MONTE CARLO RENORMALIZATION GROUP TRAJECTORIES
IN LATTICE GAUGE THEORIES

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A B S T R A C T

A Monte Carlo method for evaluating block renormalized coupling constants in a lattice gauge theory is formulated. Additionally a new blocking procedure such that each site link enters into exactly one block link is described. As a test problem the Ising gauge system in three dimensions is considered. The trajectory of the plaquette coupling is calculated and the fixed point and critical exponent are determined. The value for the critical exponent is consistent with standard results.

1 - PROLEGOMENA

Probably the most natural way to study lattice gauge theories [1] is with the real space renormalization group [2]. By dealing directly with the scaling properties of the system, the renormalization group permits the approach to the continuum limit to be handled in an economical fashion. Because of the complexity of typical lattice gauge theories, it seems likely that the renormalization must be performed with numerical techniques (such as Monte Carlo methods [3]). This combination of ideas in fact originally led to the so-called Monte Carlo renormalization group, or MCRG [4-10].

The MCRG has been quite useful in the extraction of the critical behavior of statistical systems and so presumably is useful for understanding the continuum limit of a lattice theory. A complete analysis of such a system with the renormalization group however requires that its coupling constant trajectories be calculable. These trajectories in turn determine the fixed points of the lattice action, and thus enumerate all possible continuum limits of the field theory.

Recently a way to calculate the trajectories of the block renormalized coupling constants with Monte Carlo methods has been developed. The method was first applied successfully to the two-dimensional Ising model [8]. Because the procedure is quite generally applicable, it was also applied to a ϕ^4 field theory [9]. The further utility of this new technique for lattice gauge theories is demonstrated explicitly below in a MCRG study of $Z(2)$ lattice gauge theory in three spacetime dimensions. It is of course important to point out that an entirely different (and quite clever) way to calculate block renormalized coupling constants in the Ising spin model has also been published recently [10].

Another novelty in the method presented below is the new definition of block links. The new blocking procedure possesses the twin features that

- Each site link (i.e., original, unrenormalized link) appears in one and only one block link;
- The points and axes of the block lattice are also points and axes in the site lattice. Thus matter fields can be added in a straightforward fashion, and the blocking procedure is relatively easy to visualize.

Other definitions of block links have been used in previous MCRG calculations [6,7].

The following presentation is divided into several parts. First, the general procedure for calculating block renormalized couplings is briefly reviewed. After this outline, the new definition of block links is described. The discussion is then specialized to the case of $Z(2)$ lattice gauge theory, and a calculation in three dimensions of a coupling constant trajectory is performed. The fixed point and its thermal exponent are also determined.

2 - DESCRIPTION OF THE PROCEDURE

The method used here to evaluate block renormalized couplings has been introduced and described in detail previously [8,9]. For completeness, a brief outline of the technique is presented here.

The basic idea of the real space renormalization group is that the original or "site" system is divided into sets (or "blocks") of site variables. A smaller number of block variables are defined by averaging in some fashion over the site variables. The thermodynamic description of the system can be made either in terms of the site variables (governed by a site action) or in terms of the block variables (governed by a block action). When this block action is determined, the critical properties of the system can be deduced by observing how the site action "flows" into the block action as the site degrees of freedom are systematically integrated out.

The discussion can be formulated quantitatively. Consider a system of site links $\{U\}$ governed by a site action $S\{U\}$. A block action for the system $S'\{U'\}$ is extracted by the use of an appropriate projection operator $P[\{U'\},\{U\}]$:

$$e^{-S'\{U'\}} = \text{Tr}_{\{U\}} P[\{U'\},\{U\}] e^{-S\{U\}}, \quad (1)$$

where the trace (or functional integral) is over only the site links $\{U\}$. The requirement that the renormalization group preserve the partition function (i.e., the vacuum-to-vacuum transition amplitude) of the system imposes the constraint:

$$\text{Tr}_{\{U'\}} P[\{U'\}, \{U\}] = 1 \quad (2)$$

The projection operator is otherwise arbitrary.

Both the site and block actions can be expanded in terms of appropriate basis functions $\{S_i\}$ and coupling constants $\{\beta_i\}$ and $\{\beta'_i\}$ as follows:

$$S'_i \{U'\} = \sum_i \beta'_i S_i \{U'\} , \quad (3a)$$

$$S \{U\} = \sum_i \beta_i S_i \{U\} . \quad (3b)$$

Imagine that all of the $\{U'_i, i \neq 0\}$ are held fixed, except for one (denoted by U'_0), and furthermore that they are fixed in such a fashion that all but one, S_0 , of the $\{S_i\}$ involving U'_0 are identically zero. Then the probability that the lone fluctuating variable assumes the value U'_0 is proportional to

$$g(U'_0) \propto \exp[\beta'_0 S'_0 \{U'\}] , \quad (4)$$

which depends only upon the chosen coupling β_0 and the known function $S_0 \{U'\}$. It is thus a relatively simple matter to determine β'_0 by measuring (via a Monte Carlo simulation) the function $p(U'_0)$. In this fashion all of the block couplings of the theory can be evaluated systematically.

It is of course not always possible to choose a single block configuration in which all of the $\{S_i\}$ but one are cancelled. However, by suitable combinations of the results from several block configurations it is possible to separate and evaluate all of the $\{\beta'_0\}$. Additionally in practice only a small number of couplings are large, and it is often possible to separate these with a single block configuration.

3 - DESCRIPTION OF THE PROJECTION OPERATOR

In order to turn the exercise of imagination in the last section into concrete reality, it is necessary to construct a projection operator

which allows all but one of the $\{U'\}$ to be held fixed. General methods for accomplishing this aim were given in [8] and [9]; for a discrete gauge group (or subgroup) the task is simplified somewhat.

The projection operator is taken as

$$P[\{U'\}, \{U\}] = \prod_i \delta[U'_i - g_i\{U\}], \quad (5)$$

where the $g_i\{U\}$ are functions described in detail below. By using this projection operator it is possible to fix the block links in a discrete group or subgroup. The constraints are enacted inside the routine which updates the lattice; if one of the generated tentative new site links threatens to violate a constraint it is simply not accepted. No such constraint is enforced for the link U_0' . Instead, this link is identified with the function $g_0\{U\}$ which in turn is measured at appropriate intervals.

If it is desirable to work with a continuous gauge group the projection operator Eq. (5) can be replaced with a product of (narrow) Gaussian distribution functions:

$$P[\{U'\}, \{U\}] = N(\beta) \prod_i \exp[-\beta(U'_i - g_i\{U\})^2], \quad (6)$$

where $N(\beta)$ is a constant chosen so that Eq. (2) holds. The mean of each $g_i\{U\}$ ($i \neq 0$) is constrained by the quadratic form in the exponential, which is included in the action [8,9]. The remaining block U_0' is generated by a Gaussian distribution of mean $g_0\{U\}$.

It remains to describe the $\{g_i\}$. First divide the lattice into hypercubical blocks with sides three lattice sites long. An example of this division is shown in Fig. 1a. The points of the block lattice are shown as heavy dots, and the block links connect these block points. For example, a block link connects site A with site D, and is denoted by U_{AD}' . In a like fashion the site link going from B to C is denoted by U_{BC} .

Each of the $\{g_i\}$ is set equal to a block link. As an example, take $g_{AD}\{U\}$, which is to equal U_{AD}' . Both $g_{AD}\{U\}$ and U_{AD}' must therefore transform in the same fashion under gauge rotations. This

common covariance suggests that $g_{AD}\{U\}$ is in some sense an average of several paths leading from point A to point D. By "path" a product of links leading from A to D is meant. Of course, each site link should appear in exactly one g_i , so not all possible paths between A and D can appear in $g_{AD}\{U\}$.

In two dimensions for a $Z(2)$ theory it is relatively easy to display such a set of $\{g_i\}$. An example is illustrated in Fig. 1a:

$$g_{AD} = \text{sign} \left\{ W_1 \times U_{AB} U_{BC} U_{CD} + W_2 \times [U_{AB} U_{BE} U_{EF} U_{FC} U_{CD} + U_{AB} U_{Bq} U_{qH} U_{HC} U_{CD}] \right\} \quad (7a)$$

$$= \text{sign} \left\{ U_{AB} U_{BC} U_{CD} [W_1 + W_2 (U_{BE} U_{EF} U_{FC} U_{CB} + U_{CB} U_{Bq} U_{qH} U_{HC})] \right\} \quad (7b)$$

(7c)

$$\equiv \text{sign} \left\{ U_{AB} U_{BC} U_{CD} [W_1 + W_2 P_{BC}] \right\} .$$

Recall that for a $Z(2)$ gauge theory, each link equals plus or minus one.

Of course, the sign of the sum of the paths can be replaced by a more versatile function in a more sophisticated theory. In Eqs. (7), W_1 and W_2 are weights, which can be adjusted in some fashion so as to optimize the recursion relation. (Some convention must be chosen if the argument of the sign function can vanish). The new variable P_{BC} is simply a gauge-invariant sum of all plaquettes which contain the link U_{BC} . Separating out P_{BC} in the fashion of Eq. (7c) is possible only for an Abelian group; however, this separation is done for convenience and is not essential in the sequel.

It is easy to see that the prescription Eqs. (7) for the $\{g_i\}$ ensures that each site link occurs in one and only one g_i . Nor is it difficult to generalize the procedure to three or higher dimensions. In three dimensions, the set of paths connecting two nearest-neighbor block points contain paths like that of Fig. 1b, which can be separated into a straight line times a "chair". If the sum of all chairs containing the link U_{BC} is denoted by C , then the projection operator

$$g_{AD} = \text{sign} \left\{ U_{AB} U_{BC} U_{CD} [W_1 + W_2 P_{BC} + W_3 C_{BC}] \right\} \quad (8)$$

is a reasonable choice in three dimensions. Note that with the projection operator Eq. (8) each site link occurs in exactly one of the g_i .

The entire procedure may seem more systematic after a closer re-examination of Fig. 1a. In d dimensions there are $2d$ "body" links (such as U_{AB}) for each block site (such as A). The remaining links are "face" links and can be identified with a particular face of a hypercube. For example, the links U_{BE} and U_{BC} in Fig. 1a clearly both belong to the face between the hypercubes about points A and D . Each of the paths whose weighted average comprises one of the $\{g_i\}$ begins with a body link, follows this link to a face, and then takes one of a complete set of ways through the face in order to reach the nearest body link in the next hypercube. Since every link in the lattice is either a body link or a face link (while each face link belongs to only one face), this procedure automatically includes every site link in the lattice in exactly one of the $\{g_i\}$. In higher dimensions more complicated objects than plaquettes and chairs must be included.

4 - THE CALCULATION

As an example of the efficacy of the above procedure a sample calculation with $Z(2)$ lattice gauge theory was performed. By duality arguments [11] this model possesses the same critical exponents as the three dimensional Ising spin system, whose critical behavior is known to high accuracy [11,12]. The calculation described below was performed on a periodic 15^3 lattice using the projection operator presented above [Eqs.(5) and (8)] with (W_1, W_2, W_3) equal to $(1, 1, 0)$ respectively. The speed of the routine was significantly enhanced by retaining plaquettes as well as links in memory. Additionally, it was sometimes necessary to update pairs of links in parallel in order that the blocking constraints of the system did not prevent certain links from being updated.

The discussion surrounding Eq.(4) explains why it is necessary to choose a background configuration for the block links $\{U'\}$ surrounding the lone fluctuating variable U_0' . The point is to cancel as many interactions as possible in the block system. The configuration used here is now described. It has the feature of cancelling a large fraction of the interactions involving the link U_0' . In particular, all of the two-by-one rectangles (hereafter referred to as "tables") and all of the chair configurations are cancelled.

The notation is chosen in such a fashion that the lone fluctuating link is in the vertical direction, while the other two orthogonal directions are called horizontal. All of the block links are set to unity except for the ones displayed in Fig. 1c and the vertical link immediately above the fluctuating link. This upper block link is set to minus one in order to cancel (in pairs above and below) the tables involving the fluctuating link. Four horizontal links (denoted by horizontal lines in Fig. 1c) are also set to minus one in order to cancel two thirds of the chairs; the other third are cancelled by the pattern of vertical links in Fig. 1c. Note that only three of the four vertical nearest neighbor links are set to plus one in order that enormous statistics are not needed in order to calculate the renormalized plaquette coupling.

The number of times N_+ [N_-] that the fluctuating link takes on the value $+1$ [-1] is measured, and the value of the block renormalized single-plaquette coupling is then given by

$$\beta' = \frac{1}{4} \ln \left(\frac{N_+}{N_-} \right) \quad (9)$$

The derivatives of β' with respect to the coupling constants in the original action can in principle be determined by analytic differentiation of Eq. (9):

$$\frac{\partial \beta'}{\partial \beta_i} = \frac{1}{4} \left[\langle S_i \rangle_+ - \langle S_i \rangle_- \right] \quad (10)$$

In Eq.(10) the notation of Eqs.(3) has been followed. The expectation value $\langle \dots \rangle_+$ [$\langle \dots \rangle_-$] signifies the average taken only over configurations in which U_0 equals $+1$ [-1].

Given the renormalized couplings and their derivatives, it is a simple matter to locate the fixed points of the theory using the first order Taylor series expansion:

$$\beta'_i - \beta_i^* \approx \frac{\partial \beta'_i}{\partial \beta_j} (\beta_j - \beta_j^*) \quad (11)$$

The known values of β_i , β_i' and their derivatives are substituted and Eq. (11) is solved for β^* . The procedure is repeated at the new (approximate) value for β^* until convergence is achieved, typically within one or two iterations from a reasonable starting point. The thermal eigenvalue is then determined [2] from the largest eigenvalue λ of the matrix of derivatives at the fixed point β^* :

$$y_T = \ln \lambda / \ln 3 \quad (12)$$

The factor 3 arises because it is the linear scale factor of the block transformation.

At this point it is appropriate to discuss a practical matter. Because of statistical fluctuations the derivatives of the renormalized couplings found via Eq. (10) are not known as accurately as the renormalized couplings themselves determined with Eq. (9). This difference suggests the following alternate procedure for extracting these derivatives once the fixed point is known moderately well. First the value of β' is tabulated for several values of β . Then a least squares fit of the straight line Eq. (11) is made to this data. The fixed point and critical exponent then follow in the standard fashion from the parameters of the fit.

This alternate procedure is more accurate than the direct application of Eq. (10) for several reasons. First, as stated above, the data for the renormalized coupling is in general more reliable than the data for its derivative, which involves the calculation of a correlation. Secondly, by making a least squares fit to a concatenation of several separate runs, the information of these various statistically separate events is combined. Additionally since not all the runs take place at the fixed point of the recursion relation (where fluctuations are presumably largest) the accuracy of the result is further enhanced.

A plot of β' versus β (i.e., block plaquette coupling versus site plaquette coupling) is shown in Fig. 2a. The plot was constructed from N (where N means 5,000 or 20,000 or 25,000 for different points as shown in the legend) iterations on a 15^3 lattice after equilibration for N/10 iterations during which no measurements were made. The place where the recursion $\beta'(\beta)$ crosses the line $\beta'=\beta$ indicates the fixed point β^* . The least squares fit mentioned above is displayed in Fig. 2b. The fixed point and thermal exponent (with only the one derivative of

the plaquette coupling taken into account) are equal to $\beta^* \approx 0.96$ and $\gamma \approx 1.1$. If the two points with 20000 iterations are omitted, the result is $\beta^* \approx 0.96$ and $\gamma \approx 1.9$. The expected [12] critical exponent with all couplings included is approximately 1.6. The values of the block renormalized chair and table couplings were also measured; the chair coupling was the largest but both were much less than 0.1.

The above analysis was also applied to the same system blocked by means of the procedure described in [7]. The plaquette coupling fixed point for this blocking procedure is however much larger (perhaps about 1.5). At such a large value of β the required statistical accuracy to determine critical exponents proved computationally beyond our means. Possibly this surprisingly large value of β^* is related to the fact that each site link is included in quite a large number of block links in the scheme of Ref. [7].

The method applied in Ref. [7] (hereafter called the "correlation method") to extract the derivatives of coupling constants was also used in conjunction with our blocking scheme. Rather unexpected results were obtained. In order to present these results it is first necessary to review the correlation method.

The first step in evaluating derivatives by the correlation method is to measure the correlations,

$$C_{ab} = \langle (S'_a - \langle S'_a \rangle)(S_b - \langle S_b \rangle) \rangle \quad (13a)$$

and

$$C'_{ab} = \langle (S'_a - \langle S'_a \rangle)(S'_b - \langle S'_b \rangle) \rangle \quad (13b)$$

where the notation of Eqs. (3) is used. In practice, only a small number of the elements of the matrices C and C' are measured, i.e., a and b range from one to some number N_{corr} .

The derivatives of the block renormalized couplings are determined by the solution of the matrix equation

$$C = DC' \quad (14)$$

where

$$D_{ab} = \frac{\partial k_a}{\partial k_b} \quad (15)$$

The critical eigenvalues are then determined by the diagonalization of the matrix D , using only the first $(N_{\text{eigen}})^2$ elements. Obviously N_{eigen} cannot be greater than N_{corr} .

Results for the thermal eigenvalue γ [cf. Eq. (12)] obtained with our blocking procedure and the correlation method are given in the Table. For these calculations measurements were taken over sets of 5,000 iterations after the usual "warm-up" of 500 iterations. The lattice was of size 15^3 , while β was set to 0.76113, which is the critical value for the $Z(2)$ gauge model. Note that this value is different from the actual fixed point of our blocking procedure. Chairs and tables were taken as the additional elements of the block renormalized action, i.e., made up elements two and three respectively of the matrix in Eq. (13).

As can be seen in the Table, if N_{eigen} is kept equal to N_{corr} and both are increased, the value of the critical exponent is reasonably good and improves steadily towards its exact value $\gamma \approx 1.6$. However if N_{eigen} is less than N_{corr} the results are much poorer.

The reason for this difference may be the following. The fixed point of the system with our blocking procedure is at β of about 0.96. Thus the poor results obtained for γ when N_{eigen} is not equal to N_{corr} are in fact in accord with what might be expected; after all, the system is not at its true fixed point. What is remarkable rather is how such good results ($\gamma \approx 1.3$ for $N_{\text{eigen}} = N_{\text{corr}} = 3$) can be obtained at this value of β . Evidently the correlation method actually produces derivatives of effective interactions which include additional information about the system, albeit in a totally unknown and uncontrolled fashion. Thus whereas all elements of the derivative matrix constructed by the correlation method may not be reliable its eigenvalues at the fixed point might still give good results.

On the other hand if it is desired to calculate a particular derivative with this method it seems likely that the proper way to do it is to increase N_{corr} until the derivative in question is stable against further increases. Presumably the larger N_{corr} is the more interactions are separated out and the less "effective" the extracted interactions

are. We have in fact observed that derivatives calculated with the correlation method rapidly approach those calculated with our method, Eq. (10), for both this $Z(2)$ gauge theory and for the Ising spin model [8] when N_{corr} is increased.

5 - CONCLUSIONS

A method for the direct evaluation of block renormalized couplings for a lattice gauge theory has been presented. Another result of this paper is a new definition of block links in a gauge system. With this new blocking procedure, each site link occurs in exactly one block link. Also the blocking is performed entirely on a hypercubic lattice, so that matter fields can be included in a straightforward manner.

The above general procedure was applied to a calculation of the renormalization group trajectories, fixed points, and critical exponents of the Ising gauge model in three spacetime dimensions. The results were consistent with known information, and the method is currently being applied to a combined gauge field/scalar system. An analysis of the Abelian Higgs model [13] is planned. As discussed in [9] such a calculation may have phenomenological applications.

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N_{corr}			
N_{eigen}			
	1	2	3
1	1.1	0.5	-1.4
2	-	1.2	-0.5
3	-	-	1.3

TABLE

Thermal exponents y at $\beta = 0.76113$ calculated with the correlation method for various values of N_{corr} and N_{eigen} . (Exact value is $y \sim 1.6$).

FIGURE CAPTIONS

- Fig. 1a : Diagram of the new blocking scheme in two dimensions. The boundaries of the blocks are denoted by broken lines, and the block sites by heavy dots. Site blocks appear as light dots.
- Fig. 1b : Diagram showing how a certain link path can be separated into a straight line of links times a gauge-invariant "chair" configuration.
- Fig. 1c : Diagram showing part of the pattern of block links chosen to extract the plaquette coupling β' . The sole fluctuating (vertical) link is denoted by 0; all other vertical links in its (horizontal) plane are assigned values of plus or minus one as shown. Four horizontal links in this plane (denoted by dotted lines) are also set to minus one, as is the vertical block link immediately above the fluctuating one. All others are unity.
- Fig. 2a : Plot of the block renormalized plaquette coupling β' versus β . The legend is as follows: triangles: 25,000 iterations; circles: 20,000 iterations; crosses: 5,000 iterations. A smooth curve is drawn through the points to aid the eye. The line $\beta' = \beta$ is also plotted to illustrate the fixed point.
- Fig. 2b : Detail of the recursion relation. The legend is as follows: triangles: 25,000 iterations; circles : 20,000 iterations. The straight line fit to the points (with $\beta^*=0.96$, $y = 1.1$) is shown, as is the straight line $\beta' = \beta$.

Fig. 1a

Fig. 1b

Fig. 1c

Fig. 2a