hat{\delta}_{\varepsilon} A_{\hat{a}\hat{b}\hat{c}} = \delta_{\varepsilon} A_{\hat{a}\hat{b}\hat{c}} + \mathcal{D}_{\hat{c}\hat{a}} \Lambda_{\hat{b}\hat{c}} + 4 \xi^{\hat{m}} F_{\hat{m}\hat{a}\hat{b}\hat{c}} \quad (4.10b)$$

$$\hat{\delta}_{\varepsilon} \psi_{\hat{a}} = \delta_{\varepsilon} \psi_{\hat{a}} + D_{\hat{a}}^{SO(2,3) \otimes SO(8)} \varepsilon' \quad (4.10c)$$

and we require that $\delta_{\varepsilon} \mathcal{H}_{\hat{a}\hat{b}}, \delta_{\varepsilon} A_{\hat{a}\hat{b}\hat{c}}, \delta_{\varepsilon} \psi_{\hat{a}}$ satisfy the gauge conditions (3.20). This determines the explicit form of the compensating massive transformations (4.9) and the supersymmetry rules of the "physical modes" can now be extracted from δ_{ε} . Once the SUSY rules are given from them we are able to extract the relations among bosonic and fermionic harmonics anticipated in Table 1 of Section 2, and the associated mass relations.

For this purpose, we study in this section the bosonic transformation rules.

1) - Scalars 0^+

From Eq. (3.21), we see that the two types of scalars, those associated to the eigenfunctions of the Lichnerowicz operator $Y_{(\alpha\beta)}$ and those associated to eigenfunctions of the Laplacian operator Y are both contained in the fluctuation $\mathcal{H}_{\alpha\beta}$. Hence from the transformation of this object we shall get the x-space SUSY rules we are interested in. From (3.21d) and (4.5), we obtain:

$$\begin{aligned} \delta_\varepsilon \mathcal{H}_{\alpha\beta} &= i \bar{\varepsilon} \Gamma_\alpha \psi_\beta = i \bar{\varepsilon}^M \gamma_5 \lambda_\tau^{\{R_i\} \otimes \{\Lambda\}} \bar{\eta}^M \tau_{\alpha} \square_{\beta}^{\{R_i\} \otimes \{\Lambda\}} \\ &+ i \bar{\varepsilon}^M \gamma_5 \lambda_L^{\{R_i\} \otimes \{\Lambda\}} \bar{\eta}^M \tau_{\alpha} \left[\nabla_{\beta} \square^{\{R_i\} \otimes \{\Lambda\}} \right]^{3/2} \end{aligned} \quad (4.11)$$

where the capital Latin indices are $O(N)$ indices, carried by the supersymmetry parameters ε^M , by the Killing spinor on the internal manifold η^M and also by the spinor harmonics E_α and E which have to be in some irreducible $O(N)$ representation denoted by $\{R_i\}$. The capital Greek indices, on the other hand, are $\mathcal{S}'$ indices and are carried only by the harmonics E_α and E . The objects:

$$\bar{\eta} \tau_\alpha \square, \quad \bar{\eta} \tau_\alpha \mathcal{D} \square$$

are bosonic functions on the internal manifold and will be decomposed into eigenfunctions of the bosonic mass operators. The harmonics one obtains will be in the same $\mathcal{S}'$ representation as E_α and E and in one of the $O(N)$ representations which are contained in the direct product

$$\square \otimes \{R_i\} = ? \oplus ? \oplus ? \oplus \dots \oplus ? \quad (4.12)$$

Which of the (4.12) $O(N)$ representations are actually allowed and which are identical zero depends on the specific properties of the internal manifold considered and cannot be decided at this level of generality. However, the interesting point is that the eigenvalue of the relevant $\square_z$ operator is equal for all the question-marked representations (4.12) and depends only on the eigenvalue of the spinorial $\square_z$ either on E_α or E . This means that although the $O(N)$ assignments depend on the specific coset, the mass relations are universal for all Freund-Rubin solutions with even a different number of supersymmetries (provided $N \geq 1$).

To emphasize this point and also to simplify the notation from now on, we shall omit the $O(N)$ and $\mathcal{S}$ representation indices which are understood and can be restored any time they are explicitly needed. Hence we rewrite Eq. (4.11) as:

$$\delta_\varepsilon \mathcal{H}_{\alpha\beta} = i \bar{\varepsilon} \gamma_5 \lambda_\tau \bar{\eta} \tau_{\alpha} \square_{\beta} + i \bar{\varepsilon} \gamma_5 \lambda_\tau \bar{\eta} \tau_{\alpha} [\nabla_{\beta} \square]^{3/2} \quad (4.13)$$

a) Lichnerowicz scalars

Recalling Eq. (3.21d) and noting that, due to the irreducibility of $\mathbb{E}_\alpha$, $\bar{\eta} \tau_\alpha \mathbb{E}_\beta$ is automatically traceless, we can identify the harmonic $Y_{(\alpha\beta)}$ of the Lichnerowicz scalar with the transverse part of $\bar{\eta} \tau_{\alpha} \mathbb{E}_\beta$:

$$Y_{(\alpha\beta)}^T = [\bar{\eta} \tau_{\alpha} \square_{\beta}]^T \quad (4.14)$$

The longitudinal part of $\bar{\eta} \tau_{\alpha} \mathbb{E}_\beta$, being the symmetrized derivative of a vector, can be compensated by a general co-ordinate transformation and we can forget about it. Our task is now that of determining the transverse part (4.14) and of evaluating the corresponding eigenvalue of the Lichnerowicz operator once we know that

$$\tau^\alpha \square_\alpha = \mathcal{D}^\alpha \square_\alpha = 0 \quad (4.15)$$

$$\mathcal{D}^2 \square_\alpha = \left(M_{(3/2)(1/2)^2} + 5 \right) \square_\alpha \quad (4.16)$$

Our procedure is the following. We consider the following transverse vector:

$$\xi_\alpha^T = \bar{\eta} \square_\alpha \quad (\mathcal{D}^\alpha \xi_\alpha^T = 0) \quad (4.17)$$

and taking its symmetrized derivative, we obtain a traceless longitudinal harmonic $Y_{(\alpha\beta)}^L$ with the following form:

$$Y_{(\alpha\beta)}^L = \mathcal{D}_{(\alpha} \xi_{\beta)}^T = -\bar{\eta} \tau_{\{\alpha} \square_{\beta\}} + \bar{\eta} \mathcal{D}_{\{\alpha} \square_{\beta\}} \quad (4.18)$$

The transverse harmonic $Y_{(\alpha\beta)}^T$ is therefore an independent linear combination of the same two objects appearing on the right-hand side of Eq. (4.18).

Setting

$$Y_{(\alpha\beta)}^T = a \bar{\eta} \tau_{\{\alpha} \square_{\beta\}} + b \bar{\eta} \mathcal{D}_{\{\alpha} \square_{\beta\}} \quad (4.19)$$

and computing the divergence $\mathcal{D}^{\alpha} Y_{(\alpha\beta)}^T$ we find that it is zero if and only if:

$$a/b = - \left(M_{(3/2)(1/2)^2} + 13 \right) \quad (4.20)$$

Therefore if we normalize the transverse harmonic in the following way:

$$a = \left(M_{(3/2)(1/2)^2} + 13 \right) / \left(M_{(3/2)(1/2)^2} + 12 \right) \quad (4.21)$$

factorizing $Y_{(\alpha\beta)}^T$ on both sides of Eq. (4.13), we obtain the following x-space transformation rule:

$$\mathcal{D}_\varepsilon \varphi = i \bar{\varepsilon} \gamma_5 \lambda_T \quad (4.22)$$

The question we have still to answer is what is the mass of ϕ in terms of the mass of λ_T . We answer this question by computing the eigenvalue of the Lichnerowicz operator on $Y_{(\alpha\beta)}^T$. Recalling Eq. (2.11e) we have, by definition:

$$\square_z^{(2)(0)^2} Y_{(\alpha\beta)} = (\square + 40) Y_{(\alpha\beta)} - 4 \zeta^{\alpha\lambda\beta\mu} Y_{(\lambda\mu)} \quad (4.23)$$

and computing $\square^{(2)(0)^2}$ separately on the two terms of (4.19), we find the following (2x2) matrix:

$$\begin{pmatrix} \square_{(2)(0)^2}^z \begin{pmatrix} \bar{\eta}_{\alpha} \tau_{\beta\gamma} \square_{\beta\gamma} \\ \bar{\eta}_{\alpha} \mathcal{D}_{\beta\gamma} \square_{\beta\gamma} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (M_{3/2} + 6)^2 & , & -4 \\ 4(M_{3/2} + 13) & , & M_{3/2}^2 + 8M_{3/2} - 20 \end{pmatrix} \begin{pmatrix} \bar{\eta}_{\alpha} \tau_{\beta\gamma} \square_{\beta\gamma} \\ \bar{\eta}_{\alpha} \mathcal{D}_{\beta\gamma} \square_{\beta\gamma} \end{pmatrix} \quad (4.24)$$

Using (4.24) we easily see that the transverse linear combination defined by Eq. (4.20) is an eigenstate of $\square_{(2)(0)^2}^z$ with eigenvalue:

$$M_{(2)(0)^2} = (M_{(3/2)(1/2)^2} + 4)(M_{(3/2)(1/2)^2} + 8) \quad (4.25)$$

Before posing for a moment to discuss the relevance of Eq. (4.25), we note that the derivation of Eq. (4.24) is straightforward although lengthy.

Essential tools are the following two identities:

$$\square \square_{\alpha} = (\square \square - 42) \square_{\alpha} + R_{\mu\nu}^{\alpha\beta} \tau^{\mu\nu} \square_{\beta} \quad (4.26a)$$

$$\mathcal{D}_{\alpha} (\square - 32) Y_{\beta\gamma} = (\square \delta_{\alpha\beta}^{\gamma\mu} - 4 \epsilon^{\alpha\lambda\beta\mu}) \mathcal{D}_{\lambda} Y_{\mu\gamma} - 8 \delta_{\alpha\beta}^{\gamma\mu} \mathcal{D}_{\mu} Y_{\gamma} \quad (4.26b)$$

Now, coming back to Eq. (4.25), we recall the field Eqs. (3.22) and (3.23) of the physical modes and we transform (4.25) into the following mass relation:

$$m_{\varphi}^2 = m_{\lambda_T} (m_{\lambda_T} - 4) \quad (4.27)$$

We see that massless transverse spinors transform into massless Lichnerowicz scalars.

b) Laplacian Scalars

The Laplacian scalars are contained in the trace of $\mathcal{H}_{\alpha\beta}$ and therefore their transformation rule is determined by

$$\hat{\nabla}_{\epsilon} \mathcal{H}_{\mu\mu} = ? \quad (4.28)$$

In order to replace the question mark on the right-hand side of Eq. (4.28) with a meaningful expression we must determine the compensating co-ordinate transformation which restores the gauge condition (3.20). Indeed we have $\delta_\varepsilon \mathcal{H}_{\alpha\beta} \delta^{\alpha\beta} = 0$ but

$$\mathcal{D}^\alpha \delta_\varepsilon \mathcal{H}_{\alpha\beta} \neq 0 \quad (4.29)$$

One piece of the compensating co-ordinate transformation has already been introduced in the previous subsection at the moment when the longitudinal part $Y_{(\alpha\beta)}^L$ of $\bar{\eta}\tau_{\{\alpha}\Xi_{\beta\}}$ has been thrown away. This piece is irrelevant to our purposes because, being traceless, does not contribute to $\mathcal{H}_{\mu\mu}$. The second piece, relevant to us now, must compensate the second term of Eq. (4.13). Hence we must find a vector ξ_μ obeying the following relation:

$$\begin{aligned} \mathcal{D}_{\{\alpha} \xi_{\beta\}} - \frac{1}{7} \delta_{\alpha\beta} \mathcal{D}^\mu \xi_\mu &= -i \bar{\varepsilon} \gamma_5 \lambda_L \bar{\eta} \tau_{\{\alpha} (\nabla_{\beta\}} \Xi) \Big)^{3/2} = \\ &= -i \bar{\varepsilon} \gamma_5 \lambda_L \left\{ \bar{\eta} \tau_\alpha \mathcal{D}_\beta \Xi - \frac{1}{7} \delta_{\alpha\beta} \bar{\eta} \not{\mathcal{D}} \Xi \right\} \end{aligned} \quad (4.30)$$

Such a vector is easily determined to be:

$$\xi_\alpha = -i \bar{\varepsilon} \gamma_5 \lambda_L (\bar{\eta} \tau_\alpha \Xi) \quad (4.31)$$

and this implies that, under the combined transformation supersymmetry plus co-ordinate, we have:

$$\begin{aligned} \hat{\delta}_\varepsilon \mathcal{H}_{\mu\mu} &= \mathcal{D}^\nu \xi_\nu = -i \bar{\varepsilon} \gamma_5 \lambda_L (\bar{\eta} \not{\mathcal{D}} \Xi - \bar{\eta} \tau^\mu \tau_\mu \Xi) = \\ &= -i \bar{\varepsilon} \gamma_5 \lambda_L M_{(\frac{1}{2})^3} \bar{\eta} \Xi \end{aligned} \quad (4.32)$$

where we used the information that Ξ is an eigenstate of $\not{\Xi}_{(\frac{1}{2})^3}$ with eigenvalue $M_{(\frac{1}{2})^3}$.

We can now identify the harmonic of the Laplacian scalars with $\bar{\eta}\Xi$. We fix a normalization which will turn out useful later:

$$Y = \frac{1}{7} \bar{\eta} \Xi \quad (4.33)$$

Recalling Eq. (3.21) and comparing it with (4.32) and (4.33), we obtain:

$$\begin{aligned} (6 - \sqrt{M_{(0)}^3 + 36}) \delta_\varepsilon S + (6 + \sqrt{M_{(0)}^3 + 36}) \delta_\varepsilon \bar{Z} = \\ = M_{(1/2)}^3 i \bar{\varepsilon} \gamma_5 \lambda_L \end{aligned} \quad (4.34)$$

An elementary calculation now yields the eigenvalue of $\square_{(0)}^3 = \square$ on the harmonic (4.33). We obtain

$$\square Y = M_{(0)}^3 Y \quad (4.35a)$$

$$M_{(0)}^3 = M_{(1/2)}^3 (M_{(1/2)}^3 + 12) \quad (4.35b)$$

Using (4.35b) into (4.34) we obtain:

$$\begin{aligned} -M_{(1/2)}^3 \delta_\varepsilon S + (M_{(1/2)}^3 + 12) \delta_\varepsilon \bar{Z} = M_{(1/2)}^3 i \bar{\varepsilon} \gamma_5 \lambda_L \quad (\text{if } M_{(1/2)}^3 \geq -6) \\ (M_{(1/2)}^3 + 12) \delta_\varepsilon S - M_{(1/2)}^3 \delta_\varepsilon \bar{Z} = M_{(1/2)}^3 i \bar{\varepsilon} \gamma_5 \lambda_L \quad (\text{if } M_{(1/2)}^3 < -6) \end{aligned} \quad (4.36)$$

In order to write the explicit form of the $\delta_\varepsilon S$ and $\delta_\varepsilon \Sigma$ transformation one needs now to consider the $\delta_\varepsilon A_{abc}$ variation which yields an independent equation $\alpha \delta_\varepsilon S + \beta \delta_\varepsilon \Sigma = \gamma i \bar{\varepsilon} \gamma_5 \lambda_L$. We need not do this since the mass relations are already available at this level. From (4.35) and (3.23) we get:

$$m_S^2 = m_{\lambda_L} (m_{\lambda_L} - 4) ; m_{\bar{Z}}^2 = (m_{\lambda_L} + 24)(m_{\lambda_L} + 20) \quad (\text{if } m_{\lambda_L} \geq -10) \quad (4.37a)$$

$$m_S^2 = (m_{\lambda_L} + 24)(m_{\lambda_L} + 20) ; m_{\bar{Z}}^2 = m_{\lambda_L} (m_{\lambda_L} - 4) \quad (\text{if } m_{\lambda_L} < -10) \quad (4.37b)$$

Equations (4.37) tell us that the partners of the massless scalars are spinors which are either massless or have the critical masses

$$m_{1/2} = \begin{cases} 4 \\ -24 \\ -20 \end{cases} \quad (4.38)$$

It is encouraging to note that the second eigenvalue of the mass operators for the SU(3) and SU(2) gauginos on M^{pqr} spaces is indeed $m_{\frac{1}{2}} = -20$ [see Tables in Ref. 3)]. Moreover, the value $m_{\frac{1}{2}} = -24$ is realized by the mode in the $(10, 3)$ of SU(3)×SU(2) which seems therefore the partner of the scalar zero-modes found in the same irrep¹⁸⁾.

2) - Pseudoscalars 0^-

The pseudoscalars $\pi(x)$ are contained in the three-index photon with internal indices $A_{\alpha\beta\gamma}(x, z)$ and are proportional to the transverse harmonic $Y_{\alpha\beta\gamma}^T(z)$.

From Eqs. (4.5b) and (3.21), we obtain:

$$\begin{aligned} \delta_\varepsilon A_{\alpha\beta\gamma} &= \delta_\varepsilon \pi Y_{[\alpha\beta\gamma]}^T + \text{longitudinal terms} \\ &= \bar{\varepsilon} \lambda_T \bar{\eta} \tau_{[\alpha\beta} \Xi_{\gamma]} + \bar{\varepsilon} \lambda_L \bar{\eta} \tau_{[\alpha\beta} (\nabla_{\gamma]} \Xi)^{3/2} \end{aligned} \quad (4.39)$$

As before, we have now to extract the transverse parts of the functions $\bar{\eta} \tau_{[\alpha\beta} \Xi_{\gamma]}$ and $\bar{\eta} \tau_{[\alpha\beta} (\nabla_{\gamma]} \Xi)^{3/2}$ and then to calculate their eigenvalues under the mass operator $\Box_z^{(1)3}$. In this task we have a considerable advantage from the following observation. By its very definition (2.11d) the operator $\Box_z^{(1)3}$ has eigenvalue zero on every longitudinal harmonic ($\Box_z$ annihilates every exact three-form). Therefore if we know that both the longitudinal and transverse harmonics we want to construct are a linear combination of the same

two objects then it is sufficient to evaluate the 2×2 matrix of the $\boxtimes_z^{(1)^3}$ operator on these two building blocks. This matrix will have a vanishing eigenvalue and a finite one: the eigenvector corresponding to the zero eigenvalue is the longitudinal one, while the eigenvector belonging to the non-zero eigenvalue is the transverse harmonic we are looking for. Let us begin with:

a) Contribution from the transverse spinors

In this case we see that the longitudinal and transverse harmonics are linear combinations of the following type:

$$Y_{\alpha\beta\gamma}^L = \mathcal{D}_{[\alpha} \bar{\eta} \tau_{\beta} \square_{\gamma]} = -\bar{\eta} \tau_{[\alpha} \square_{\beta} \gamma] - \bar{\eta} \tau_{\alpha} \mathcal{D}_{\beta} \square_{\gamma]} \quad (4.40a)$$

$$Y_{\alpha\beta\gamma}^T = \alpha \bar{\eta} \tau_{[\alpha} \square_{\beta} \gamma] + \beta \bar{\eta} \tau_{\alpha} \mathcal{D}_{\beta} \square_{\gamma]} \quad (4.40b)$$

Recalling the definition (2.11d) by an explicit evaluation, we get:

$$\boxtimes_z^{(1)^3} \begin{pmatrix} \bar{\eta} \tau_{[\alpha} \square_{\beta} \gamma] \\ \bar{\eta} \tau_{\alpha} \mathcal{D}_{\beta} \square_{\gamma]} \end{pmatrix} = \frac{1}{24} \begin{pmatrix} -6[M_{3/2}+2], & 12 \\ 6[M_{3/2}+2], & -12 \end{pmatrix} \begin{pmatrix} \bar{\eta} \tau_{[\alpha} \square_{\beta} \gamma] \\ \bar{\eta} \tau_{\alpha} \mathcal{D}_{\beta} \square_{\gamma]} \end{pmatrix} \quad (4.41)$$

where $M_{(3/2)(1/2)^2}$ is the eigenvalue of $\boxtimes_{(3/2)(1/2)^2}$ on E_{α} . From Eq. (4.41) we recover that the longitudinal harmonic (4.40a) has indeed an eigenvalue zero while we find that with

$$\alpha/\beta = - \left(1 + \frac{1}{2} M_{(3/2)(1/2)^2} \right) \quad (4.42)$$

the harmonic (4.40b) is transverse and has an eigenvalue:

$$M_{(1)^3} = -\frac{1}{4} \left(M_{(3/2)(1/2)^2} + 4 \right) \quad (4.43)$$

Recalling Eqs. (3.23) from Eq. (4.43), we obtain the following mass relation:

$$m_{\pi}^2 = m_{\lambda_T} (m_{\lambda_T} + 4) \quad (4.44)$$

from which we see that massless transverse spinors transform into massless pseudoscalars.

We can now normalize the transverse harmonic in such a way that the transverse part of $\bar{\eta}\tau_{[\alpha\beta}^{\Xi}\gamma]$ has coefficient 1, namely:

$$\beta = \frac{2}{M_{(3/2)}(1/2)^2 + 4} ; \alpha = \frac{M_{(3/2)}(1/2)^2 + 2}{M_{(3/2)}(1/2)^2 + 4} \quad (4.45)$$

and we shall have

$$\delta \pi = \Xi \lambda_T + (\lambda_L - \text{term}) \quad (4.46)$$

b) Contribution from the longitudinal spinors

In this case we set

$$Y_{\alpha\beta\gamma}^L = \mathcal{D}_{\alpha} \bar{\eta} \tau_{\beta\gamma}^{\Xi} \Xi = -\bar{\eta} \tau_{\alpha\beta\gamma}^{\Xi} \Xi + \bar{\eta} \tau_{\alpha\beta}^{\Xi} \mathcal{D}_{\gamma} \Xi \quad (4.47a)$$

$$Y_{\alpha\beta\gamma}^T = \alpha' \bar{\eta} \tau_{\alpha\beta\gamma}^{\Xi} \Xi + \beta' \bar{\eta} \tau_{\alpha\beta}^{\Xi} \mathcal{D}_{\gamma} \Xi \quad (4.47b)$$

and we obtain:

$$\boxtimes_z^{(1)^3} \begin{pmatrix} \bar{\eta} \tau_{\alpha\beta\gamma}^{\Xi} \Xi \\ \bar{\eta} \tau_{\alpha\beta}^{\Xi} \mathcal{D}_{\gamma} \Xi \end{pmatrix} = \begin{pmatrix} \frac{3}{4}, -\left[1 + \frac{1}{4}(M_{(1/2)}^3 + 7)\right] \\ \frac{3}{4}, -\left[1 + \frac{1}{4}(M_{(1/2)}^3 + 7)\right] \end{pmatrix} \begin{pmatrix} \bar{\eta} \tau_{\alpha\beta\gamma}^{\Xi} \Xi \\ \bar{\eta} \tau_{\alpha\beta}^{\Xi} \mathcal{D}_{\gamma} \Xi \end{pmatrix} \quad (4.48)$$

Therefore the harmonic (4.46b) with the condition

$$\alpha'/\beta' = - (M_{(1/2)}^3 + 11)/3 \quad (4.49)$$

is a transverse one with eigenvalue:

$$M_{(1)^3} = -\frac{1}{4} \left(M_{(1/2)^3} + 8 \right) \quad (4.50)$$

Comparing (4.50) with Eq. (3.23) we obtain the new mass relation

$$m_{\pi}^2 = m_{\lambda_L} (m_{\lambda_L} + 4) \quad (4.51)$$

which is formally analogue to Eq. (4.44).

The question now is whether the same π -field can have contributions, in its transformation rule, from both the transverse and longitudinal spinors. This can happen only if there are transverse and longitudinal spinors in the same representation of the gauge group and with their masses in the relation

$$m_{\lambda_L} (m_{\lambda_L} + 4) = m_{\lambda_T} (m_{\lambda_T} + 4) \quad (4.52)$$

This question can be decided only by an analysis of the specific manifold.

3) - Vectors 1⁻ (from two-forms)

This type of spin-1 fields is contained in the photon $A_{m\alpha\beta}$. Using Eq. (4.5b), we obtain:

$$\nabla_{\varepsilon} A_{m\alpha\beta} = \frac{1}{3} \bar{\varepsilon} \Gamma_{\alpha\beta} \psi_m + \frac{2}{3} \bar{\varepsilon} \Gamma_{m\alpha} \psi_{\beta} \quad (4.53)$$

Then, using Eq. (3.21) and the following identities:

$$\bar{\eta} \tau_{[\alpha} \left\{ \nabla_{\beta]} \bar{\zeta} \right\}^{3/2} = \mathcal{D}_{\beta} \left(\bar{\eta} \tau_{\alpha} \bar{\zeta} \right) - \frac{1}{7} \left(M_{(1/2)^3} + 4 \right) \bar{\eta} \tau_{\alpha\beta} \bar{\zeta} \quad (4.54a)$$

$$\bar{\varepsilon} \left[\mathcal{D}_m \lambda_L \right]^{3/2} = \mathcal{D}_m \bar{\varepsilon} \lambda_L + \frac{1}{4} \left(M_{(1/2)^3} + 8 \right) \bar{\varepsilon} \gamma_5 \gamma_m \lambda_L \quad (4.54b)$$

we can rewrite (4.53) as:

$$\begin{aligned} \delta_\varepsilon A_{m\alpha\beta} = & \left\{ \frac{1}{3} \bar{\varepsilon} \chi_m + \frac{\frac{4}{7} M_{(1/2)^3} + 8}{3(M_{(1/2)^3} + 8)} \mathcal{D}_m (\bar{\varepsilon} \lambda_L) \right\} \bar{\eta} \tau_{\alpha\beta} \square \\ & - \frac{2}{3} \bar{\varepsilon} \gamma_5 \gamma_m \lambda_T \bar{\eta} \tau_{[\alpha} \square_{\beta]} - \frac{2}{3} \bar{\varepsilon} \gamma_5 \gamma_m \lambda_L \mathcal{D}_{[\beta} \bar{\eta} \tau_{\alpha]} \square \end{aligned} \quad (4.55)$$

Now we observe that the last term in Eq. (4.55) is a pure gauge transformation of parameter

$$\Lambda_{m\alpha} = - \frac{2}{3} \bar{\varepsilon} \gamma_5 \gamma_m \lambda_L \bar{\eta} \tau_\alpha \square \quad (4.56)$$

and can therefore be suppressed.

Also the third term in the curly bracket comes from a gauge transformation of parameter

$$\Lambda_{\alpha\beta} = \text{const} \times \bar{\varepsilon} \lambda_L \bar{\eta} \tau_{\alpha\beta} \square \quad (4.57)$$

The point now is that after supersymmetry transformation the $A_{\alpha\beta\gamma}$ field, already analyzed in the previous subsection, no longer satisfies the gauge condition

$$\mathcal{D}^\alpha A_{\alpha\beta\gamma} = 0 \quad (4.58)$$

Indeed, as we have shown, $\delta_\varepsilon A_{\alpha\beta\gamma}$ always decomposes into a transverse $\oplus$ a longitudinal part. Recalling Eq. (4.39), we have:

$$\delta_\varepsilon A_{\alpha\beta\gamma} = \lambda_T \text{-term} + \bar{\varepsilon} \lambda_L \left\{ \bar{\eta} \tau_{[\alpha\beta} \mathcal{D}_{\gamma]} \square - \frac{1}{7} (M_{(1/2)^3} + 7) \bar{\eta} \tau_{\alpha\beta\gamma} \square \right\} \quad (4.59)$$

Recalling now Eqs. (4.49) and (4.47a) we see that we can write

$$\begin{aligned} \delta_\varepsilon A_{\alpha\beta\gamma} = & \lambda_T \text{-term} + \\ & + \bar{\varepsilon} \lambda_L \left\{ \frac{3 M_{(1/2)^3}}{7(M_{(1/2)^3} + 8) \beta'} Y_{\alpha\beta\gamma}^T + \frac{\frac{4}{7} M_{(1/2)^3} + 8}{M_{(1/2)^3} + 8} Y_{\alpha\beta\gamma}^L \right\} \end{aligned} \quad (4.60)$$

The gauge transformation which restores the gauge condition (4.58) has a parameter

$$\Lambda_{\beta\gamma}^L = - \frac{\frac{4}{7} M_{(1/2)}^3 + 8}{M_{(1/2)}^3 + 8} \bar{\varepsilon} \lambda_L \bar{\eta} \tau_{\beta\gamma} \square \quad (4.61)$$

This transformation, on the other hand, affects also $A_{m\alpha\beta}$. Indeed its contribution:

$$\delta_\Lambda A_{m\beta\gamma} = \frac{1}{3} \mathcal{D}_m \Lambda_{\beta\gamma}^L \quad (4.62)$$

exactly cancels the third term in the curly bracket of Eq. (4.55). The conclusion is that the longitudinal spinors never appear in the Z_m transformation rule. Before proceeding, we have to consider a similar mechanism for the λ_T -term. Recalling Eqs. (4.40) and (4.42), we can write:

$$\delta_\varepsilon A_{\alpha\beta\gamma} = - \frac{2}{M_{(3/2)}(1/2)^2 + 4} \bar{\varepsilon} \lambda_T \left[\frac{1}{\beta} Y_{\alpha\beta\gamma}^T + Y_{\alpha\beta\gamma}^L \right] + \lambda_L \text{-term} \quad (4.63)$$

where $Y_{\alpha\beta\gamma}^T$ and $Y_{\alpha\beta\gamma}^L$ are given in Eqs. (4.40). The gauge transformation which restores the gauge condition (4.58) has a parameter

$$\Lambda_{\alpha\beta}^T = - \frac{2}{M_{(3/2)}(1/2)^2 + 4} \bar{\varepsilon} \lambda_T \bar{\eta} \tau_{\alpha\beta} \square \quad (4.64)$$

As in the previous case, it also contributes to $A_{m\alpha\beta}$ with the following term:

$$\delta_\Lambda A_{m\alpha\beta} = \frac{1}{3} \mathcal{D}_m \Lambda_{\alpha\beta}^T \quad (4.65)$$

Hence the final outcome of our analysis is that under the combined action of supersymmetry + restoring gauge transformation, we have:

$$\begin{aligned} \delta_\varepsilon A_{m\alpha\beta} &= \delta_\varepsilon Z_m Y_{\alpha\beta}^T = \frac{1}{3} \bar{\varepsilon} \chi_m \left[\bar{\eta} \tau_{\alpha\beta} \square \right]^{(\text{Transverse})} \\ &- \frac{2}{3(M_{(3/2)}(1/2)^2 + 4)} \left[\bar{\varepsilon} \mathcal{D}_m \lambda_T + (M_{(3/2)}(1/2)^2 + 4) \bar{\varepsilon} \chi_m \lambda_T \right] \left(\bar{\eta} \tau_{[\alpha} \square_{\beta]} \right)^{(\text{Transverse})} \end{aligned} \quad (4.66)$$

At this point, as usual, we must find the explicit form of the transverse parts indicated in (4.66) and evaluate the eigenvalue of the bosonic mass operator. Let us begin with the spin- $\frac{1}{2}$ contribution.

a) Spin- $\frac{1}{2}$ contribution

In the present case, the mass operator $\frac{(1)^2(0)}{z}$ is the Hodge de Rham operator which automatically has eigenvalue zero on longitudinal harmonics. Hence we set

$$Y_{[\alpha\beta]}^L = \mathcal{D}_{[\alpha} \bar{\eta}_{\beta]} \square_{\beta]} = -\bar{\eta}_{[\alpha} \tau_{\beta]} \square_{\beta]} + \bar{\eta}_{\beta]} \mathcal{D}_{[\alpha} \square_{\beta]} \quad (4.67a)$$

$$Y_{[\alpha\beta]}^T = \alpha \bar{\eta}_{[\alpha} \tau_{\beta]} \square_{\beta]} + \beta \bar{\eta}_{\beta]} \mathcal{D}_{[\alpha} \square_{\beta]} \quad (4.67b)$$

Recalling the definition (2.11c) and setting $M_{(3/2)(1/2)^2} = M_{3/2}$ after some lengthy algebra, we obtain:

$$\square_{\frac{(1)^2(0)}{z}} \begin{pmatrix} \bar{\eta}_{[\alpha} \tau_{\beta]} \square_{\beta]} \\ \bar{\eta}_{\beta]} \mathcal{D}_{[\alpha} \square_{\beta]} \end{pmatrix} = \begin{pmatrix} M_{3/2}^2 + 13M_{3/2} + 36, -M_{3/2} - 4 \\ M_{3/2}^2 + 13M_{3/2} + 36, -M_{3/2} - 4 \end{pmatrix} \begin{pmatrix} \bar{\eta}_{[\alpha} \tau_{\beta]} \square_{\beta]} \\ \bar{\eta}_{\beta]} \mathcal{D}_{[\alpha} \square_{\beta]} \end{pmatrix} \quad (4.68)$$

Hence the transverse harmonic is defined by the following condition:

$$\alpha/\beta = - \left(M_{3/2}^2 + 13M_{3/2} + 36 \right) / \left(M_{3/2} + 4 \right) \quad (4.69)$$

and has an eigenvalue:

$$M_{(1)^2(0)} = \left(M_{(3/2)(1/2)^2} + 8 \right) \left(M_{(3/2)(1/2)^2} + 4 \right) \quad (4.70)$$

Comparing with Eqs. (3.23) we obtain the mass relation:

$$m_Z^2 = m_{\lambda_T} (m_{\lambda_T} - 4) \quad (4.71)$$

which is identical to Eq. (4.27). This shows that the vector and the scalars which sit in the same multiplet as the transverse spinors have the same mass, in particular zero if $m_{\lambda_T} = 0$. The pseudoscalars, on the other hand, have a different mass, as is apparent from Eq. (4.52).

b) Spin $3/2$ contribution

We repeat the same analysis as in the previous case and we obtain:

$$Y_{\alpha\beta}^L = \mathcal{D}_{[\alpha} \bar{\eta} \tau_{\beta]} \xi = -\bar{\eta} \tau_{\alpha\beta} \xi - \bar{\eta} \tau_{[\alpha} \mathcal{D}_{\beta]} \xi \quad (4.72a)$$

$$Y_{\alpha\beta}^T = \alpha' \bar{\eta} \tau_{\alpha\beta} \xi + \beta' \bar{\eta} \tau_{\alpha} \mathcal{D}_{\beta} \xi \quad (4.72b)$$

and the harmonic (4.72b) with

$$\alpha'/\beta' = -\frac{1}{2} \left(M_{(1/2)^3} + 2 \right) \quad (4.73)$$

is transverse with eigenvalue:

$$M_{(1)^2(0)} = M_{(1/2)^3} \left(M_{(1/2)^3} + 8 \right) \quad (4.74)$$

This yields the mass relation:

$$m_Z^2 = m_\chi (m_\chi + 8) \quad (4.75)$$

From Eq. (4.75), one might conclude that there exists a massless Z-vector communicating with the massless gravitini: this is not true, however, because in this case the transverse harmonic for Z vanishes identically. Indeed, $m_\chi = 0 \rightarrow E = \eta$ and $M_{(1/2)^3} = 0 \rightarrow \alpha'/\beta' = -1$, so that

$$Y_{\alpha\beta}^T (m_\chi = 0) = \bar{\eta} \tau_{\alpha\beta} \eta - \bar{\eta} \tau_{\alpha} \mathcal{D}_{\beta} \eta$$

Hence, Formula (4.75) is meaningful only for $m_\chi \neq 0$.

4) - Vectors 1^+ (from 1-forms Y_α)

They are contained in $\mathcal{H}_{m\beta}$ whose transformation under supersymmetry is:

$$\delta_\varepsilon \mathcal{H}_{m\beta} = \frac{i}{2} \left(\bar{\varepsilon} \Gamma_m \psi_\beta + \bar{\varepsilon} \Gamma_\beta \psi_m \right) \quad (4.76)$$

Before utilizing (4.76) to obtain the transformation rule of the vector field A_m , we must remember that in order to restore the gauge (3.20b) we have performed a general co-ordinate transformation in seven dimensions which affects not only $\mathcal{H}_{\alpha\beta}$ but also $\mathcal{H}_{m\beta}$. The parameter of this general co-ordinate transformation is the sum of two terms, one is given by Eq. (4.31) and accounts for the λ_L contribution, the other is such as to compensate the longitudinal part of the term

$$i \bar{\varepsilon} \gamma_5 \lambda_T \bar{\eta} \tau_{\alpha} \square_{\beta}$$

analyzed in the subsection of the Lichnerowicz scalars. Recalling Eqs. (4.18), (4.19) and (4.20), we find

$$\left(i \bar{\varepsilon} \gamma_5 \lambda_T \bar{\eta} \tau_{\alpha} \square_{\beta} \right)^L = \frac{i}{M_{3/2}(1/2)^2 + 12} \bar{\varepsilon} \gamma_5 \lambda_T \mathcal{D}_{\alpha} \bar{\eta} \square_{\beta} \quad (4.77)$$

and therefore we conclude that the complete parameter of the compensating co-ordinate transformation is:

$$\xi_\alpha = - \frac{i}{M_{3/2} + 12} \bar{\varepsilon} \gamma_5 \lambda_T \bar{\eta} \square_{\alpha} - i \bar{\varepsilon} \gamma_5 \lambda_L \bar{\eta} \tau_{\alpha} \square \quad (4.78)$$

Hence the δ_ε transformation of $\mathcal{H}_{m\beta}$ is:

$$\begin{aligned} \hat{\delta}_\varepsilon \mathcal{H}_{m\beta} &= \frac{i}{2} \left(\bar{\varepsilon} \Gamma_m \psi_\beta + \bar{\varepsilon} \Gamma_\beta \psi_m \right) \\ &\quad + \frac{1}{2} \mathcal{D}_m \xi_\alpha \end{aligned} \quad (4.79)$$

Substituting Eqs. (3.21) and (4.78) into (4.79), we obtain:

$$\begin{aligned} \hat{\mathcal{S}}_{\varepsilon} \mathcal{H}_{m\beta} = & -\frac{i}{2} \left\{ \frac{1}{M_{3/2}+12} \left[\bar{\varepsilon} \gamma_5 D_m \lambda_T - (M_{3/2}+12) \bar{\varepsilon} \gamma_m \lambda_T \right] \right\} (\bar{\eta} \square_{\beta})^{(transv.)} \\ & + \frac{i}{2} \left\{ \bar{\varepsilon} \gamma_m - \frac{1}{M_{1/2}+8} \left[\frac{3}{7} M_{1/2} \bar{\varepsilon} \gamma_5 D_m \lambda_L + \left(\frac{5}{7} M_{1/2} + 8 \right) (M_{1/2}+8) \bar{\varepsilon} \gamma_m \lambda_L \right] \right\} (\bar{\eta} \tau_{\beta} \square)^{(transv.)} \end{aligned} \quad (4.80)$$

where, as usual ()^{transv} denotes the transverse part.

Now we note that $\bar{\eta} \square_{\beta}$ is already transverse and we have

$$\square_{(1)(0)^2} \bar{\eta} \square_{\beta} = M_{(1)(0)^2} \bar{\eta} \square_{\beta} \quad (4.81)$$

with

$$M_{(1)(0)^2} = M_{3/2} (M_{3/2} + 8) \quad (4.82)$$

which yields the following mass relations:

$$m_A^2 = m_{\lambda_T}^2 - 20m_{\lambda_T} + 96 \quad (\text{if } m_{\lambda_T} \geq 4) \quad (4.83a)$$

$$m_A^2 = m_{\lambda_T} (m_{\lambda_T} + 4) \quad (\text{if } m_{\lambda_T} < 4) \quad (4.83b)$$

From Eq. (4.83b) it seems that a transverse massless spinor λ_T transforms into a massless vector A_a : there is, however, a subtlety which we must not forget and which is apparent from Eq. (4.82). The massless vector A_a would correspond to a closed one-form Y_{α} and not to a Killing vector since it is a zero of the Hodge-de Rham operator. Now on a coset space, these forms do not exist since $B_1 = 0$. This means that the harmonic $\bar{\eta} \square_{\beta}$ has to vanish identically when $M_{3/2} = -8$:

$$\bar{\eta} \square_{\beta} \equiv 0 \iff \text{if } M_{3/2} = -8 \quad (4.84)$$

Hence the relations (4.83) hold only if $m_{\lambda_T} \neq 0$. Next we consider the contributions due to the spin- $3/2$ field and to the longitudinal spinor λ_L . We obtain the following result for the longitudinal and transverse parts of the harmonic:

$$Y_\alpha^L = \partial_\alpha \bar{\eta} \Xi = -\bar{\eta} \tau_\alpha \Xi + \bar{\eta} \partial_\alpha \Xi \quad (4.85a)$$

$$Y_\alpha^T = a \bar{\eta} \tau_\alpha \Xi + b \bar{\eta} \partial_\alpha \Xi \quad (4.85b)$$

$$a/b = - (M_{(1/2)}^3 + 13) \quad (4.85c)$$

and on the transverse part we find

$$M_{(1)(0)^2} = M_{(1/2)}^2 + 16 M_{(1/2)} + 48 \quad (4.86)$$

which yields the mass relation

$$m_A^2 = m_{\lambda_L} (m_{\lambda_L} - 4); m_W^2 = m_{\lambda_L}^2 - 28 m_{\lambda_L} + 192 \quad (\text{if } m_{\lambda_L} \leq 8) \quad (4.87a)$$

$$m_A^2 = m_{\lambda_L}^2 - 28 m_{\lambda_L} + 192; m_W^2 = m_{\lambda_L} (m_{\lambda_L} - 4) \quad (\text{if } m_{\lambda_L} \geq 8) \quad (4.87b)$$

$$m_A^2 = m_\chi (m_\chi + 4); m_W^2 = m_\chi^2 + 28 m_\chi + 192 \quad (\text{if } m_\chi \geq -8) \quad (4.87c)$$

$$m_A^2 = m_\chi^2 + 28 m_\chi + 192; m_W^2 = m_\chi (m_\chi + 4) \quad (\text{if } m_\chi \leq -8) \quad (4.87d)$$

We see again that the massless spinors transform into massless vectors. This time, however, the massless vectors come from Killing vectors since they correspond to $M_{(1)(0)^2} = 48$ and therefore they exist. The transformation rule of the fields W_m, A_m is then found by identifying either $\bar{\eta} \Xi_\alpha$ or Y_α^T with the harmonic sitting on the left-hand side of Eq. (4.80) and comparing with the similar equation for $\delta A_{ab\gamma}$.

5) - Gravitons

From Eq. (4.5a) we get

$$\delta_{\varepsilon} \mathcal{H}_{ab} = i \bar{\varepsilon} \gamma_{(a} \psi_{b)} \quad (4.88)$$

and in order to find the transformation rule of the graviton we are interested only in the traceless, transverse part of the right-hand side of Eq. (4.88). Comparing with Eqs. (3.21), we see that this is due only to the spin- $3/2$ field χ_a . Hence we get:

$$\delta_{\varepsilon} h_{ab} = 7 i \bar{\varepsilon} \gamma_{(a} \chi_{b)} \quad (4.89)$$

where the number 7 is due to the already chosen normalization (4.33) of the Y harmonic. The mass relation follows from a comparison of Eq. (4.35b) with Eq. (3.23):

$$m_h^2 = m_{\chi} (m_{\chi} + 12) \quad (4.90)$$

Hence the massless gravitini transform into massless gravitons but massive gravitini transform into gravitons of different mass.

We can now summarize our results in the following way. All the multiplets of the four-dimensional theory are contained in the range between spin-2 and spin-0 and depending on N they can start at some intermediate spin and stop at some other intermediate spin. Schematically, we have:

In Eq. (4.91) the arrows depart from fermions and go into bosons. This means that the specified fermion field appears in the transformation rule of the specified bosonic field.

The interesting outcome of the analysis we have carried through is that to each arrow of the diagram (4.91) there is an associated mass-relation between the bosonic and fermionic field which is universal, independent of the coset S/R and of the number of supersymmetries, provided $N \geq 1$.

5. - MASSLESS, QUASI-MASSLESS AND MATTER MULTIPLETS

The Kaluza-Klein theory is based on the idea that the massive modes emerging from compactification can be discarded and that an effective four-dimensional theory may be obtained by retaining only the massless modes. This procedure turns out to be in conflict with supersymmetry. Indeed, de Sitter supermultiplets contain, in general, fields of different masses so that, by retaining only the massless modes one does not, necessarily, retain an entire multiplet¹⁶⁾. This is an important ununderstood problem posed by Kaluza-Klein supergravity, which we already emphasized in the Introduction. We have nothing to add on this point. Nevertheless, it is clear that there are some supermultiplets which have a special status and should certainly be retained in the four-dimensional effective theory: they are those whose masses are all zero (massless multiplets). Other multiplets one might also retain are those which have a spin content not exceeding spin- $\frac{1}{2}$ (matter multiplets)^{*)} and which therefore exclude the trouble-maker massive spin-1 fields. Also interesting are the "quasi-massless" multiplets having one massless state among massive ones. Indeed one may ask the question whether quasi-massless multiplets are always matter multiplets. The answer seems, however, to be "not always". The nice point is that while the matter multiplets have not yet a clear interpretation the massless ones are linked to the essential properties of the manifold and can be studied in general. They are:

*) On this point we are indebted to S. Ferrara for important suggestions.

- a) The gravitational multiplet which is due to the existence of the Killing spinors on the manifold $\mathcal{S}/\mathcal{R}$. Its spin content is the usual one for the three cases $N = 1$, $N = 2$ and $N = 3$, namely $[2, \frac{3}{2}]$, $[2, \frac{3}{2}, 1]$ and $[2, \frac{3}{2}, 1, \frac{1}{2}]$, respectively.
- b) The Killing vector multiplet is due to the existence of the Killing vectors of $\mathcal{S}'$. Its content is that of a vector multiplet. In the three cases $N = 1$, $N = 2$ and $N = 3$, this content is $[1, \frac{1}{2}]$, $[1, \frac{1}{2}, 0^+, 0^-]$, $[1, \frac{1}{2}, 0^+, 0^-]$, respectively.
- c) The Betti vector multiplet is due to the existence of a closed harmonic two-form on $\mathcal{S}/\mathcal{R}$ (Betti number $B_2 \neq 0$). Its content is that of a vector multiplet and it is the same as for the Killing multiplet.

The "quasi-massless" scalar multiplet is a more intriguing object. It emerges because of the form of the mass formulae for the scalar field S , the pseudoscalar field π and the spinor field λ_L [see Eqs. (3.23)]

$$m_S^2 = M_{(0)3} + 176 - 24 \sqrt{M_{(0)3} + 36} \quad (5.1a)$$

$$m_\pi^2 = 16 (M_{(1)3} - 2) (M_{(1)3} - 1) \quad (5.1b)$$

$$m_{\lambda_L} = - (M_{(1/2)3} + 16) \quad (5.1c)$$

which are to be coupled with the relations among the spectra of the $M_{(0)3}$, $M_{(1)3}$, $M_{(1/2)3}$ eigenvalues given in Table I and repeated here for convenience:

$$M_{(0)3} = M_{(1/2)3} (M_{(1/2)3} + 12) \quad (5.2a)$$

$$M_{(1)3} = -\frac{1}{4} (M_{(1/2)3} + 8) \quad (5.2b)$$

From these equations, we see that there is a massless scalar S if the internal Laplacian has either a $M_{(0)3} = 64$ or an $M_{(0)3} = 160$ mode. From Eq. (5.2a) we see that for each choice of $M_{(0)3}$ there are two possible choices of $M_{(1/2)3}$ and which is actually realized depends on the properties of the manifold $\mathcal{S}/\mathcal{R}$. Explicitly, we find:

$$M_{(0)^3} = 64 \implies \begin{cases} M_{(1/2)^3} = -16 \implies m_{\lambda_L} = 0; m_{\pi}^2 = 0 \\ M_{(1/2)^3} = 4 \implies m_{\lambda_L} = -20; m_{\pi}^2 = 320 \end{cases}$$

$$M_{(0)^3} = 160 \implies \begin{cases} M_{(1/2)^3} = -20 \implies m_{\lambda_L} = 4; m_{\pi}^2 = 32 \\ M_{(1/2)^3} = 8 \implies m_{\lambda_L} = -24; m_{\pi}^2 = 480 \end{cases} \quad (5.3)$$

Now it is remarkable that the mass sum rule¹⁹⁾

$$m_{0+}^2 + m_{0-}^2 - 2 m_{1/2}^2 = 0 \quad (5.4)$$

which obtains in the most general coupling of an $N = 1$ $[0^+, 0^-, \frac{1}{2}]$ multiplet to supergravity is satisfied both in the first and the second case by the upper choice of $M_{(\frac{1}{2})^3}$ but not by the lower one. This has the implication that in order for a "quasi-massless" multiplet to be a matter multiplet, the first choice should be realized by the manifold. However, in the case of the $N = 2$ $SU(3) \times SU(2) \times U(1)$ theory, it is the value $m_{\lambda_L} = -24$ which is realized, as we shall discuss in a separate paper¹⁸⁾. It seems therefore that in this case the "quasi-massless" multiplet is not a "matter multiplet". It would be interesting to check whether the squashed sphere behaves differently.

After this introduction, we discuss the massless multiplets. To do this we need to write also the transformation rule of the λ_L and λ_T fields. They are contained in the irreducible part of ψ_α . Hence we write the irreducible part of the $\delta_\epsilon \psi_\alpha$ rule which is obtained from Eq. (4.5c) after some long and cumbersome manipulations:

$$\begin{aligned} [\delta_\epsilon \psi_\alpha]^{3/2} &\equiv \square_\alpha \delta_\epsilon \lambda_T + [\nabla_\alpha \square]^{3/2} \delta_\epsilon \lambda_L = \\ &\left\{ \frac{9}{7} \gamma_5 \gamma_m \epsilon \partial_m S - \frac{3}{7} (\sqrt{M_{(0)^3} + 36} - 6) \epsilon S \right\} \Lambda_\alpha [Y] + \\ &\left\{ \frac{9}{7} \gamma_5 \gamma_m \epsilon \partial_m \tilde{Z} + \frac{3}{7} (\sqrt{M_{(0)^3} + 36} + 6) \epsilon \tilde{Z} \right\} \Lambda_\alpha [Y] + \\ &\left\{ \frac{1}{2} \gamma_5 \gamma_m \epsilon \partial_m \varphi \Lambda_\alpha^1 [Y_{\square}] - \frac{1}{2} \epsilon \varphi \Lambda_\alpha^2 [Y_{\square}] \right\} + \\ &+ \frac{3}{14} M_{(1)^3} \left\{ \frac{i}{3} \gamma_5 \epsilon \pi + \frac{i \gamma_m}{4 M_{(1)^3}} \epsilon \partial_m \pi \right\} \Lambda_\alpha \left[Y_{\square} \right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{15}{28} i \gamma_5 \gamma_{mn} \varepsilon \partial_m Z_n \Lambda_\alpha^{(1)} [Y_\square] + \frac{3}{14} i \gamma_m \varepsilon Z_m \Lambda_\alpha^{(2)} [Y_\square] \\
 & + \gamma_{mn} \varepsilon \partial_m A_n \Lambda_{(+)}^{(1)} [Y_\square] + \gamma_5 \gamma_m \varepsilon A_m \Lambda_{(+)}^{(2)} [Y_\square] \\
 & + \gamma_{mn} \varepsilon \partial_m W_n \Lambda_{(-)}^{(1)} [Y_\square] + \gamma_5 \gamma_m \varepsilon W_m \Lambda_{(-)}^{(2)} [Y_\square] \quad (5.5)
 \end{aligned}$$

where the irreducible spinors $\Lambda_\alpha[Y\dots]$ are listed below

$$(i) \quad \Lambda_\alpha [Y] = \eta \partial_\alpha Y - \frac{1}{6} \tau_{\alpha\beta} \eta \partial_\beta Y \quad (5.6a)$$

$$(ii) \quad \Lambda_\alpha^{(1)} [Y_\square] = \tau_\mu \eta Y_{(\mu\alpha)}; \quad \Lambda_\alpha^{(2)} [Y_\square] = -\tau_{\mu\nu} \eta \partial_\mu Y_{(\nu\alpha)} \quad (5.6b)$$

$$(iii) \quad \Lambda_\alpha [Y_\square] = \tau_{\alpha\mu\nu\rho} \eta Y_{[\mu\nu\rho]} - 4 \tau_{\mu\nu} \eta Y_{[\alpha\mu\nu]} \quad (5.6c)$$

$$(iv) \quad \Lambda_\alpha^{(1)} [Y_\square] = \tau_\mu \eta Y_{[\mu\alpha]} + \frac{1}{5} \tau_{\alpha\mu\nu} \eta Y_{[\mu\nu]} \quad (5.6d)$$

$$(v) \quad \Lambda_\alpha^{(2)} [Y_\square] = \tau_{\mu\nu} \eta \partial_{[\alpha} Y_{\mu\nu]} - \frac{1}{4} \tau_{\alpha\mu\nu\rho} \eta \partial_\mu Y_{[\nu\rho]} \quad (5.6e)$$

$$\begin{aligned}
 (vi) \quad \Lambda_{(\pm)}^{(1)} [Y_\square] = & \left\{ \mp \frac{1}{2} \left(\sqrt{M_{(1)(0)}^2 + 16} \mp 4 \right) \eta Y_\alpha + \frac{2}{3} \tau_\mu \eta \partial_{[\mu} Y_{\alpha]} \right. \\
 & \left. + \frac{1}{6} \tau_{\alpha\mu\nu} \eta \partial_\mu Y_\nu \right\}^{3/2} \quad (5.6f)
 \end{aligned}$$

$$\begin{aligned}
 (vii) \quad \Lambda_{(\pm)}^{(2)} [Y_\square] = & \left\{ \mp \frac{1}{2} \left(\sqrt{M_{(1)(0)}^2 + 16} \mp 4 \right) \tau_\mu \eta \partial_\mu Y_\alpha + \right. \\
 & \left. + \left(Y_\alpha \eta - \frac{1}{2} \tau_{\alpha\beta} \eta Y_\beta \right) \left(\frac{M_{(1)(0)}^2}{3} + 8 \mp 2 \sqrt{M_{(1)(0)}^2 + 16} \right) \right\}^{3/2} \quad (5.6g)
 \end{aligned}$$

In Eqs. (5.6f) and (5.6g) the symbol $()^{3/2}$ denotes, as usual, the τ traceless part which we have not written explicitly.

Using Eqs. (5.6) and the information provided by Section 4, we can obtain all our results.

As a preliminary computation we focus on Eq. (5.6e) which yields the harmonic associated to the contribution of the pseudoscalars. To study the relation between the three-form $Y_{[\alpha\beta\gamma]}$ and the fermionic harmonics, we consider the following linear combination

$$\Omega_\alpha = a \tau_{\alpha\mu\nu\rho} \eta Y_{\mu\nu\rho} + b \tau_{\mu\nu} \eta Y_{\alpha\mu\nu} + c \tau_{\mu\nu\rho} \eta \mathcal{D}_\alpha Y_{\mu\nu\rho} \quad (5.7)$$

and we find that Ω_α is transverse and irreducible ($\tau^\alpha_{\Omega_\alpha} = \mathcal{D}^\alpha_{\Omega_\alpha} = 0$) if the following equations are satisfied:

$$4a + b - 4 M_{(1)^3} c = 0 \quad (5.8)$$

From Eq. (5.4), we obtain:

$$a = (4 M_{(1)^3} - 6) c \quad (5.9a)$$

$$b = 12(2 - M_{(1)^3}) c \quad (5.9b)$$

Then we compute the matrix associated to the operator $\mathcal{D}$ on the three basis vectors used in (5.3) and we get

$$\mathcal{D} \begin{pmatrix} \tau_{\alpha\mu\nu\rho} \eta Y_{\mu\nu\rho} \\ \tau_{\mu\nu} \eta Y_{\alpha\mu\nu} \\ \tau_{\mu\nu\rho} \eta \mathcal{D}_\alpha Y_{\mu\nu\rho} \end{pmatrix} = \begin{pmatrix} -1, & 12 M_{(1)^3}, & 1 \\ \frac{4}{3} M_{(1)^3}, & 3, & \frac{1}{3} \\ -12, & -48, & -4(M_{(1)^3} + 1) \end{pmatrix} \begin{pmatrix} \tau_{\alpha\mu\nu\rho} \eta Y_{\mu\nu\rho} \\ \tau_{\mu\nu} \eta Y_{\alpha\mu\nu} \\ \tau_{\mu\nu\rho} \eta \mathcal{D}_\alpha Y_{\mu\nu\rho} \end{pmatrix} \quad (5.10)$$

One can then check that the transverse and irreducible combination corresponding to the conditions (5.9) is an eigenstate of (5.10) with the following eigenvalue:

$$\mathcal{D} \Omega_\alpha^\top = -(4 M_{(1)^3} - 1) \Omega_\alpha^\top \quad (5.11)$$

Equation (5.11) yields the relation [see Ref. 3) for conventions]

$$M_{(3/2)(1/2)^2} = -4 \left(M_{(1)^3} + 1 \right) \quad (5.12)$$

which was already anticipated in Table I.

The longitudinal part of the combinations (5.7) is the following derivative

$$\Omega_\alpha^L = \mathcal{D}_\alpha \square; \quad \square = \tau_{\alpha\mu\nu} \gamma_{[\alpha\mu\nu]} \quad (5.13)$$

An easy calculation yields:

$$\mathcal{D} \square = - \left(1 + 4 M_{(1)^3} \right) \square \quad (5.14)$$

which provides the second relation anticipated in Table I:

$$M_{(1/2)^3} = -4 \left(M_{(1)^3} + 2 \right) \quad (5.15)$$

This being done, we can now discuss the various multiplets. For each multiplet we begin with a table where the harmonics and the states are summarized and then we discuss its properties. We write formulae which hold for $N = 1$, $N = 2$ and $N = 3$, with the proviso that objects with a number of antisymmetric indices larger than N are zero.

1) - Gravitational multiplet (Table VI)

This multiplet is generated by the Killing spinor η^M , which, corresponding to $M_{(\frac{1}{2})^3} = 0$, yields a massless spin- $3/2$ particle. The associated graviton is also massless and its harmonic is given by Eq. (4.33)

$$Y^{M \otimes N} = \frac{1}{7} \bar{\eta}^M \square^N = \frac{1}{7} \bar{\eta}^M \eta^N = \frac{1}{7} \delta^{MN} \quad (5.16)$$

The harmonic of the vector is found from Eqs. (4.80) and (4.85). There we see that the spin- $3/2$ particle goes into a vector of harmonic

$$\begin{aligned} Y_{\alpha}^{M \otimes N} &= \left\{ - (M_{(1/2)}^3 + 13) \bar{\eta}^M \tau_{\alpha}^N \square + \bar{\eta}^M \mathcal{D}_{\alpha} \square^N \right\} \times \text{const} = \\ &= Y_{\alpha}^{[MN]} = \bar{\eta}^M \tau_{\alpha}^N \eta^N \Rightarrow \text{const} = -\frac{1}{12} \end{aligned} \quad (5.17)$$

which is a Killing vector

$$\mathcal{D}_{\{\alpha} Y_{\beta\}}^{[MN]} = 0 \quad (5.18)$$

and generates the $SO(N)$ subgroup of $\mathcal{G} = \mathcal{G}' \times SO(N)$. Finally, the harmonic of the spin- $\frac{1}{2}$ field is obtained by insertion of the harmonic (5.17) into Eqs. (5.6f) and (5.6g). We get:

$$\Lambda^{(1)} [Y_{\alpha}^{[MN]}] R = - \left(\nabla_{\alpha} \square^{[MNR]} \right)^{3/2} \quad (5.19a)$$

$$\Lambda^{(2)} [Y_{\alpha}^{[MN]}] R = \frac{4}{3} \left[\nabla_{\alpha} \square^{[MNR]} \right]^{3/2} \quad (5.19b)$$

where

$$\square^{[MNR]} = \tau_{\mu}^M \eta^N \bar{\eta}^N \tau_{\mu}^R \eta^R \quad (5.20)$$

Using these harmonics in Eqs. (4.89), (5.5), (5.19) and (4.80), we obtain the transformation rules of the gravitational multiplet

$$\delta_{\varepsilon} h_{ab} = i \bar{\varepsilon}^M \chi_{(a} \chi_{b)}^M \quad (5.21a)$$

$$\begin{aligned} \delta_{\varepsilon} \chi_m^M &= -\frac{1}{2} \gamma_{ab} \varepsilon^M \mathcal{D}_a h_{bm} - 4 \left\{ \gamma_5 \gamma_z \mathcal{D}_{[m} A_{z]}^{MN} - 6 A_m^{MN} \right. \\ &\quad \left. + \frac{1}{4} \gamma_5 \gamma_z \mathcal{D}_{i[m} A_{z]}^{MN} + 2 \gamma_{mz} A_z^{MN} - i \varepsilon_{ijkm} \gamma_k \mathcal{D}_i A_j^{MN} \right\} \varepsilon^N \end{aligned} \quad (5.21b)$$

$$\delta_{\varepsilon} A_a^{[MN]} = \frac{i}{8} \bar{\varepsilon}^{[M} \chi^{N]} + i \frac{1}{64} \left[-\frac{48}{7} \bar{\varepsilon}^R \gamma_5 \mathcal{D}_m \chi^{[RMN]} + \frac{192}{7} \bar{\varepsilon}^R \gamma_m \chi^{[RMN]} \right] \quad (5.21c)$$

$$\delta_{\varepsilon} \chi^{[MNR]} = - \gamma_{mn} \varepsilon^{[R} \mathcal{D}_m A_n^{MN]} + \frac{4}{3} \gamma_5 \gamma_m \varepsilon^{[R} A_m^{MN]} \quad (5.21d)$$

2) - The Killing vector multiplet (Table VII)

This vector multiplet exists because the manifold S/R admits the Killing vectors k_{α}^{Λ} of the group $\mathcal{G}'$:

$$\mathcal{D}_{\alpha} k_{\beta}^{\Lambda} = -\mathcal{D}_{\beta} k_{\alpha}^{\Lambda} \quad (5.22)$$

and because of the following proposition:

Lemma 1: On the manifold S/R if k_{α}^{Λ} is a Killing vector of $\mathcal{G}'$ and η^M is a Killing spinor the following identity holds true:

$$k_{\alpha}^{\Lambda} \tau^{\alpha} \eta^M = -\frac{1}{4} \tau_{\alpha\beta} \eta^M \mathcal{D}_{\alpha} k_{\beta}^{\Lambda} \quad (5.23)$$

Proof It relies on the property of the Killing spinor η^M of being in the identity representation of $\mathcal{G}^{(3)}$. This implies:

$$\mathcal{L}_{\Lambda} \eta^M \equiv \mathcal{L}_{\vec{k}_{\Lambda}} \eta^M - \frac{1}{4} W_{\Lambda}^{\alpha\beta} \tau_{\alpha\beta} \eta^M = 0 \quad (5.24)$$

where $\mathcal{L}_{\Lambda}$ is the covariant Lie derivative along the Killing vector $\vec{k}_{\Lambda}$. On the other hand, the definition of the Manton W-matrix³⁾:

$$\mathcal{L}_{\vec{k}_{\Lambda}} \mathcal{B}^{\alpha} = W_{\Lambda}^{\alpha\beta} \mathcal{B}_{\beta} \quad (5.25)$$

yields

$$W_{\Lambda}^{\alpha\beta} = \mathcal{D}_{\Lambda}^{\beta} k_{\Lambda}^{\alpha} + k_{\Lambda}^{\gamma} \mathcal{B}_{\gamma}^{\alpha\beta} \quad (5.26)$$

where $\mathcal{B}_{\gamma}^{\alpha\beta}$ are the intrinsic components of the spin connection $\mathcal{B}^{\alpha\beta}$.

Inserting Eq. (5.22) into (5.20) we find:

$$k_{\Lambda}^{\alpha} \mathcal{D}_{\alpha} \eta^M = -\frac{1}{4} \mathcal{D}_{\Lambda}^{\alpha} k_{\Lambda}^{\beta} \tau_{\alpha\beta} \eta^M \quad (5.27)$$

and recalling the hypothesis that η^M is a Killing spinor [Eq. (2.2)], we obtain the proof of Eq. (5.19).

Using Eq. (5.19) one can easily prove that harmonics listed below are eigenstates of the corresponding invariant operators with eigenvalues corresponding to massless states:

$$Y_{\alpha}^{\Lambda} = k_{\alpha}^{\Lambda} ; M_{(1)(0)^2} = 48 \Rightarrow m_A^2 = 0$$

$$\square^{M\otimes\Lambda} = \tau^{\alpha} \eta^M k_{\alpha}^{\Lambda} ; M_{(1/2)^3} = -16 \Rightarrow m_{\lambda_L} = 0$$

$$Y^{[MN]\otimes\Lambda} = \bar{\eta}^M \tau_{\mu}^N \eta^{\mu} k_{\mu}^{\Lambda} ; M_{(0)^3} = 64 \Rightarrow m_S^2 = 0$$

$$\begin{aligned} Y_{\alpha\beta\gamma}^{[MN]\otimes\Lambda} &= \bar{\eta}^M \tau_{[\alpha\beta}^N \eta^{\gamma]} k_{\gamma}^{\Lambda} - \frac{1}{2} \bar{\eta}^{[M} \tau_{\alpha}^N \eta^{\gamma]} \partial_{\beta} k_{\gamma}^{\Lambda} = \\ &= -\frac{1}{12} \bar{\eta}^{[M} \tau_{\alpha\beta\gamma\mu\nu}^N \eta^{\mu]} \partial_{\mu} k_{\nu}^{\Lambda} ; M_{(1)^3} = 2 \Rightarrow m_{\pi}^2 = 0 \end{aligned} \quad (5.28)$$

Inserting Eq. (5.24) into Eqs. (4.33), (4.47b), (4.49), (4.85b), (4.85c), (5.6a), (5.6c), (5.6f) and (5.6g), we can moreover verify that our harmonics satisfy the algebra of the vector multiplet, namely:

$$\bar{\eta}^M \square^{N\otimes\Lambda} = \text{const } Y^{[MN]\otimes\Lambda}$$

$$\frac{5}{3} \bar{\eta}^M \tau_{\alpha\beta\gamma} \square^{N\otimes\Lambda} + \bar{\eta}^M \tau_{\alpha\beta} \partial_{\gamma} \square^{N\otimes\Lambda} = \text{const } Y_{\alpha\beta\gamma}^{[MN]\otimes\Lambda}$$

$$3 \bar{\eta}^M \tau_{\alpha} \square^{N\otimes\Lambda} + \bar{\eta}^M \partial_{\alpha} \square^{N\otimes\Lambda} = \text{const } \delta^{\alpha N} Y_{\alpha}^{\Lambda}$$

$$\bar{\eta}^M \partial_{\alpha} Y^{[PQ]\otimes\Lambda} - \frac{1}{6} \tau_{\alpha\beta} \eta^M \partial_{\beta} Y^{[PQ]\otimes\Lambda} = \text{const} \left(\delta^M{}_{\alpha} \square^{[P} \square^{Q]\otimes\Lambda} \right)^{3/2} \quad (5.29)$$

and similar equations for the remaining cases. This guarantees that the supersymmetry transformations on the fields of Table VII are closed as expected.

3) - The Betti multiplet (Table VIII)

This multiplet exists if the manifold S/R admits a closed and co-closed two-form $\Omega_{\alpha\beta}$, namely if it has a Betti number $B_2 \neq 0$.

The Betti form satisfies the following properties:

a) - It is in the identity representation of the isometry group namely it is annihilated by all the Lie derivatives along any Killing vector²⁰⁾:

$$\mathcal{L}_K \Omega_{\alpha\beta} = 0 \quad (5.30)$$

Equation (5.26) implies that $\Omega_{\alpha\beta}$ is a constant.

b) - Theorem: The Betti form $\Omega_{\alpha\beta}$ is an element of the holonomy algebra of S/R in the sense that the SO^7 Lie algebra element

$$g = \tau_{\alpha\beta} \Omega_{\alpha\beta} \quad (5.31)$$

annihilates all the Killing spinors ($g\eta = 0$)

Proof^{*)}: Consider the following spinor

$$\xi^M = \tau_{\alpha\beta} \gamma^M \Omega_{\alpha\beta} \quad (5.32)$$

Calculating the $\mathcal{D}$ operator on ξ we obtain, under the hypothesis that $\Omega_{\alpha\beta}$ is harmonic:

$$\mathcal{D}^\alpha \Omega_{\alpha\beta} = \mathcal{D}_\alpha \Omega_{\beta\gamma} = 0 \quad (5.33)$$

*) This proof of the theorem, that we had only conjectured, was found in a conversation between one of us (P.F.) and C.N. Pope whom we warmly thank for his useful suggestions.

the following result:

$$\not\partial \xi^M = 3 \xi^M \quad (5.34)$$

This implies that:

$$\square \xi^M = (\not\partial \not\partial - 42) \xi^M = -33 \xi^M \quad (5.35)$$

Therefore ξ^M has to be zero because $\square$ is a positive definite operator.
Hence

$$\tau_{\alpha\beta} \Omega_{\alpha\beta} \eta^M = 0 \quad (5.36)$$

Using this property, one can then show that the harmonics $Y_{[\alpha\beta]}$, E_α , $Y_{(\alpha\beta)}$, $Y_{[\alpha\beta\gamma]}$ listed in Table VIII are eigenstates of the proper differential operators with eigenvalues corresponding to massless states.

$$\begin{aligned} M_{(1)^2(0)} &= 0 ; \quad M_{(3/2)(1/2)} = -8 ; \quad M_{(2)(0)^2} = 0 \\ M_{(1)^3} &= 1 \end{aligned} \quad (5.37)$$

In a similar way to the case of the Killing multiplet, it is then easy to check that the harmonics transform into each other closing the algebra of the vector multiplet.

We emphasize the following corollaries.

Corollary 1

A zero mode of the Hodge de Rham operator on two forms, namely the fact that the second Betti number is different from zero, implies the existence of a zero-mode of the Rarita-Schwinger and of the Lichnerowicz operators. The converse is also true. Hence the spectrum of the Rarita-Schwinger or Lichnerowicz operators measures the cohomology of the manifold $\mathcal{I}/R$ (in seven-dimension).

Corollary 2

The $\star d$ operator on three-forms has two critical eigenvalues (2 and 1) corresponding to massless modes. The first is associated with the existence of Killing vectors, the second with the existence of harmonic two-forms, hence a mode $M_{(1)3} = 1$ again implies $B_2 \neq 0$.

The analysis we have carried through enables one to write the massless multiplets of any supersymmetric compactification of $D = 11$ in a general form and removes some of the question marks of Table II as to the geometrical meaning of the zero-modes. The analysis of matter multiplets on the other hand requires more detailed information on the manifold and can be performed only case by case.

6. - CONCLUSIONS

Kaluza-Klein supergravity has several unsolved problems but the most basic one is to define a consistent criterion of truncation from the complete theory with infinite towers of multiplets to an effective four-dimensional theory with a finite number of them. It is clear that in de Sitter space the simple idea of keeping the massless modes and discarding the massive ones is wrong. Other criteria are in want.

Indications come from an analysis of the type of multiplets one effectively obtains. It is clear that the massless multiplets corresponding, always, to gauge multiplets, must be retained and have a general structure which we have described in this paper. In principle, however, there should also be matter multiplets which, by definition, have to stop at spin- $\frac{1}{2}$ and do not contain massive vectors. It is likely that they should be retained^{*)} and could play the rôle of matter multiplets coupled to gauged supergravity. The potential they generate is unknown but it might be conceived that it has some other stationary points corresponding to Minkowski space. This is a possible solution to the cosmological constant problem.

We feel that the coupling of the matter multiplets yield by $D = 11$ supergravity should be further studied.

^{*)} We are indebted to S. Ferrara for stressing this point.

TABLE I - Relation among the spectra of the bosonic and fermionic operators due to Killing spinors

A) - Longitudinal Spinors $\rightarrow$ Bosons

$$M_{(0)^3} = M_{(1/2)^3} (M_{(1/2)^3} + 12)$$

$$M_{(1)(0)^2} = M_{(1/2)^3}^2 + 16 M_{(1/2)^3} + 48$$

$$M_{(1)^2(0)} = M_{(1/2)^3} (M_{(1/2)^3} + 8)$$

$$M_{(1)^3} = -\frac{1}{4} (M_{(1/2)^3} + 8)$$

B) - Transverse spinors → Bosons

$$M_{(2)(0)^2} = (M_{3/2}(1/2)^2 + 4)(M_{3/2}(1/2)^2 + 8)$$

$$M_{(1)^2(0)} = (M_{3/2}(1/2)^2 + 4)(M_{3/2}(1/2)^2 + 8)$$

$$M_{(1)^3} = -\frac{1}{4} (M_{3/2}(1/2)^2 + 4)$$

$$M_{(1)(0)^2} = M_{3/2}(1/2)^2 (M_{3/2}(1/2)^2 + 8)$$

C) - Three-forms $\rightarrow$ Fermions

$$\begin{array}{l}
 [1,1,1] \\
 \Downarrow \\
 Y_{[\mu\nu\rho]}
 \end{array}
 \begin{array}{l}
 \rightarrow \left[\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right] \Rightarrow \Xi_{\alpha} = a \tau_{\alpha\mu\nu\rho} \eta Y_{\mu\nu\rho} \\
 \quad + b \tau_{\mu\nu} \eta Y_{\alpha\mu\nu} + c \tau_{\mu\nu\rho} \eta \partial_{\alpha} Y_{\mu\nu\rho} \\
 \left\{ \begin{array}{l} \frac{a}{c} = (4 M_{(1)}^3 - 6) \\ \frac{b}{c} = -12(M_{(1)}^3 - 2) \end{array} \right. \\
 \rightarrow \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \Rightarrow \Xi = \tau_{\mu\nu\rho} \eta Y_{\mu\nu\rho}
 \end{array}$$

$$M_{(3/2)(1/2)^2} = -4(M_{(1)}^3 + 1)$$

$$M_{(1/2)^3} = -4(M_{(1)}^3 + 2)$$

TABLE II - Occurrence of zero-modes

Spin	Field	Critical value of $M_{[\lambda_1 \lambda_2 \lambda_3]}$	Geometrical interpretation of the harmonic
0^+ {	S^I φ^I	$\begin{cases} M_{(0)^3} = 64 \\ M_{(0)^3} = 160 \\ M_{(2)(0)^2} = 0 \end{cases}$	$\begin{matrix} ? \\ ? \\ \Rightarrow \text{zero of Lichnerowicz-like operator.} \end{matrix}$
0^-	π^I	$\begin{cases} M_{(1)^3} = 1 \\ M_{(1)^3} = 2 \end{cases}$	$\begin{matrix} ? \\ ? \end{matrix}$
1^+	A_a^I	$\begin{cases} M_{(1)(0)^2} = 48 \\ M_{(1)(0)^2} = 0 \end{cases}$	$\begin{matrix} \Rightarrow Y_\alpha = k_\alpha = \text{Killing vector} \\ \Rightarrow Y_\alpha = \text{harmonic 1-form} \\ \text{(closed 1-form)} \\ \text{Betti number } B_1 \neq 0 \end{matrix}$
1^-	Z_a^I	$M_{(1)^2(0)} = 0$	$\Rightarrow Y_{\alpha\beta} = \text{harmonic 2-form}$ Betti number $B_2 \neq 0$
$1/2$ {	λ_L^I λ_T^I	$\begin{matrix} M_{(1/2)^3} = -16 \\ M_{(1/2)^3} = -8 \end{matrix}$	$\begin{matrix} ? \\ ? \end{matrix}$
$3/2$	χ_a^I	$M_{(1/2)^3} = 0$	$\Rightarrow \bar{\Sigma} = \eta = \text{Killing spinor}$
2	h_{ab}^I	$M_{(0)^3} = 0$	$\Rightarrow Y = \text{constant function on the manifold.}$

TABLE III - Spin $3/2 \rightarrow$ bosons: mass relations

$$m_h^2 = m_\chi(m_\chi + 12)$$

$$m_Z^2 = m_\chi(m_\chi + 8)$$

$$m_A^2 = m_\chi(m_\chi + 4) ; m_W^2 = m_\chi^2 + 28m_\chi + 192 \quad (\text{if } m_\chi \geq -8)$$

$$m_A^2 = m_\chi^2 + 28m_\chi + 192 ; m_W^2 = m_\chi(m_\chi + 4) \quad (\text{if } m_\chi \leq -8)$$

TABLE IV - Spin- $\frac{1}{2}$ (transverse) $\rightarrow$ bosons: mass relations

$$m_\pi^2 = m_{\lambda_T}(m_{\lambda_T} + 4)$$

$$m_Z^2 = m_{\lambda_T}(m_{\lambda_T} - 4)$$

$$m_\varphi^2 = m_{\lambda_T}(m_{\lambda_T} - 4)$$

$$m_A^2 = m_{\lambda_T}(m_{\lambda_T} + 4) ; m_W^2 = m_{\lambda_T}^2 - 20m_{\lambda_T} + 96 \quad (\text{if } m_{\lambda_T} \geq 4)$$

$$m_A^2 = m_{\lambda_T}^2 - 20m_{\lambda_T} + 96 ; m_W^2 = m_{\lambda_T}(m_{\lambda_T} + 4) \quad (\text{if } m_{\lambda_T} \leq 4)$$

TABLE V - Spin- $\frac{1}{2}$ (longitudinal) $\rightarrow$ bosons: mass relations

$$m_{\pi}^2 = m_{\lambda_L} (m_{\lambda_L} + 4)$$

$$m_S^2 = m_{\lambda_L} (m_{\lambda_L} - 4) ; m_{Z'}^2 = (m_{\lambda_L} + 24)(m_{\lambda_L} + 20) \quad (\text{if } m_{\lambda_L} \geq -10)$$

$$m_S^2 = (m_{\lambda_L} + 24)(m_{\lambda_L} + 20) ; m_{Z'}^2 = m_{\lambda_L} (m_{\lambda_L} - 4) \quad (\text{if } m_{\lambda_L} \leq -10)$$

$$m_A^2 = m_{\lambda_L} (m_{\lambda_L} - 4) ; m_W^2 = m_{\lambda_L}^2 - 2m_{\lambda_L} + 192 \quad (\text{if } m_{\lambda_L} \leq 8)$$

$$m_A^2 = m_{\lambda_L}^2 - 2m_{\lambda_L} + 192 ; m_W^2 = m_{\lambda_L} (m_{\lambda_L} - 4) \quad (\text{if } m_{\lambda_L} \geq 8)$$

TABLE VI - The gravitational multiplet

$\mathcal{S}$ irrep. = singlet				
Spin	O(N) irrep.	Field	Harmonic	Mass
2	.	h_{ab}	$Y = \frac{1}{7}$	$m_h^2 = 0$
$\frac{3}{2}$	$\square$	χ_a^M	$\square^M = \eta^M$ (Killing spinor)	$m_\chi = 0$
1	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	A_a^{MN}	$Y_\alpha^{[MN]} = \bar{\eta}^{[M} \tau_\alpha^N \square =$ $= \bar{\eta}^{[M} \tau_\alpha^N \eta^{N]}$	$m_A^2 = 0$
$\frac{1}{2}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$\lambda_L^{[HNR]}$	$\square^{MNR} = \tau_\mu^M \eta^N \bar{\eta}^R \tau_\mu^R$	$m_{\lambda_L} = 0$

TABLE VII - The Killing vector multiplet

$\mathcal{J}$, irrep. = adjoint				
Spin	O(N) irrep.	Field	Harmonic	Mass
1^+	$\bullet$	A_a^Λ	$Y_a^\Lambda = k_a^\Lambda$ (Killing vector)	$m_A^2 = 0$
$1/2$	$\square$	$\lambda_L^{M\otimes\Lambda}$	$\square^{M\otimes\Lambda} = \tau_a \eta^M k_a^\Lambda$	$m_{\lambda_L} = 0$
0^+	$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$S^{[MN]\otimes\Lambda}$	$Y^{[MN]\otimes\Lambda} = \bar{\eta}^{[M} \tau_\mu \eta^{N]} k_\mu^\Lambda$	$m_S^2 = 0$
0^-	$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$\pi^{[MN]\otimes\Lambda}$	$Y_{\alpha\beta\gamma}^{[MN]\otimes\Lambda} = \bar{\eta}^{[M} \tau_{\alpha\beta} \eta^{N]} k_\gamma^\Lambda$ $- \frac{1}{2} \bar{\eta}^{[M} \tau_\alpha \eta^{N]} \mathcal{D}_\beta k_\gamma^\Lambda$	$m_\pi^2 = 0$

TABLE VII - The Betti multiplet

$\mathcal{J}$, irrep. = singlet				
Spin	O(N) irrep.	Field	Harmonic	Mass
1^-	$\bullet$	Z_a	$Y_{[\alpha\beta]} = \Omega_{\alpha\beta}$ (Harmonic 2-form) $\mathcal{D}^\alpha \Omega_{\alpha\beta} = \mathcal{D}_{[\alpha} \Omega_{\beta\gamma]} = 0$	$m_Z^2 = 0$
$1/2$	$\square$	λ_τ^M	$\mathcal{L}_\alpha^M = \Omega_{\alpha\beta} \tau_\beta \eta^M$	$m_{\lambda_\tau} = 0$
0^+	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\varphi^{[MN]}$	$Y_{(\alpha\beta)}^{[MN]} = \frac{1}{2} \{ \Omega_{\alpha\gamma} \bar{\eta}^M \tau_{\gamma\beta} \eta^N$ $+ (\alpha \leftrightarrow \beta) \}$	$m_S^2 = 0$
0^-	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\pi^{[MN]}$	$Y_{\alpha\beta\gamma}^{[MN]} = \bar{\eta}^M \tau_\alpha \eta^N \Omega_{\beta\gamma}$	$m_\pi^2 = 0$

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