CLASSICAL SOLUTIONS OF CP^{N-1} MODELS IN 2+1 DIMENSIONS

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ABSTRACT

We present a method to generate time-dependent solutions of the non-linear CP^{N-1} models in 2+1 dimensions. We exhibit some finite energy solutions and give a qualitative analysis of their properties.

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1. - INTRODUCTION

It is generally believed that non-Abelian gauge theories are likely to play an important role in any field theoretical description of the theory of elementary particles. In particular, weak and electromagnetic interactions are described by such a theory, and it is generally felt that this is also the case for strong interactions. As non-Abelian gauge theories are strongly non-linear, it is very important to study the effects of these nonlinearities. A very remarkable feature of spontaneously broken gauge theories is the existence of topological excitations (vertices, monopoles). These excitations are static finite energy solutions of the classical equations of motion. In addition, non-Abelian gauge theories in four Euclidean dimensions possess instanton solutions, which have finite action and non-trivial topology. Recently, considerable progress has been made in finding such monopole and instanton solutions. In this task, one is helped by the topology of the fields; one solves only the first-order differential equations (so-called "self-duality" equations), and the topology guarantees that these solutions are also solutions of the equations of motion. However, this method does not say anything about the solutions of the equations of motion which are not self-dual. It has turned out to be very difficult to find such solutions; however, their existence has been recently demonstrated by Taubes¹⁾, who used the saddle point technique of functional analysis to establish the existence of a solution describing a static interacting monopole-antimonopole system, and which lies outside the self-dual or antiself-dual sector of the theory. This result is very interesting, because it shows that one can extract non-trivial information from the theory without solving the equations themselves.

Given the complexity of gauge theories in four dimensions, it is natural that some people started looking at models in lower dimensions which exhibit some features of the four-dimensional theory. Such a class of models in two (Euclidean) dimensions are the non-linear CP^{N-1} models²⁾. These models also have a first-order system of equations analogous to the four-dimensional self-duality equations, the solutions of which are also

analogous to the instantons of gauge theories. The instantons of the CP^{N-1} models are also the finite action solutions of the "self-duality" equations. Their topological stability is guaranteed by

$$\pi_2 \left(\frac{U(N)}{U(N-1) \times U(1)} \right) = \mathbb{Z}.$$

Up to now, all topologically non-trivial solutions in Minkowski space were static. Therefore, it is of considerable interest to try to find non-trivial, time-dependent solutions. Perhaps the most interesting aim would be to find solutions in 3+1 dimensions corresponding to the scattering of monopoles. In some 1+1-dimensional models, such time-dependent solutions exist, and they provide an important insight into the properties of these models. These finite energy solutions - "lumps" in the terminology of Coleman³⁾ - happen to retain their shape even after collision (they are solitons in the strict sense). However, not much is known about the scattering of "lumps" in 2+1 dimensions. The problem is considerably more complicated, as the "lumps" of the theory have a greater degree of freedom of motion. In contrast to 1+1-dimensional models, one can expect the existence of genuine scattering.

The aim of this paper is to investigate finite-energy time-dependent solutions of 2+1-dimensional CP^{N-1} models. As is well known, the finite action Euclidean "instanton" solutions in D dimensions correspond to static finite energy "lumps" in $D+1$ -dimensional Minkowski space-time. Thus, CP^{N-1} models possess static "lumps", and so it is very interesting to determine what behaviour they exhibit when they move. Can they scatter on each other? Do they preserve their shape, or is their interaction so strong that they lose their identity? In this paper, we make a first step towards answering these and other questions. In fact, we present a method which allows us to generate large classes of finite energy, non-trivial time-dependent solutions. Unfortunately, we cannot use the method of inverse scattering, which works very well in 1+1 dimensions, since we do not know whether our 2+1-dimensional models possess an associated linear system.

Our method gives local solutions in terms of arbitrary functions. We do not have a criterion, as yet, to select global solutions from the local ones, apart from a tedious case-by-case analysis. However, we have succeeded in finding a whole class of global solutions which seem to correspond to interacting lumps. We have analyzed their properties in some detail, but

in order to have a more quantitative description, one would probably have to resort to some numerical work.

The plan of this paper is as follows. In the next Section, we give a detailed description of our solution-generating techniques for 2+1-dimensional CP^{N-1} models. Let us remark here that our method can easily be generalized to 3+1 dimensions, but for CP^{N-1} models such a generalization would only be of academic interest. In Section 3, we analyze the simplest example which is generated by a solution of a quadratic equation. As the total energy of this solution is infinite, we are forced to consider solutions of a cubic equation. It is possible to arrange parameters in such a way that the total energy is finite for the solutions of this equation; the main difficulty here is to find a root of this equation which is a single-valued function of (x,y,t) . In Section 4, we give a qualitative discussion of a prototype time-dependent finite energy solution based on the cubic equation of Section 3. The paper ends with two appendices which contain some more technical details.

2. - GENERAL CONSTRUCTION OF SOLUTIONS

The action for the CP^{N-1} model in 2+1 dimensions is given by:

$$S = \int dx dy dt \, \overline{D_\mu z_\alpha} \cdot D^\mu z_\alpha, \quad (2.1)$$

where z_α is a complex field transforming according to the fundamental representation of $SU(N)$, which in addition satisfies

$$\overline{z_\alpha} \cdot z_\alpha = 1, \quad (2.2)$$

while the covariant derivative is defined by

$$D_\mu \Psi_\alpha = \partial_\mu \Psi_\alpha - (\overline{z_\beta} \partial_\mu z_\beta) \Psi_\alpha. \quad (2.3)$$

The classical equations of motion are given by

$$D_\mu D^\mu z_\alpha = \lambda z_\alpha, \quad \overline{z_\alpha} \cdot z_\alpha = 1 \quad (2.4)$$

where λ is an arbitrary function of $x^\mu = (t, x, y)$. It is convenient to introduce

$$P_{\alpha\beta} = Z_\alpha Z_\beta^+ . \quad (2.5)$$

Then $P_{\alpha\beta}$ is a projector and so it satisfies

$$P^+ = P , \quad P^2 = P . \quad (2.6)$$

Moreover, the equations of motion (2.4), when written in terms of P , take a simple form

$$[\square P, P] = 0 , \quad (2.7)$$

where, as usual, $\square = \partial_\mu \partial^\mu = \partial_t^2 - \partial_x^2 - \partial_y^2$.

We shall seek solutions to the equations of motion (2.4) or equivalently (2.7) by making an assumption that

$$Z_\alpha = Z_\alpha(u, \bar{u}) ,$$

where u is a complex function of x^μ . Then a few lines of algebra show that

$$\begin{aligned} \square P = & \square u \partial_u P + \square \bar{u} \partial_{\bar{u}} P + (\partial_\mu u \partial^\mu u) \partial_u^2 P + \\ & + \partial_\mu \bar{u} \partial^\mu \bar{u} \partial_{\bar{u}}^2 P - 2(\partial_\mu u \cdot \partial^\mu \bar{u}) \partial_{u\bar{u}}^2 P , \end{aligned} \quad (2.8)$$

where

$$\partial_u P = \frac{\partial P}{\partial u} \quad \text{and} \quad \partial_{\bar{u}} P = \frac{\partial P}{\partial \bar{u}} . \quad (2.9)$$

Thus, if we choose u to satisfy

$$\square u = 0 \quad (2.10)$$

i.e., the wave equation, and in addition also

$$\partial_\mu u \partial^\mu u = 0 \quad (2.11)$$

we see that

$$[\square P, P] = 0 \quad \Leftrightarrow \quad [\partial_{u\bar{u}}^2 P, P] = 0. \quad (2.12)$$

This reduces the problem to an effective two-dimensional model whose solutions have been studied in detail⁴⁾.

A class of solutions of

$$[\partial_{u\bar{u}}^2 P, P] = 0 \quad (2.13)$$

is given by

$$Z_\alpha = \frac{f_\alpha}{|f|}, \quad f_\alpha = f_\alpha(u). \quad (2.14)$$

In addition to these "instanton-like" solutions, further classes of solutions are generated from f_α , applying the operator defined by

$$P_+^k g_\alpha = P_+ (P_+^{k-1} g_\alpha),$$

and

$$P_+ g_\alpha = \partial_u g_\alpha - \frac{g_\alpha (\bar{g}_\beta \partial_u g_\beta)}{\bar{g}_\gamma \cdot g_\gamma}. \quad (2.15)$$

Hence, as can be easily checked (or see Appendix 1)

$$Z_\alpha^1 = \frac{f_\alpha}{|f|}, \quad Z_\alpha^2 = \frac{P_+ f_\alpha}{|P_+ f|}, \quad \dots \quad Z_\alpha^k = \frac{P_+^{k-1} f_\alpha}{|P_+^{k-1} f|}, \quad \dots \quad (2.16)$$

all solve the equations of motion (2.4), provided that u solves Eqs. (2.10) and (2.11). Thus, the problem has been reduced to that of solving Eqs. (2.10) and (2.11). The initial problem has thus been greatly simplified, as it now involves solving a wave equation for one scalar field together with a constraint, instead of coupled non-linear partial equations for f_α .

To solve Eqs. (2.10) and (2.11), we proceed as follows. First of all, we assume that $u = u(\mu, \gamma)$ where $\mu = \mu(x^\mu)$ and $\gamma = \gamma(x^\mu, \mu(x^\mu))$. We take

$$\gamma = -t + \mu x_+ + \frac{x_-}{\mu}, \quad (2.17)$$

where $x_\pm = \frac{1}{2}(x \pm iy)$. Then, observing that

$$\begin{aligned} u_t &= u_\mu \mu_t + u_\gamma \left[-1 + x_+ \mu_t - \frac{x_-}{\mu^2} \mu_t \right] \\ u_+ &= u_\mu \mu_+ + u_\gamma \left[\mu + x_+ \mu_+ - \frac{x_-}{\mu^2} \mu_+ \right] \\ u_- &= u_\mu \mu_- + u_\gamma \left[\frac{1}{\mu} + x_+ \mu_- - \frac{x_-}{\mu^2} \mu_- \right], \end{aligned} \quad (2.18)$$

where

$$\mu_+ = \partial_+ \mu = \frac{\partial \mu}{\partial x_+}, \quad \mu_- = \frac{\partial \mu}{\partial x_-}$$

we see that

$$\partial_\mu u \partial^\mu u = u_+^2 - u_+ u_- = 0 \quad (2.19)$$

for any dependence of u on μ and γ if μ satisfies

$$\begin{aligned} \mu_t &= -\mu \mu_- \\ \mu_+ &= -\mu \mu_t. \end{aligned} \quad (2.20)$$

The choices of $\gamma(\mu)$ (2.17) and Eqs. (2.20) for μ were motivated by the soliton theoretical methods for generating static self-dual monopoles in three dimensions⁵⁾. There the Euclidean analogue of $\gamma(\mu)$ ($t \rightarrow it$) naturally emerges, and also the analogue of Eqs. (2.20). Moreover, as Eqs. (2.20) imply that μ satisfies

$$\square \mu = (\partial_t^2 - \partial_+ \partial_-) \mu = 0 \quad (2.21)$$

we see that the wave equation for u (2.10) is also automatically fulfilled. Hence, the whole problem is reduced to that of finding solutions to the Eqs. (2.20).

Here we are helped by our judicious choice of $\gamma(\mu, x^\mu)$. In fact, as we shall show below, any smooth function providing a relation between μ and γ is a solution of (2.20), i.e., if we take

$$H = H(\gamma, \mu) = 0 \quad (2.22)$$

and interpret this relation as an implicit equation for μ , we automatically solve Eqs. (2.20). To show this, we observe that

$$\mu_+ = \frac{\partial H}{\partial \gamma} \left[\frac{\partial H}{\partial \mu} + \frac{\partial H}{\partial \gamma} \frac{\partial \gamma}{\partial \mu} \right]^{-1}$$

while

$$\mu_+ = -\mu \frac{\partial H}{\partial \gamma} \left[\frac{\partial H}{\partial \mu} + \frac{\partial H}{\partial \gamma} \frac{\partial \gamma}{\partial \mu} \right]^{-1} \quad (2.23)$$

and

$$\mu_- = -\frac{1}{\mu} \frac{\partial H}{\partial \gamma} \left[\frac{\partial H}{\partial \mu} + \frac{\partial H}{\partial \gamma} \frac{\partial \gamma}{\partial \mu} \right]^{-1}$$

thus proving relations (2.20) for any choice $H(\mu, \gamma)$. We see therefore that to find classical solutions of the equations of motion (2.4), we first of all have to make a choice of the function H in (2.22), and then, using the explicit form of γ [Eq. (2.17)], solve it for $\mu(x^\mu)$ and $\gamma(x^\mu)$. Then we take any complex function u of these μ and γ , and further consider any complex vector f_α which is a function of this u (and through μ and γ a function of x^μ). Then, any vector z_α^i from the sequence (2.16) solves the equations of motion.

Presented this way, we see a large degree of arbitrariness [we could think of almost any function $H(\mu, \gamma)$ or $u(\mu, \gamma)$ or $f(u)$]. In practice, however, some degree of this arbitrariness is reduced by imposing some additional requirements on the behaviour of the function $f_\alpha((\mu(x, y, t), \gamma(x, y, t)))$. Some of these requirements are more mathematical in nature, others come from physics. In particular, our procedure gives us local (in x^μ) solutions of the equations of motion; in most applications that we can envisage, we are

interested in solutions defined globally, thus valid either for all $x^\mu \in \mathbb{R}^3$, or perhaps only over some regions, but with periodic or antiperiodic boundary conditions. We shall see in the next sections that these conditions of single-valuedness of all functions for all $x^\mu \in \mathbb{R}^3$ are very restrictive. Moreover, one of the "reasons d'être" we had in mind when we decided to look for time-dependent classical solutions of the equations of motion was to relate them to the time-independent solutions. We hoped to gain some insight into the time evolution of some instanton or mixed instanton-anti-instanton states, with the interpretations of instantons as the solitons of the 2+1-dimensional theory. This brings us to the consideration of the energy of our solutions, the most interesting of which would have finite energy. This "physical" requirement further reduces the arbitrariness.

In order to have a better insight into the properties of the solutions, let us look at probably the simplest form of H in Eq. (2.22), namely

$$H = \mu - b = 0. \quad (2.24)$$

Then

$$\gamma = -t + x_+ b + \frac{x_-}{b} \quad (2.25)$$

and

$$u = u(\mu, r) = u\left(-t + x_+ b + \frac{x_-}{b}\right), \quad (2.26)$$

where we have dropped the dependence of u on the constant b , and so effectively

$$f_\alpha(u) = f_\alpha\left(-t + x_+ b + \frac{x_-}{b}\right). \quad (2.27)$$

This corresponds to a Lorentz boost of the familiar time-independent anti-instanton solutions. To see this, observe that if we perform a rotation in the x, y plane

$$\begin{aligned} x' &= x \cos \phi + y \sin \phi, \\ y' &= -x \sin \phi + y \cos \phi, \end{aligned} \quad (2.28)$$

followed by a Lorentz boost along the x' axis

$$\begin{aligned} X &= \gamma x' - \gamma \beta t \\ Y &= y' \\ T &= \gamma \beta t + \gamma x' , \end{aligned} \quad (2.29)$$

where

$$\gamma = (1 - \beta^2)^{-1/2}$$

then

$$X + iY = -\gamma \beta t + \gamma (x \omega \phi + y \sin \phi) \pm i y \omega \phi \mp i x \sin \phi \quad (2.30)$$

However, this can be rewritten as

$$\begin{aligned} X + iY &= \gamma \beta \left[-t + \frac{x}{\beta} \left(\omega \phi \mp \frac{i \sin \phi}{\gamma} \right) \pm \frac{iy}{\beta} \left(\mp i \sin \phi + \frac{\omega \phi}{\gamma} \right) \right] = \\ &= \gamma \beta \left[-t + x_+ \left(\frac{\omega \phi}{\beta} \mp \frac{i \sin \phi}{\gamma \beta} \pm \frac{\omega \phi}{\beta \gamma} - \frac{i \sin \phi}{\beta} \right) + x_- \left(\frac{\omega \phi}{\beta} \mp \frac{i \sin \phi}{\beta \gamma} \mp \frac{\omega \phi}{\beta \gamma} + \frac{i \sin \phi}{\beta} \right) \right] \\ &= \gamma \beta \left[-t + x_+ \frac{e^{-i\phi}}{\beta} \left(1 \pm \frac{1}{\gamma} \right) + x_- \frac{e^{i\phi}}{\beta} \left(1 \mp \frac{1}{\gamma} \right) \right] = \gamma \beta \left[-t + x_+ b + \frac{x_-}{b} \right] , \end{aligned} \quad (2.31)$$

where

$$b = e^{-i\phi} \frac{1}{\beta} \left(1 \pm \frac{1}{\gamma} \right) .$$

Thus writing

$$b = |b| e^{i\psi}$$

we see that $\phi = -\psi$, and that

$$|b| = \frac{1}{\beta} \left(1 \pm \frac{1}{\gamma} \right) . \quad (2.32)$$

However, this last expression only makes sense if we choose

$$|b| = \frac{1}{\beta} \left(1 + \frac{1}{\gamma}\right) \quad \text{for } |b| > 1 \quad (2.33)$$

and

$$|b| = \frac{1}{\beta} \left(1 - \frac{1}{\gamma}\right) \quad \text{for } |b| < 1 ;$$

however,

$$\beta = \frac{2 |b|}{1 + |b|^2} \quad (2.34)$$

in each case.

As we can always drop the irrelevant constant factors $(\gamma\beta)$ in going from (2.31) to (2.27), we see that a constant μ indeed describes a Lorentz boost of static instantons for $|\mu| > 1$ and static anti-instantons for $|\mu| < 1$. $|\mu| = 0$ or $|\mu| = \infty$ corresponds to the state of rest, while $|\mu| \rightarrow 1$ describes a boost with velocity close to the velocity of light. The direction of the boost is determined by the phase of μ .

All other solutions given in (2.16) have a similar interpretation; they correspond to mixed instanton or anti-instanton solutions of the two-dimensional Euclidean problem appropriately Lorentz-boosted.

If we now consider more complicated forms of $H(\mu, \gamma)$, we will obtain as solutions $\mu = \mu(x^\nu)$. The above discussion suggests that one can think of such μ 's as local (position and time) dependent boost parameters. Of course, as u depends in general on both γ and μ , and $|\mu| \gg 1$ describes different boosts, such interpretations may not be very useful, but we will keep it in mind in the next sections.

3. - INFINITE AND FINITE ENERGY SOLUTIONS

As we are mostly interested in finite energy solutions, we express the energy functional in terms of $D_u z_\alpha$, $D_{\bar{u}} \bar{z}_\alpha$, u, μ . In fact, the energy of any configuration $z_\alpha(u, \bar{u})$ is given by

$$E = \int dx dy \mathcal{E} = \int dx dy \left(D_t Z_\alpha \overline{D_t Z_\alpha} + D_x Z_\alpha \overline{D_x Z_\alpha} + D_y Z_\alpha \overline{D_y Z_\alpha} \right)_{(3.1)}$$

Now

$$\mathcal{E} = D_u Z_\alpha \overline{D_u Z_\alpha} (\partial_t u \partial_t \bar{u} + \partial_x u \partial_x \bar{u} + \partial_y u \partial_y \bar{u}) = D_u Z_\alpha \overline{D_u Z_\alpha} (\nabla u \nabla \bar{u})$$

and

$$(\nabla u \nabla \bar{u}) = \frac{1}{2} (|u_+| + |u_-|)^2$$

which follows from

$$u_t^2 - u_x^2 - u_y^2 = 0.$$

Using the fact that $\gamma_+ = -\mu\gamma_t$, $\gamma_t = -\mu\gamma_-$, we get

$$\begin{aligned} |u_+| &= |u_{,\mu} \mu_t + u_{,r} \gamma_{,t}| |\mu| \\ |u_-| &= |u_{,\mu} \mu_t + u_{,r} \gamma_{,t}| \frac{1}{\mu} \end{aligned}$$

So

$$(\nabla u \nabla \bar{u}) = \frac{1}{2} \left(|\mu| + \frac{1}{|\mu|} \right)^2 |u_{,\mu} \mu_t + u_{,r} \gamma_{,t}|^2.$$

This expression can be further simplified using Eq. (2.22) for μ :

$$\mu_t = \frac{H_{,r}}{H_{,\mu} + \left(\chi_t - \frac{\chi_-}{\mu^2} \right) H_{,r}}.$$

Therefore, the final expression for the energy functional (3.1) is

$$E = \int dx dy D_u Z_\alpha \overline{D_u Z_\alpha} |J|^2 \left(|\mu| + \frac{1}{|\mu|} \right)^2, \quad (3.2)$$

where

$$J = \frac{u_{,\mu} H_{,r} - u_{,r} H_{,\mu}}{H_{,\mu} + H_{,r} \left(\chi_+ - \frac{\chi_-}{\mu^2} \right)}.$$

One should remember that $u = u(\mu, \gamma)$ and $H = H(\mu, \gamma)$ are both arbitrary functions which depend on two variables, and so one appears to be in a situation with an "embarras de richesse". Unfortunately, we do not have any systematic way of choosing the "physically interesting" functions H and u , i.e., those which correspond to solutions with finite energy. Therefore, we restrict ourselves to algebraic equations for μ . The only motivation for this choice is the relative simplicity of finding explicit expressions for $\mu(x_+, x_-, t)$. One can equally well try transcendental equations for μ , but this appears to be much more difficult to analyze. So, let us proceed with the simplest non-trivial choice for $H(\gamma, \mu)$, which is given by

$$H(\gamma, \mu) = \gamma + a\mu + \frac{b}{\mu} + c = 0, \quad (3.3)$$

where a, b, c , are arbitrary complex constants and $\gamma = x_+\mu + (x_-/\mu) - t$. By a translation $y \rightarrow y - a$, $t \rightarrow t + \text{Re}(c)$, we can always put $a = 0$, $c = -i\alpha$ where $\alpha \in \mathbb{R}$. Clearly, (3.3) is a quadratic equation for μ , and its two solutions are:

$$\mu_{1,2} = \frac{t + i\alpha \pm \sqrt{(t + i\alpha)^2 - 4x_+(x_- + b)}}{2x_+}. \quad (3.4)$$

To ensure that μ_1 and μ_2 are well defined - i.e., that μ is a single-valued function of x_+ , x_- , t - we choose $\alpha > |b|$. Then the discriminant of (3.3) cannot vanish and in our case, this guarantees that μ is single-valued. One of the roots in (3.4) is non-singular for any value of x_+ , x_- , t , whereas the other behaves like $1/x_+$ around $x_+ = 0$.

Let us now consider the behaviour of the energy integral (3.2) for large values of x and y . For any fixed t , and $\rho \rightarrow \infty$, $\mu_{1,2} \rightarrow \pm i e^{-i\phi}$ where $x_+ = \rho e^{i\phi}$, $x_- = \rho e^{-i\phi}$. Thus, for $\rho \rightarrow \infty$

$$\mathcal{E} \rightarrow \frac{|A|}{\rho^2}$$

Hence we see that the total energy diverges as $\propto \rho$ for large ρ . It is easy to find the source of this divergence. As $\gamma = -(\rho/\mu) - i\alpha$, any $u(\gamma, \mu) = \bar{u}(\mu)$, and as $\mu \rightarrow \pm i e^{-i\phi}$, as $\rho \rightarrow \infty$, $|D_u z|^2 \rightarrow \text{const}$ almost everywhere, independently of any particular form of $f_\alpha = f_\alpha(u)$. We see that the expected damping factors from $|D_u z|^2$ do not come into play, since for large ρ , $u(\gamma, \mu)$ is independent of ρ . So, in order for the finite energy solutions to exist, we must find equations for μ such that their solutions give μ 's which either go to ∞ or to a constant, as $\rho \rightarrow \infty$.

Even though the energy of our field configurations is infinite, it is interesting to look at them, and in particular to analyze their time dependence. For simplicity, let us concentrate on looking at the $CP^1 = O(3)$ model case. Even then there is a lot of arbitrariness. To resolve it, we make special simple choices: we take

$$u = u(\mu) = \mu \quad (3.5)$$

and

$$f_\alpha = (1, u).$$

Then, going over to the variables of the $O(3)$ model, we have

$$\begin{aligned} q_1 &= \frac{\mu + \bar{\mu}}{1 + |\mu|^2} \\ q_2 &= \frac{i(\mu - \bar{\mu})}{1 + |\mu|^2} \\ q_3 &= \frac{1 - |\mu|^2}{1 + |\mu|^2} \end{aligned} \quad (3.6)$$

where μ is given by (3.4) with $b = 0$.

Looking at the time dependence of (3.6), we see that at $t = 0$

$$\begin{aligned} q_1 &= \frac{2i B(x_- - x_+)}{4|x|^2 + B^2} \\ q_2 &= \frac{-2B(x_+ + x_-)}{4|x|^2 + B^2} \\ q_3 &= \frac{4|x|^2 - B^2}{4|x|^2 + B^2} \end{aligned} \quad (3.7)$$

where $B = 2 \pm \sqrt{a^2 + 4|x|^2}$ and where $|x|^2 = x_+ x_- = (1/4)(x^2 + y^2)$.

Let us observe that at small $|x|$ this field configuration resembles a one-instanton configuration, while at large $|x|$, it resembles a meron. It is this meronic-like behaviour at large $|x|$ that is responsible for the ∞ value of the energy of the configuration.

For large t ($t \rightarrow \pm \infty$),

$$\mu_1 \rightarrow \frac{t}{x_+}$$

$$\mu_2 \rightarrow \frac{x_- + b}{t}$$

and so we see that for $|x| \ll t$ the field configuration again resembles a one-instanton configuration, whose size is proportional to $|t|$. Of course, for any large but finite $|t|$, the region $|x| \gg t$ still resembles a meron.

As, for any fixed t , the field configuration changes as one increases $|x|$ from that describing an instanton to that of a meron, we may expect that the energy density may peak in this intermediate region. To see whether this is the case, we look at $\mathcal{E}$ [of (3.1)], for our solution. We find

$$\mathcal{E} \cong \frac{1}{[1 + |\mu|^2]^2} \left(|\mu|^2 + \frac{1}{|\mu|^2} \right) \frac{1}{\left| x_+ - \frac{(x_- + b)}{\mu^2} \right|^2} =$$

$$= \frac{1}{\left| \mu x_+ - \frac{(x_- + b)}{\mu} \right|^2} = \frac{1}{|(t + i\alpha)^2 - 4x_+(x_- + b)|} \quad (3.8)$$

For $b = 0$ (the simplest case to analyze), this is

$$\mathcal{E} = \frac{1}{\sqrt{(t^2 - \alpha^2 - \rho^2)^2 + 4t^2\alpha^2}},$$

where

$$x_+ = \rho e^{i\phi}.$$

Hence the energy density has a maximum at $\rho = 0$ for $t^2 - \alpha^2 \leq 0$ and at $\rho^2 = t^2 - \alpha^2$ for $t^2 \geq \alpha^2$. Of course, the total energy, as we said earlier, is infinite.

The next simplest case within algebraic equations for μ is the cubic equation. The most general cubic equation is of the following form:

$$\begin{aligned}
 H &= c\gamma\mu^2 + e\gamma\mu + a\mu^3 + b\mu^2 + d\mu + f = \\
 &= (cx_+ + a)\mu^3 + (-ct + b + ex_+)\mu^2 + (cx_- + d - et)\mu + ex_- + f = 0,
 \end{aligned} \tag{3.9}$$

where a, b, c, d, e, f are complex constants. For $\rho \rightarrow \infty$, the three roots of (3.9) behave as:

$$\mu_{1,2} \rightarrow \pm i\sqrt{\frac{x_-}{x_+}} = \pm ie^{-i\phi} ; \quad \mu_3 \rightarrow -\frac{e}{c}.$$

So μ_3 looks like a promising candidate for generating non-trivial finite energy time-dependent solutions. In fact, as we shall see in the following, it is rather difficult to ensure that μ is single-valued in the whole R^2 for any value of t . We classify our cubic equations according to the three possible asymptotic behaviours of μ_3 . If $e \neq 0$, $c \neq 0$, $\mu_3 \rightarrow -1/c$ where we rescaled our parameters by e , then by a shift in x_+ we get rid of the parameter a , so (3.9) becomes

$$\begin{aligned}
 H(\mu, \gamma) &= c\gamma\mu^2 + \mu\gamma + b\mu^2 + d\mu + f = \\
 &= cx_+\mu^3 + (-ct + b + x_+)\mu^2 + (cx_- + d - t)\mu + x_- + f = 0.
 \end{aligned} \tag{3.10}$$

If $c \neq 0$ but $e = 0$, $\mu_3 \rightarrow 0$ and (3.9) reduces to

$$\begin{aligned}
 H(\mu, \gamma) &= \mu^2\gamma + b\mu^2 + d\mu + f = \\
 &= x_+\mu^3 + (b - t)\mu^2 + (x_- + d)\mu + f = 0.
 \end{aligned} \tag{3.11}$$

If $c = 0$, $e \neq 0$, $\mu_3 \rightarrow \infty$ asymptotically and the relevant equation is

$$\begin{aligned}
 H(\mu, r) &= e\delta\mu + \mu^3 + e\delta\mu + f = \\
 &= \mu^3 + ex_+\mu^2 + e(b-t)\mu + ex_- + f = 0.
 \end{aligned}
 \tag{3.12}$$

The three roots of any cubic equation:

$$x^3 + Ax^2 + Bx + C = 0$$

are given by the textbook (Cardano) formula

$$\begin{aligned}
 x_i &= y_i - \frac{A}{3}, \quad y_{1,2} = -\frac{u+v}{2} \pm i\frac{\sqrt{3}}{2}(u-v) \\
 y_3 &= u+v,
 \end{aligned}
 \tag{3.13}$$

where

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}}, \quad v = \sqrt[3]{-\frac{q}{2} - \sqrt{Q}}.$$

Q is the discriminant of the cubic equations, and it is given by

$$\begin{aligned}
 Q &= \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \quad \text{where} \\
 p &= -\frac{A^2}{3} + B, \quad q = \frac{2}{27}A^3 - \frac{AB}{3} + C.
 \end{aligned}
 \tag{3.14}$$

As formulae (3.13) and (3.14) show, the dependence of the roots on the coefficients of the equation is rather complicated. In our case, the main difficulty lies in the fact that we have to ensure the differentiability of the root $\mu = \mu(x_+, x_-, t)$ with respect to x_+, x_-, t . In general, the roots are not single-valued functions of (x_+, x_-, t) , and therefore they cannot give rise to finite energy solutions of the CP^{N-1} models on R^2 . In general, the vanishing of the discriminant already indicates that some roots are not single-valued, since in that case at least two roots coincide. In other words, the multi-valuedness of the two roots is usually connected to the fact that when they collide, they "get confused", that is, we cannot tell which is which

after the collision. In the light of this argument, it would be desirable to guarantee the non-vanishing of the discriminant for any t, x_+, x_- . As we saw earlier for the quadratic equation (3.3), the possible zeros of the discriminant are very sensitive to the values of the parameters a, b, c . Therefore, one can hope that for the cubic equations (3.10), (3.11) and (3.12), one can arrange by a suitable choice of parameters that Q is not zero for any x_+, x_-, t . We analyzed this problem numerically and the result gave a strong indication that Q always has zeros. We do not have proof of this, but we could not find a single non-trivial example. We analyzed in detail the nature of the zeros of Q ; unfortunately, $Q \approx x_+$ near the zeros. Therefore, $\sqrt{Q}$ is not single-valued, which implies that u and v are not single-valued either. However, $u+v$ is smooth when $Q \rightarrow 0$, and so $\mu_3 = u+v - (A/3)$ is a possible choice. We must still ensure that $u+v$ is single-valued, i.e., that its phase is well defined. [We refer to $\mu_3 = u+v - (A/3)$ as the symmetric root.] In the case of Eq. (3.10), the symmetric root is not the interesting one for us, that is, μ_3 does not tend to $-1/c$ as $\rho \rightarrow \infty$. Therefore, no root of Eq. (3.10) can be used to generate finite energy, time-dependent solutions. Fortunately, the symmetric roots of Eqs. (3.11) and (3.12) behave quite differently for large ρ from that of (3.10). For (3.11), $\mu_3 \xrightarrow{\rho \rightarrow \infty} -f/x_-$ and for (3.12), $\mu_3 \xrightarrow{\rho \rightarrow \infty} -x_+/c$.

This is encouraging enough for us to analyze further the behaviour of μ_3 of (3.11) and (3.12). We made a thorough analysis of these symmetric roots, and although we do not have a strict proof, it is extremely likely that they are single-valued functions of x_+, x_-, t for a suitable choice of the parameters. For details, we refer to Appendix 2.

4. - PROPERTIES OF OUR SOLUTIONS

In this section, we discuss some properties of our solutions, i.e., those corresponding to the symmetric root of Eq. (3.12) for large $|t|$. The leading $|t|$ behaviour of these solutions is given by

$$\mu_1 \sim \sqrt{et} \quad , \quad \mu_2 \sim -\sqrt{et} \quad , \quad \mu_3 \sim \frac{ex_- + f}{et} \quad (4.1)$$

respectively. To check which is the behaviour of our solution, we return to the Cardano formula (3.13) and find that our solution corresponds to μ_3 .

Now we fix t and vary $|x_+|$. For small $|ex_+ + f|$, our solution behaves as

$$\mu_3 \sim \frac{-ex_- - f}{e(b-t)} \quad (4.2)$$

while for large $|x_+|$ it gives

$$\mu_3 \sim -ex_+ \quad (4.3)$$

We are now ready to make some qualitative statements about the behaviour of our solutions. For simplicity, as in the previous section, we choose the $CP^1 \simeq O(3)$ model and assume that

$$u = u(\mu) = \mu$$

and let

$$f_x = (1, \mu). \quad (4.4)$$

Then our results (4.2) and (4.3) show that, at fixed finite t , the field configuration resembles a one-anti-instanton configuration for small $|ex_+ + f|$, while for large $|x_+|$ it looks like that of one instanton. As $|t|$ increases, so does the region of the approximate anti-instanton configuration, and we see that all t dependence of the solution is associated with this movement of the intermediate region between one instanton and one anti-instanton. Hence we expect the energy density to peak in this intermediate region and gradually decrease in the region of small $|ex_+ + f|$. A few lines of calculation show that this is indeed the case. We have

$$\mathcal{E} \sim \frac{1}{(1+|\mu|^2)^2} \left(|\mu| + \frac{1}{|\mu|} \right)^2 |\mu_+|^2 = |\mu_-|^2, \quad (4.5)$$

which shows that

$$\mathcal{E} \sim \frac{1}{|t|^2} \quad (4.6)$$

for fixed $|x_+|$ and large $|t|$. For finite $|t|$ the energy density exhibits an approximate ring structure. The size of the ring augments and its height decreases with an increase in $|t|$.

A similar discussion can be given for our solutions of Eq. (3.11). We find that for large $|t|$ our solution behaves like

$$\mu_3 \sim - \frac{b-t}{x_+}, \quad (4.7)$$

which is also its behaviour for small x_+ . For large x_+ , our solution goes as

$$\mu_3 \rightarrow - \frac{f}{x_+ + d}. \quad (4.8)$$

We see that this case closely resembles the one discussed above, except that this time the roles of instantons and anti-instantons are interchanged. The interpretation is very similar and the energy density exhibits a similar behaviour.

Of course, we can consider other relations between u and μ ; in particular we could choose

$$u = \mu^2. \quad (4.9)$$

In this case, the field configuration changes its resemblance from one instanton and one anti-instanton to two instantons and two anti-instantons respectively. The energy density is also modified. It now reads

$$\mathcal{E} \sim \frac{|\mu|^2 (1+|\mu|^2)^2}{(1+|\mu|^4)^2} |\mu_t|^2. \quad (4.10)$$

The appearance of the additional factors increase the convergence of the energy integral and changes somewhat the shape of the energy density and its time evolution. It is difficult to analyze in any detail the behaviour of (4.10) without doing some numerical work. However, we expect qualitatively to see again the effective ring structure observed for (4.6).

We do not have a clear interpretation of our solutions. One can argue that they describe instantons and anti-instantons located close to each other, but of different sizes. The time evolution affects their sizes, thus effectively changing their relative importance, and thus shifting the position of the local maximum of the energy functional. It is worth recalling that a somewhat similar ring structure of the energy density was found for static magnetic monopoles lying very close to each other⁵⁾. There, there was no parameter which could describe their sizes and the system was static. Here we effectively have such parameters, and as they become time-dependent, the ring structure exhibits time-dependence.

We are not sure whether this interpretation is correct - it may necessitate some numerical work - but we feel that the interesting structure seen here merits these preliminary remarks.

Of course, one can also consider even more complicated functions $u(\mu, \gamma)$. For example, we can take

$$u = \mu^n \gamma. \quad (4.11)$$

There it is easy to check that as $|x_+| \rightarrow \infty$, $\mu \rightarrow -ex_+$, $\gamma \rightarrow -ex_+^2$, and so

$$u \rightarrow (-)^{n+1} e^{n+2} (x_+)^{n+2}. \quad (4.12)$$

Taking once again $f_\alpha = (1, u)$, we see that the energy density

$$\begin{aligned} \mathcal{E} &\sim \frac{1}{(1+|u|^2)^2} \left(|\mu| + \frac{1}{|\mu|} \right)^2 \left| \frac{u_{,\mu} H_{,\gamma} - u_{,\gamma} H_{,\mu}}{\frac{dH}{d\mu}} \right|^2 \sim \\ &\sim \frac{1}{|u|^2} \frac{|en\mu^n \gamma - \mu^n (e\gamma + 3\mu^2 + eb)|}{|3\mu^2 + 2\mu x_+ e + e(b-t)|^2} \sim \frac{1}{|x_+|^4} \end{aligned} \quad (4.13)$$

guaranteeing the convergence of the total energy. The field configuration for large $|x_+|$ resembles that of $n+2$ instantons. As for fixed $|x|$, large $|t|$:

$$\begin{aligned}\mu &\rightarrow \frac{ex_- + f}{et} \\ \gamma &\rightarrow \frac{-f}{ex_- + f}\end{aligned}\tag{4.14}$$

we see that

$$u = \gamma \mu^n \rightarrow -(et)^{-n} f (ex_- + f)^{n-1}\tag{4.15}$$

which suggests (for $n \neq 1$) that the field configuration contains an ever-increasing region which resembles $n-1$ anti-instantons. This is not dramatically different from the cases studied earlier.

In all the cases studied so far, we have restricted ourselves to looking only at the simplest solutions, i.e., those corresponding to $z_\alpha^1 = f_\alpha/|f|$ of Eq. (2.16). Further, interesting solutions are obtained by considering other vectors from the sequence z_α^K of (2.16). We have not yet looked at them in much detail, as we feel it is more important to understand various properties of the simplest solutions first. Of course, guided by our experience with static solutions (2.16), where the further vectors z_α^i described various mixtures of instantons and anti-instantons, we expect our time-dependent solutions to mix in a similar way the field configurations describing the the simplest time-dependent configurations described above.

5. - CONCLUSIONS

In this paper, we have set up a framework for deriving time-dependent classical solutions of CP^{N-1} models in 2+1 dimensions. Our procedure reduces the problem to finding solutions of a large class of implicit algebraic equations for an auxiliary complex function $\mu(x,y,t)$. As the procedure guarantees only the existence of solutions locally, the requirement of their global existence imposes some constraints on the arbitrariness involved (some solutions are not globally defined). Further conditions stem from the requirement of the finiteness of the energy of the solution. Unfortunately, we have not found any convenient way of imposing these constraints; in our discussion we were forced to look at each case by itself. We found that the simplest possible cases corresponded to the static two-dimensional solutions

appropriately boosted. The next simplest class involved genuine time-dependent solutions. They have interesting properties, but their energy diverges logarithmically. The following class involved functions which are solutions of the cubic equations. In this class, we found some solutions which seem to be well defined over the whole space R^2 and which have finite energy. we then analyzed their properties in some detail and suggested their interpretation. To obtain further solutions, one can now look at solutions of higher order algebraic equations, or even transcendental equations. They all give local solutions. To check whether they are globally single-valued and/or whether their energy is finite is a very difficult task and would require a large dose of mathematical analysis. We have not found any simple way of satisfying these constraints, and so we have not looked in much detail at any equations of order higher than 3.

We finish by pointing out that although we have always spoken about solutions of CP^{N-1} models, our procedure is not restricted to these models alone. As it reduces the problem to solving one non-linear equation for μ , the procedure provides solutions to all Grassmannian models and, with some reality restrictions imposed on the solutions, also to $O(2N+1)$ models.

APPENDIX 1

For completeness, we repeat the demonstration that

$$Z_{\alpha}^{k+1} = \frac{P_+^k f_{\alpha}}{|P_+^k f|} \quad k=0, \dots, N-1 \quad (A1.1)$$

solves the equations of motion (2.4) [or, equivalently, (2.7)] if $\partial_u f_{\alpha} = 0$.

Of the various proofs that have been given, perhaps the simplest one is that given by Sasaki⁶⁾, which we repeat here, adapted to our case. The proof proceeds in several steps. First of all, one observes that the vector z_{α}^1 can be thought of as the i^{th} vector e^i in the sequence of orthonormal vectors obtained from the vectors $(f_{\alpha}, \partial_u f_{\alpha}, \partial_u^2 f_{\alpha}, \partial_u^3 f_{\alpha}, \dots, \partial_u^L f_{\alpha}, \dots)$ via the Gramm-Schmidt orthonormalization procedure.

We have to show that the projector $P = e_K e_K^+$ satisfies

$$[\partial_{u\bar{u}}^2 P, P] = 0. \quad (A1.2)$$

To prove this, we first of all construct two auxiliary projectors

$$R = \sum_{i=1}^{k-1} e_i e_i^+ \quad (A1.3)$$

and

$$Q = \sum_{i=1}^k e_i e_i^+.$$

Then

$$Q = P + R \quad (A1.4)$$

and

$$P \cdot R = R \cdot P = 0.$$

Moreover, as

$$\partial_{\bar{u}} e_i = \sum_{p=1}^i e_p (e_p^+ \partial_{\bar{u}} e_i) \quad (\text{A1.5})$$

we see that

$$\begin{aligned} \partial_{\bar{u}} R R &= 0 \\ \partial_{\bar{u}} Q Q &= 0. \end{aligned} \quad (\text{A1.6})$$

In addition, as

$$\partial_u e_\ell = \sum_{p=1}^{\ell+1} e_p (e_p^+ \partial_u e_\ell) \quad (\text{A1.7})$$

it also follows that

$$Q \partial_u R = \partial_u R \quad (\text{A1.8})$$

and so, by Hermitian conjugation

$$\partial_{\bar{u}} R = \partial_{\bar{u}} R Q = \partial_{\bar{u}} R (P+R) = \partial_{\bar{u}} R P. \quad (\text{A1.9})$$

Combining (A1.9), (A1.6) and (A1.4), we see that

$$0 = \partial_{\bar{u}} (P+R) (P+R) = \partial_{\bar{u}} P P + \partial_{\bar{u}} R P + \partial_{\bar{u}} P R = \partial_{\bar{u}} P P + \partial_{\bar{u}} R. \quad (\text{A1.10})$$

where we used (A1.5) to show that

$$\partial_{\bar{u}} P R = - P \partial_{\bar{u}} R = 0 \quad (\text{A1.11})$$

Taking the Hermitian conjugate of (A1.10)

$$P \partial_u P + \partial_u R = 0 \quad (A1.12)$$

we see that $\partial_u(A1.10) - \partial_{\bar{u}}(A1.12)$ gives us

$$\partial_{u\bar{u}}^2 P P - P \partial_{u\bar{u}}^2 P = [\partial_{u\bar{u}}^2 P, P] = 0 \quad (A1.13)$$

which shows that z_α^K solves the equations of motion.

APPENDIX 2

In this Appendix, we collected the relevant formulae for the analysis of the cubic equations. First of all, the discriminant Q (3.12) is expressed in terms of the coefficients A, B, C , as

$$Q = \frac{1}{4 \cdot 27} [4A^3C - A^2B^2 - 18ABC + 4B^3 + 27C^2]. \quad (A2.1)$$

The necessary and sufficient condition for the coincidence of two roots is the vanishing of Q . For example, in the case of Eq. (3.11),

$$Q = \frac{1}{4 \cdot 27 \cdot x_+^4} [-4f(t+i\alpha)^3 - (x_-+d)(t+i\alpha)^2 + 18(t+i\alpha)f x_+(x_-+d)^3 + 27f^2 x_+^2], \quad (A2.2)$$

where we have written $b = -i\alpha$. Taking the real and imaginary parts of (A2.2), we get a cubic and a quadratic equation in t (with real coefficients). These two equations have a common (real) solution along a curve in R^2 . In fact, for (3.10) and (3.12), the situation is exactly the same; $Q = 0$ along some (quite complicated) curve in R^2 . In the case of Eq. (3.10), $Q = 0$ corresponds to a quartic and a cubic equation in t . We made a thorough numerical analysis of the real roots of $Q = 0$, and in all three cases we concluded that for any choice of the parameters we cannot arrange Q to be non-vanishing for all real t .

For Eq. (3.10), the interesting root goes to $-1/c$ for both large ρ and large t . In fact, if $b - dc + fc^2 = 0$, it is exactly equal to $-1/c$, and the other two are like (3.4). It is easy to verify that the symmetric root has the "wrong" asymptotic behaviour: for large t , $\mu_3 \rightarrow t/x_+$. Now for Eq. (3.11) the asymptotic behaviour of the symmetric root can also be easily computed from A, B, C :

$$\begin{matrix} p \rightarrow \infty \\ t \text{ fixed} \end{matrix} \quad A \rightarrow -\frac{i\omega+t}{x_+} ; B \rightarrow \frac{x_-}{x_+} ; C \rightarrow \frac{t}{x_+} \text{ so } \mu_3 \rightarrow -\frac{t}{x_-} \quad (\text{A2.3})$$

$$\begin{matrix} t \rightarrow \infty \\ p \text{ fixed} \end{matrix} \quad A \rightarrow -\frac{t}{x_+} ; B \rightarrow \frac{x_-+d}{x_+} ; C \rightarrow \frac{t}{x_+} \text{ so } \mu_3 \rightarrow \frac{t}{x_+} ,$$

while for Eq. (3.12),

$$\begin{matrix} p \rightarrow \infty \\ t \text{ fixed} \end{matrix} \quad A \rightarrow ex_+ ; B \rightarrow -e(t+i\alpha) ; C \rightarrow ex_- ; \text{ so } \mu_3 \rightarrow -ex_+$$

$$\begin{matrix} t \rightarrow \infty \\ p \text{ fixed} \end{matrix} \quad A \rightarrow ex_++b ; B \rightarrow -e(t+i\alpha) ; C \rightarrow ex_-+f \text{ so } \mu_3 \rightarrow \frac{f+ex_-}{e} . \quad (\text{A2.4})$$

In the view of (A2.3) and (A2.4), the symmetric root of (3.11) and (3.12) is a reasonable candidate. The remaining problem is to show that the symmetric root is well defined, i.e., single-valued. In the rest, we argue that this is indeed the case. First of all, we have to be sure that there are no triple roots. These occur only if $u = v = 0$, that is

$$p = q = 0 . \quad (\text{A2.5})$$

For Eq. (3.11), we can ensure that $p \neq 0$ if $\alpha > 3|d|$ and a similar condition for Eq. (3.12) guarantees that p and q have no simultaneous zeros. Then a problem can only arise if the phase of u and v is not well defined for a certain region in R^2 . It is sufficient to show that u is single-valued at $t = 0$, because then v is also single-valued. If u is not single-valued, it can only change to $u\varepsilon_1$ or $u\varepsilon_2$, where $\varepsilon_1, \varepsilon_2$ are cubic roots of unity. Then v must change to $v\varepsilon_2$ or $v\varepsilon_1$ respectively, because one must always have three roots. At $t = 0$, it is not hard to convince oneself that when $\alpha > |d|$, μ_3 is single-valued. For non-zero t , μ_3 cannot become multi-valued, since it is a solution of the wave equation with well-defined Cauchy data given on the whole space. (It can be verified that $\mu_3|_{t=0}$ is also well defined.) It is known that, in this case, the solution of the wave equation is unique and well defined. The other way to show that μ_3 is single-valued for any t , given that $\mu_3|_{t=0}$ and $\mu_3|_{t=0}$ are well defined, is to look directly at μ_3 expressed in terms of the infinite volume Green function⁷⁾. That also shows that μ_3 is a single-valued function.

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