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ON THE $U(N)$ GROUP INTEGRAL WITH VARIABLES
IN THE ADJOINT REPRESENTATION AND THE TWISTED CHIRAL MODEL

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ABSTRACT

We consider the generalized one link group integral for group variables and external sources in the adjoint representation of $U(N)$. We develop the strong and weak couplings of this class of integral. Special attention is paid to the case of a source matrix proportional to the identity, which we treat by a saddle point method. The case of twisted chiral models is reconsidered, including the effects of non-trivial maxima. They are argued not to affect the large N limit of the model and to exist also in the standard lattice chiral model.

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1. INTRODUCTION

Some years ago, it was shown that the single link group integrals was important for lattice gauge theory in two dimensions¹⁾; they also enter in a crucial way in the mean field approach²⁾ of gauge and chiral models on the lattice. Therefore, they received much attention and are now solved in the case where the group variables are in the fundamental representation of the group $U(N)$ or $SU(N)$ ³⁾.

More recently, these group integrals for variables in the adjoint representation have entered in the analysis of the twisted reduced models⁴⁾ and of the mixed action of lattice gauge theory⁵⁾.

We still are not able to solve the general case for an arbitrary source in the adjoint representation of $U(N)$ with the known techniques (Schwinger-Dyson equation, character expansion, large N methods) except in very few particular cases.

In this paper, we would like to present some progress that we have done in several directions. In the first section, we study the case corresponding to a source matrix proportional to the identity using the technique of large N saddle point which is a non-perturbative one.

The second section is devoted to the twisted chiral model, paying much attention to the non-perturbative contribution due to the presence of non-trivial maxima⁶⁾. We also point out that such analogous extrema are present in the ordinary lattice chiral models.

In the third section, we exhibit the first few terms in strong and weak coupling expansion of the integral (1) when the source matrix is expanded as a linear combination of the so-called Gonzalez-Arroyo-Okawa basis⁷⁾; the advantage of this basis is that the elements obey simple (anti)commutation relations, as a consequence, the different group invariants appearing in the integration can be written explicitly. The source is kept as general as allowed by certain technical problems.

Before starting we will present the problem, introduce our notations and quote a few particular cases which have already been considered.

The integrals we are interested in can be written in two different ways:

$$Z_N = \int_{U(N)} dU \exp \sum_{\alpha, \beta=1}^{N^2-1} c_{\alpha\beta} U_{\beta\alpha}^A \quad (1.1)$$

$$= \int_{U(N)} dU \exp \sum_{\substack{a,b \\ c,d=1}}^N c_{abcd} U_{ab} U_{cd}^+ \quad (1.2)$$

where dU denotes the Haar measure on $U(N)$, U_{ab} and $U_{\alpha\beta}^A$ are $U(N)$ matrix elements respectively in the fundamental and the adjoint representation. The two tensors $c_{\alpha\beta}$ and c_{abcd} can be related by using a complete set of normalized and Hermitian generators of $U(N)$: T_{ij}^α

$$T_{ij}^\alpha T_{ji}^\beta = \delta_{\alpha\beta} \quad , \quad \sum_{\alpha=0}^{N^2-1} T_{ij}^\alpha T_{lm}^\alpha = \delta_{im} \delta_{jl} \quad (1.3)$$

$$U_{\alpha\beta}^A = \frac{1}{N} (T^\alpha U T^\beta U^+) \quad , \quad \alpha, \beta = 1, \dots, N^2-1 \quad (1.4)$$

Then

$$c_{abcd} = \sum_{\alpha, \beta=0}^{N^2-1} T_{da}^\alpha T_{bc}^\beta c_{\beta\alpha} \quad (1.5)$$

The integrals above have been considered already in several particular cases.

- (i) In the case $N = 1$, it is trivial and in the case $N = 2$ it is solved in Refs. 8) and 9).

- (ii) If $c_{abcd} = V_{bc} V_{da}^+$ (V being an arbitrary matrix) then the integral can be computed in character expansion¹⁰⁾.
- (iii) If $c_{abcd} = \sum_{\mu=1}^d (D_{\mu})_{bc} (D_{\mu}^+)_{da}$, $[D_{\mu}, D_{\nu}] = 0$. Then the integral corresponds to the quenched chiral model and a semi-classical form is given in Ref. 11).

2. SOURCE MATRIX PROPORTIONAL TO THE IDENTITY

In this particular case, the integral (1.1) can be written

$$Z_N(\beta) = \int dU \exp \beta \operatorname{tr} U \operatorname{tr} U^+ \quad (C_{\mu\nu} = \beta \cdot \delta_{\mu\nu}) \quad (2.1)$$

and can be treated by using large N saddle point methods. The large N limit of the integral (2.1) can be obtained by using the technique developed in the study of mixed action⁵⁾; however, it fails to produce the $1/N^2$ correction because taking the limit of vanishing β_F in the mixed action introduces a supplementary $U(1)$ symmetry which is related to a zero mode in the $1/N$ expansion¹²⁾. It is then necessary to modify the usual method. The starting point consist (like in the mixed action problem) in introducing an auxiliary complex scalar field in the integral (2.1)

$$Z_N = \left\{ \int dU \exp \beta \operatorname{tr} U \operatorname{tr} U^+ \right\} \beta N^2 \int da da^* \exp -\beta (Na - \operatorname{tr} U)(Na^* - \operatorname{tr} U^*) \quad (2.2)$$

$$= \beta N^2 \int da da^* \exp -\beta N^2 aa^* + N w(a\beta \mathbb{1}, a^*\beta \mathbb{1})$$

where the function w denotes the standard one link integral.

For our purpose, we will use the $1/N^2$ expansion of w discussed in Ref. 9)

$$\begin{aligned}
 \frac{W(A, A^+)}{N} &= \frac{1}{N^2} \ln \int dU \exp N \operatorname{tr} (U A^+ + U^+ A) \\
 &= \frac{2}{N} \sum_{i=1}^N \lambda_i^{1/2} - \frac{1}{2N^2} \sum_{i,j=1}^N \ln (\lambda_i^{1/2} + \lambda_j^{1/2}) - \frac{3}{4} \quad (2.3) \\
 &\quad - \frac{1}{8N^2} \ln \left(1 - \frac{1}{2N} \sum_i \lambda_i^{-1/2} \right) + \dots = O\left(\frac{1}{N^4}\right); \left(\sum \frac{1}{\sqrt{\lambda_i}} < 2N \right)
 \end{aligned}$$

where λ_i are the eigenvalues of the matrix AA^+ .

The integral (2.2) depends only on the modulus of a ; then, using the formula (2.3) and the new co-ordinates $a = \rho e^{i\theta}$ leads to the new integral

$$Z_N = \beta N^2 2\pi \int d\rho \exp N^2 f(\rho) \quad (2.4)$$

$$\begin{aligned}
 f(\rho) &= -\beta \rho^2 + 2\beta \rho - \frac{1}{2} \ln 2\beta \rho + \frac{1}{N^2} \left(\ln \rho - \frac{1}{8} \ln \left(1 - \frac{1}{2\beta \rho} \right) \right) \\
 &\quad + O\left(\frac{1}{N^4}\right) \quad ; \quad \beta \rho > \frac{1}{2}
 \end{aligned} \quad (2.5)$$

In order to estimate the integral (2.4), we will use the large N saddle point technique, i.e., solve the saddle point equation

$$\frac{\partial f}{\partial \rho} = 0 = -2\beta \rho + 2\beta - \frac{1}{2\rho} + \frac{1}{N^2} \left(\frac{1}{\rho} - \frac{1}{8\rho(2\beta\rho - 1)} \right) + O\left(\frac{1}{N^4}\right) \quad (2.6)$$

for which the solution takes the form

$$\rho_{\text{saddle}} \equiv \bar{\rho} = \rho_0 + \frac{1}{N^2} \rho_1 + O\left(\frac{1}{N^4}\right) \quad (2.7)$$

$$\rho_0 = \frac{1}{2} \left(1 + \sqrt{1 - \frac{1}{\beta}} \right) \quad \beta \gg 1 \quad (2.8a)$$

$$\int_1 = - \frac{\rho_0 (9 - 16 \beta \rho_0)}{4 (1 - 2 \beta \rho_0) \cdot (1 - 4 \beta \rho_0^2)} \quad \beta > 1 \quad (2.8b)$$

The condition $\beta \rho > \frac{1}{2}$ is respected on the domain of validity of the solution (2.8). Then, one can perform an $1/N^2$ expansion of (2.1) using (2.4) and considering the fluctuation around the saddle point (i.e., $\rho = \bar{\rho} + \xi$) recollecting all terms of the same order in N , we have obtained

$$\begin{aligned} \frac{\ln Z_N}{N^2} = & -\beta \rho_0^2 + 2\beta \rho_0 - \frac{1}{2} \ln \beta \rho_0 - \frac{3}{4} \\ & + \frac{1}{N^2} \left[\ln \beta \rho_0 - \frac{1}{8} \ln \left(1 - \frac{1}{2\beta \rho_0}\right) - \frac{1}{2} \ln \left(2\beta - \frac{1}{2\rho_0^2}\right) \right] \quad (2.9) \\ & + \frac{1}{N^4} \left[-\frac{1}{2(4\beta \rho_0^2 - 1)^2} + O\left(\frac{1}{\beta^3}\right) \right] \end{aligned}$$

It is clear that the $1/N^2$ correction becomes singular when β approaches the critical value ($\beta_c = 1/2$). This is a general feature of the expansion (2.3) early remarked by Goldschmidt¹³⁾. However, if one supposes β to be large and expands ρ_0 in powers of β^{-1} then, one finds

$$\left. \frac{\ln Z_N}{N^2} \right|_{\beta \gg 1} = \beta + \frac{N^2 - 1}{N^2} \left\{ -\frac{1}{2} \ln \beta + \frac{1}{16\beta} + \frac{N^2 + 2}{N^2} \frac{1}{64\beta^2} + O\left(\frac{1}{\beta^3}\right) \right\} \quad (2.10)$$

the formula (2.10) is in complete agreement with the $1/\beta$ expansion of the integral (2.1); it also agrees with the known cases $N = 1$ and $N = 2$.

The method developed above cannot be used to predict the function (2.1) for β small. In fact, it only predicts that in the large N limit the function $\ln Z_N/N$ is zero. However, using the elementary group integrations of Ref. 14), we have computed directly the strong coupling expansion of (2.1). The result is the following

$$Z_N^{SC}(\beta) = 1 + \beta + \beta^2 + \dots \beta^N + \beta^{N+1} \cdot R_N \quad (2.11)$$

where R_N is a complicated function of N and β , as an example for $N = 3$

$$Z_3^{sc}(\beta) = 1 + \beta + \beta^2 + \beta^3 + \beta^4 \frac{2070}{2160} + o(\beta^5) \quad (2.12)$$

Knowing (2.11), it is obvious to compute the large N limit

$$Z_\infty^{sc} = \frac{1}{1-\beta}, \quad \ln Z_\infty = -\ln(1-\beta) \gg 0 \quad \text{as } \beta < 1 \quad (2.13)$$

the quantity $\ln Z_\infty^{sc} / N^2$ vanishes in the large N limit, as expected from previous considerations⁴⁾.

The method used above can be successfully applied to the more general following case:

$$Z_N = \int dU \exp \beta \operatorname{tr} A U \operatorname{tr} A^+ U^+ \quad (2.14)$$

Using the invariance of the Haar measure the source matrix corresponding to (2.14) can be put in the canonical form

$$C_{\alpha\beta} = \beta \sum_{j,k} a_j a_k \operatorname{tr} T_{jk}^\alpha T_{kj}^\beta \quad (2.15)$$

where a_j^2 are the eigenvalues of AA^+ . If the basis T^α is the standard Gell-Mann basis, then it is easy to see that the matrix $C_{\alpha\beta}$ is diagonal (but not the most general diagonal matrix). Using the method developed above we have found

$$\begin{aligned} \frac{\ln Z_N}{N^2} = & -\beta \rho_0^2 + 2\beta \rho_0 \frac{\tilde{F}_N A}{N} - \frac{1}{2} \ln \beta \rho_0 - \frac{1}{2N^2} \sum_{i,j} \ln(a_i + a_j) \\ & - \frac{3}{4} + \frac{1}{N^2} \left\{ \ln \beta + \ln \rho_0 - \frac{1}{8} \ln \left(1 - \frac{1}{2\beta \rho_0} \frac{\tilde{F}_N A^{-1}}{N} \right) \right. \\ & \left. - \frac{1}{2} \ln \left(2\beta - \frac{1}{2\rho_0^2} \right) \right\} + o\left(\frac{1}{N^4}\right) \end{aligned} \quad (2.16)$$

with $\rho_0 = \frac{1}{2} \left(\frac{\tilde{F}_N A}{N} + \sqrt{\left(\frac{\tilde{F}_N A}{N} \right)^2 - \frac{1}{\beta}} \right)$; $\tilde{F}_N A^\epsilon \equiv \sum_{\ell=1}^N a_{\ell}^\epsilon$, ($\epsilon = \pm 1$)

3. THE TWISTED CHIRAL MODEL REVISITED

The twisted chiral model (TCM) was proposed¹⁵⁾ after the discovery of a reduction scheme¹⁶⁾ for lattice gauge theory. It is described by the following partition function

$$Z_{\text{TCM}}(\beta) = \int dU \exp -S(U) \quad (3.1)$$

$$S(U) = -\beta N \sum_{\mu=1}^2 \frac{1}{N} \left(U \gamma_{\mu} U^{\dagger} \gamma_{\mu}^{\dagger} + U \gamma_{\mu}^{\dagger} U \gamma_{\mu} - 2 \right) \quad (3.2)$$

$$\gamma_1 \gamma_2 = \exp \frac{2i\pi}{N} \gamma_2 \gamma_1 \quad (3.3)$$

where γ_1 and γ_2 play the role of translation operator on the lattice and are chosen as the P and Q matrices introduced by 't Hooft¹⁷⁾. The model (3.1) is believed to be the equivalent to the ordinary chiral model on the lattice in the large N limit. Clearly the function (3.1) enters in the class (1.1) but it is not known how to treat it in a non-perturbative way (like in the previous section). So far, weak⁴⁾ and strong coupling expansions (in β) can be performed and shown to be those of the chiral model in the large N limit.

A Monte Carlo analysis of TCM¹⁸⁾ revealed an unsuspected behaviour which was soon interpreted⁶⁾ as an instanton effect. Here, we will reconsider the set of configurations

$$\mathcal{C} = \left\{ D(m, m) = \gamma_1^m \gamma_2^m ; \quad 0 \leq m, n \leq N-1 \right\} \quad (3.4)$$

and classify the maxima, minima and saddle points.

To this aim, we expand the action (3.2) around a given configuration

$$U = D(m, m) e^{iA}, \quad A = \sum_{p \neq 0} a(p) G(p), \quad G(p) \equiv \gamma_1^{\vec{p}} \gamma_1^{-\vec{p}} \quad (3.5)$$

$$A = A^\dagger \Longleftrightarrow a(p) = \exp \frac{2i\pi}{N} p_1 p_2 a^*(-p) \quad (3.6)$$

Using the Hermiticity condition (3.6), the expanded action takes the following form:

$$S = S_0 + \sum_{q \neq 0} \beta \mathcal{M}(m, n; q) \cdot |a(q)|^2 \quad (3.7)$$

$$S_0 = 4\beta N^2 \left\{ \sin^2 \frac{\pi m}{N} + \sin^2 \frac{\pi n}{N} \right\} \quad (3.8a)$$

$$\mathcal{M}(m, n; q) = 4 N^2 \left\{ \cos \frac{2\pi m}{N} \sin^2 \frac{\pi q_2}{N} + \cos \frac{2\pi n}{N} \sin^2 \frac{\pi q_1}{N} \right\} \quad (3.8b)$$

slightly different from the form obtained in Ref. 6). However, from the mass matrix (3.8b) the nature of the extremum appears obviously according to the signs of the two cosine. For values of m and n fixed and such that $m, n \ll N$, the classical weight (3.8a) stays finite in the large N limit.

The fundamental vacuum corresponds to the choice $(m, n) = (0, 0)$, then the next-to-leading ones are respectively $(m, n) = (\pm 1, 0)$ and $(0, \pm 1)$; $(1, 1), \dots$, etc.

For large but finite N , some concrete information about the shape of the function $\exp - S(U)$ can be obtained:

- (i) The Euclidean distance^{1*} between any couple of elements in ξ is fixed to be $\sqrt{2}$, independently of N . So two different maxima configurations remain far away from each other in the group.

(1*)

The Euclidean distance on $U(N)$ is obtained by embedding the group into $GL(N, \mathbb{C})$ where one uses the usual distance
 $d(A, B) = [N^{-1} \operatorname{tr}((A-B)(A-B)^\dagger)]^{\frac{1}{2}} \rightarrow d(\mathbb{1}, 0) = 1).$

- (ii) All the saddle point and minima configurations have a classical weight of order of $\exp - \beta N^2$, i.e., vanishing for large N .
- (iii) If we approximate $e^{-S(U)}$ by a sum of Gaussians (for weak coupling), then the Gaussian around the principal maximum and the next-to-leading one are well separated. The variance is (at worse) of the order of β^{-1} and we have explicitly checked that the fundamental Gaussian tail was of order of $\exp -4\beta N\pi^2$ in the neighbourhood of the next-to-leading maxima (of order of $\exp -4\beta\pi^2$).

It is not clear to us how the presence of several maxima affect the integral (3.1) for fixed β and N . In the large N limit (and β large), it is obvious that the principal maximum will dominate; furthermore, we have a straightforward (but not rigorous) argument that the other maxima will not affect the large N limit of TCM.

Let us approximate e^{-S} by a sum of Gaussians centered around the surviving maxima:

$$Z_{TCM} \simeq \sum_{|m|, |n| \leq 2^{-p} N} \exp -S_0(m, n) \beta^{-\frac{1}{2}(N-1)} \left[\det_q \mathcal{M}(m, n) \right]^{-\frac{1}{2}} \quad (3.9)$$

$3 \leq p \leq \infty$

In this approximation, we have neglected a lot of maxima because, around them, the function e^{-S} tends uniformly to zero in the large N limit. As a consequence, the limit commutes with the group integral, and their contributions vanish. Suitable majorations in (3.9) lead to

$$Z_{TCM} \leq \beta^{-\frac{N-1}{2}} \left(\cos \frac{2\pi}{2^p} \right)^{-\frac{N-1}{2}} \left[\det_q \mathcal{M}(0,0) \right]^{-\frac{1}{2}} \sum_{m,n} \exp -S_0(m, n) \quad (3.10)$$

$$\lesssim \beta^{-\frac{N-1}{2}} \left(\cos \frac{2\pi}{2^p} \right)^{-\frac{N-1}{2}} \left[\det_q \mathcal{M}(0,0) \right]^{-\frac{1}{2}} \frac{1}{4\beta\pi} \quad (3.11)$$

$$\frac{\ln Z}{N^2} \text{TCM} \leq -\left(\frac{N^2-1}{2N^2}\right) \left(\ln \beta - \ln \cos \frac{2\pi}{2^p}\right) - \frac{\ln 4\beta\pi}{N^2} \quad (3.12)$$

obviously, the second term in (3.12) vanishes in the large N limit. On the other hand, p being arbitrarily large, the only surviving term is the well-known $-\frac{1}{2} \ln \beta$. So, at least the leading terms in the weak coupling expansion of TCM are not affected by the maxima in the large N limit.

The presence of non-trivial maxima in the twisted chiral model naturally raises the question of finding their analogues in the standard lattice chiral model. We have found such configurations that we present briefly here: consider the two-dimensional chiral model in U(N) on a finite periodic lattice of volume $V = L^2$. The partition function is

$$Z(\beta) = \int_{\mathbf{U} \in \Lambda} \prod_x dU_x \exp -S(U) \quad (3.13)$$

$$S(U) = -\beta N \sum_{\mu, x} \text{tr} (U_x U_{x+\mu}^\dagger + U_x^\dagger U_{x+\mu} - 2) \quad (3.14)$$

The classical equations of motion are satisfied by the ansatz $U_x = A^{x_1} B^{x_2}$ ($A, B \in U(N)$) but only in the case when A and B are proportional to the identity we have been able to show that the mass matrix was positive definite. So, including the periodicity, our solutions take the form

$$U_x^{(0)} = \exp \left(\frac{2i\pi}{L} \right) (m_1 x_1 + m_2 x_2) \mathbb{1}, \quad -\frac{L}{4} < m_1, m_2 < \frac{L}{4} \quad (3.15)$$

$$S_0(m_1, m_2) = 4\beta N^2 V \left(\sin^2 \frac{\pi m_1}{L} + \sin^2 \frac{\pi m_2}{L} \right) \quad (3.16)$$

The classical value of the energy (3.16) clearly indicates that some configurations remain action finite in the large volume limit; in complete analogy with the large N limit of TCM. The two-point correlation functions computed around a maximum configuration labelled (m_1, m_2)

$$\frac{t_n}{2N} \langle U_0 U_x^\dagger + U_0^\dagger U_x \rangle_{(m_1, m_2)} = \cos \frac{2\pi}{L} (m_1 x_1 + m_2 x_2) + o\left(\frac{1}{\beta}\right) \quad (3.17)$$

also supports the comparison with the result of Ref. 6).

Considering e^{-S} in (3.13) as a sum of Gaussians centered at the fundamental maxima and the next-to-leading ones, gives for the mean action:

$$\bar{S} = \frac{\beta^{-1} \sum_{m,n} S_0(m,n) \exp -S_0(m,n)}{\sum_{m,n} \exp -S_0(m,n)} \quad (3.18)$$

$$\simeq 16 \pi^2 N^2 \exp -4 \pi^2 \beta N^2$$

This number roughly represents the shift of the mean action due to the non-trivial maxima in the weak coupling regime.

4. GENERAL SOURCE MATRIX

Now, we present the first few terms of the strong coupling expansion of the integral (1.1). It is convenient to express the source matrix c in terms of the matrices $D(m,n)$ used in the previous section:

$$c_{abcd} = N^{-1} \sum_{p,q} c_{pq} D(p_1, p_2)_{bc} D(q_1, q_2)_{da}^\dagger \quad p \equiv (p_1, p_2) \quad (4.1)$$

$$c_{p,q} = \exp \frac{2i\pi}{N} (q_1 q_2 - p_1 p_2) c_{-p, -q}^* \quad (4.2)$$

The choice (4.1) is clearly the most general one while the relation (4.2) ensures that the exponent is real and that the integral makes sense.

From the general properties of the $D(p)$ matrices, one can see that the constants $c_{p,0}$ and $c_{0,p}$ are useless because their coefficients are zero; also the c_{00} coefficient can be factorized out of the integral. In the formula above, we have used the group integration of Ref. 14);

all the summations (including the traces) have to be considered for p and q different from zero.

$$\begin{aligned} \bar{Z}_N^{s.c.}(c) = \exp C_{00} \left\{ 1 + \frac{1}{2(N^2-1)} \text{tr} CC^* + \frac{1}{3N^2(N^2-1)(N^2-4)} \left((N^2-2)I_1 + 2I_2 \right) \right. \\ \left. + \frac{1}{4N^2(N^2-1)(N^2-4)(N^2-9)} \left[(N^4-8N^2+6) \left(\frac{(\text{tr} CC^*)^2}{2} + \frac{J_1}{N^2} \right) \right. \right. \\ \left. \left. + (2N^2-3) \frac{J_2}{N^2} + (N^4+6) \left(\text{tr} CC^*CC^* + \frac{J_3}{N^2} \right) + 2(-N^2+4)K_1 - 5K_2 \right] \right\} \quad (4.3) \end{aligned}$$

The invariants I, J, K have been computed to be respectively

$$I_i = \sum_{\substack{p,q \\ n,s}} C_{pq}^* C_{ns}^* C_{p+n,q+s} \varphi^{<p/n>} \begin{pmatrix} \varphi^{-<q/s>} \\ \varphi^{-<s/q>} \end{pmatrix} \quad i=1,2 \quad (4.4)$$

$$J_i = \sum_{\substack{p,q, \\ n,s, \\ v,w}} C_{pq}^* C_{ns}^* C_{vw}^* C_{p+n+v,q+s+w} \varphi^{<p/n/v>} \begin{pmatrix} \varphi^{-<q/s/w>} \\ \{ \varphi^{-<s/q/w>} + \varphi^{-<w/q/s>} + \\ \varphi^{-<q/w/s>} + \varphi^{-<s/w/q>} \\ \varphi^{-<w/s/q>} \} \end{pmatrix} \quad i=1,2,3$$

(4.5)

$$K_i = \sum_{\substack{p,q \\ n,l}} \left[(cc^\dagger)_{pq} (cc^\dagger)_{ln} + (c^\dagger c)_{pq} (c^\dagger c)_{ln} \right] \\ \delta(p+l-q-n) \varphi^{<p|n>} \begin{pmatrix} \varphi^{<q|l>} \\ \varphi^{<l|q>} \end{pmatrix}, \quad i=1,2 \quad (4.6)$$

with the notations

$$\varphi \equiv \exp \frac{2i\pi}{N}, \quad <p|q> = p_1 q_2, \quad (4.7)$$

$$<p|q|n> = p_1 q_2 + p_1 n_2 + q_1 n_2$$

Finally, the weak coupling of (1.1) has been computed in the case of a diagonal matrix C_{pq} (we have not succeeded to invest the mass matrix in the general case). Up to one-loop, the result takes the form

$$\ln Z_N^{wc}(c) = \sum_q c(q) - i^{-1} \ln \pi \{m_i\} \\ + \sum_{i,j \neq 0} \left(\frac{1 + \cos \frac{2\pi}{N} (i_1 j_2 - i_2 j_1)}{m_i m_j} \right) \sum_q \left(\operatorname{Re} c(q) \sin^2 \frac{\pi}{N} q \cdot i \right. \\ \left. \times \sin^2 \frac{\pi}{N} q \cdot j \right) + o\left(\frac{1}{c^2}\right) \quad (4.8)$$

$$m_i = 4 \sum_q c(q) \sin^2 \frac{\pi}{N} (i \cdot q) \quad i \equiv (i_1, i_2) \quad (4.9) \\ i \cdot q = i_1 q_1 + i_2 q_2$$

The strong coupling formula (4.3) is tedious but simplifies in some particular cases. As an example, in the TCM the source matrix takes the form

$$c_{pq} = \beta N^2 \delta_{pq} \quad \text{if } p = (0,1), (1,0), (0,N-1), (N-1,0) \\ = 0 \quad \text{otherwise} \quad (4.10)$$

and the result of the strong coupling can be obtained after a little algebra:

$$\ln Z_{TCM}(\beta) = 2 \beta^2 \frac{N^4}{N^2-1} + \frac{\beta^4 N^4}{(N^2-1)^2 (N^2-4) (N^2-9)} \cdot (2 N^6 - 13 N^4 - N^2 - 36 \\ + 6(N^2-1) \cdot (N^2-4) \cdot (\varphi + \varphi^*) + (N^2-1)(N^2+6)(\varphi^2 + \varphi^{*2}) \quad (4.11)$$

the large N limit of which agrees with the weak coupling expansion of the d-dimensional chiral model¹⁹⁾ and of the twisted chiral model

$$\frac{\ln Z_{CM}}{V N^2} = \frac{\ln Z_{TCM}}{N^2} \\ = d \beta^2 + d(d-1) \beta^4 + d(d-1) \left(\frac{16d-26}{3} \right) \beta^6 + o(\beta^8) \quad (4.12)$$

We do not write the weak coupling expansion, it is discussed in Ref. 4).

To conclude, we stress that the problem stated in Eq. (1.1) or (1.2) is difficult and far from solved. In the particular case of a source proportional to the identity matrix, the technique developed in Section 2 furnishes a $1/N^2$ expansion. The knowledge of this expansion is very important for the mean field approach of lattice spin and gauge model when the variable is in the fundamental representation¹³⁾. Moreover, this kind of expansion is crucial for the improved mean field calculation²⁰⁾ and we can hope that it is also the case whenever the group variables are in the adjoint representation.

Especially in two dimensions, the non-trivial maxima of the TCM are in one-to-one correspondence with local maxima of a $U(N)$ chiral model on a periodic lattice. These should be put in evidence by anyone who wants to perform a high precision numerical study of such a model.

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