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STOCHASTIC IDENTITIES IN SUPERSYMMETRIC THEORIES

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A B S T R A C T

The correspondence between stochastic quantization and supersymmetry is reobtained for the quantum mechanics case by use of "stochastic identities". The method is easily generalized to $N = 1$ super Yang-Mills theory in the light cone gauge.

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1. Supersymmetric theories exhibit many interesting features which make them unique among renormalizable field theories.

A very important property of such theories, first suggested by Nicolai¹⁾, is the fact that, after integration of the fermionic fields, the resulting functional integral becomes "Gaussian" in a new set of bosonic variables (Nicolai map).

Conversely, Parisi and Sourlas²⁾ have shown that a supersymmetry naturally emerges from considering stochastic equations in real space (i.e., without introducing an extra time).

The Nicolai-Parisi-Sourlas approach works perfectly in supersymmetric quantum mechanics and in two-dimensional field theories, where the new bosonic variables are actually local functions of the old ones. However, for field theories in more dimensions, and in particular for gauge theories, the standard procedure appears to encounter difficulties.

The aim of this note is to introduce a new, simple procedure which can be easily employed in four-dimensional gauge theories. It is based on the existence of certain "stochastic identities".

We shall first illustrate our method in the known case of supersymmetric quantum mechanics where we show the equivalence with the standard, functional approach. We then apply the procedure to N=1 gauge supersymmetry in the light cone gauge.

2. The starting point of the discussion is given by the Euclidean supersymmetric action for a system of commuting $x_\alpha(t)$ and anticommuting $\lambda_\alpha(t)$ fields, $\alpha = 1, \dots, N$ ³⁾:

$$I = \int ds \left\{ \frac{1}{2} \dot{x}_\alpha^2 + \lambda_\alpha \dot{\omega}_\alpha^+ \right\}, \quad ds \equiv i dt \quad (1)$$

where

$$T_\alpha = \frac{dX_\alpha}{ds} + \frac{\partial \mathcal{W}}{\partial X_\alpha} \quad (2)$$

$$\omega_\alpha^+ = (\delta_{\alpha\beta} d/ds + \partial_\alpha \partial_\beta \mathcal{W}) \lambda_\beta^+ = \delta I / \delta \lambda_\alpha \quad (3)$$

The form (1) of the action is equivalent to the usual one up to a partial integration. The action given by Eqs. (1), (2) and (3) is invariant under the supersymmetric transformations

$$\delta X_\alpha = \lambda_\alpha^+ \epsilon ; \quad \delta \lambda_\alpha = T_\alpha \epsilon ; \quad \delta \lambda_\alpha^+ = 0 \quad (4)$$

from which, using Eqs. (2) and (3), one has

$$\delta T_\alpha = \omega_\alpha^+ \epsilon ; \quad \delta \omega_\alpha^+ = 0 \quad (5)$$

The Nicolai procedure is well known¹⁾. If one integrates over the fermionic variables, one obtains the effective action

$$Z_{\text{eff}} = \int d\Omega(\tau) \exp \left\{ -\frac{1}{2} \int ds T_\alpha^2 \right\} \quad (6)$$

with the functional measure

$$d\Omega(\tau) = \prod_{\alpha, s} dT_\alpha(s) = \prod_{\alpha, s} dX_\alpha(s) \det \left(\frac{\delta T_\beta}{\delta X_\alpha} \right) \quad (7)$$

Our point of view is to consider as fundamental the stochastic identities that follow from this treatment. These are

$$\langle T_\alpha(s) \rangle = 0 \quad (8.a)$$

$$\langle T_{\alpha_1}(s_1) T_{\alpha_2}(s_2) \rangle = \delta_{\alpha_1, \alpha_2} \delta(s_1 - s_2) \quad (8.b)$$

and, in general,

$$\begin{aligned} & \langle T_{\alpha_1}(s_1) T_{\alpha_2}(s_2) \dots T_{\alpha_n}(s_n) \rangle = \\ & = \sum_{r,s} \delta_{\alpha_r \alpha_s} \delta(s_r - s_s) \langle \prod_{i \neq r,s} T_{\alpha_i}(s_i) \rangle \end{aligned} \quad (9)$$

We offer a proof of Eqs. (8) and (9) which starts from the supersymmetric action (1) without resorting to the Nicolai transformation. Although obvious in the present one-dimensional case, this proof is instrumental to the derivation of the stochastic identities in the case of $N=1$ gauge theory.

We start from the fact that, if supersymmetry is not spontaneously broken, then we have

$$\langle \delta_\epsilon F \rangle = 0 \quad (10)$$

where F is any functional of x_α and λ_α and δ_ϵ is the supersymmetric transformation defined in Eqs. (4). Taking $F = \lambda_\alpha$, since $\delta_\epsilon \lambda_\alpha = T_\alpha \epsilon$, Eq. (8.a) follows immediately. If we take $F = \lambda_\alpha(s_1) T_\beta(s_2)$, then

$$\delta_\epsilon \left\{ \lambda_\alpha(s_1) T_\beta(s_2) \right\} = \left\{ T_\alpha(s_1) T_\beta(s_2) + \lambda_\alpha(s_1) \omega_\beta^+(s_2) \right\} \epsilon \quad (11)$$

Using the second part of Eq. (3), we get

$$\langle \lambda_\alpha(s_1) \omega_\beta^+(s_2) \rangle = - \delta_{\alpha\beta} \delta(s_1 - s_2) \quad (12)$$

which follows from

$$\int d\Omega(\lambda, \lambda^+, x) \frac{\delta}{\delta \lambda_\beta(s_2)} \lambda_\alpha(s_1) \exp(-I) = 0 \quad (13)$$

and one is immediately led to Eq. (8.b). The more general identities (9) can be similarly derived by considering $F = \lambda_{\alpha_1}(s_1) T_{\alpha_2}(s_2) \dots T_{\alpha_n}(s_n)$.

3. We now consider N=1 super Yang-Mills theory. The action, written in Minkowski space and in the Wess-Zumino gauge, reads

$$I = - \int d^4x \left\{ \frac{1}{4} F_{\mu\nu}^2 + i \lambda^\alpha \not{D}_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \right\} \quad (14.a)$$

$$I = \int d^4x \left\{ \frac{1}{2} \vec{F}_3^2 + \frac{1}{2} \vec{F}_+ \cdot \vec{F}_- - \lambda^\alpha \omega_\alpha^+ \right\} \quad (14.b)$$

Equation (14.a) is the standard action, with gauge indices suppressed. As in the quantum mechanical case, we shall rather employ the form (14.b) [equivalent to (14.a) up to topological terms^{*)}] where we have introduced [σ_i are the Pauli matrices and $\sigma^{\mu\nu}$ are defined as in Ref. 5)]

$$\begin{aligned} \vec{F}_{\mu\nu} (\sigma^{\mu\nu})_\alpha^\beta &= (\sigma_i)_\alpha^\beta F_i \quad ; \quad F_i = - (F_{0i} + \frac{i}{2} \epsilon_{ijk} F_{jk}) = -E_i - iH_i \\ \vec{F}_\pm &= \vec{F}_1 \pm i \vec{F}_2 \quad ; \quad \omega_\alpha^+ = i \not{D}_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \end{aligned} \quad (15)$$

Defining for later convenience the components of the vector potential A_μ as

$$A_L = A_0 - A_3 \quad ; \quad A_R = A_0 + A_3 \quad ; \quad A_\pm = A_1 \pm i A_2 \quad (16)$$

and using the matrix representation, we find immediately the relation between the infinitesimal variations of the F's and of the A's to be

*) For possible non-perturbative difficulties, see Ref. 6).

$$\begin{aligned}
 \delta F_+ &= -D_- \delta A_+ + D_+ \delta A_- \\
 2\delta F_3 &= -D_- \delta A_+ + D_R \delta A_L - D_L \delta A_R + D_+ \delta A_- \\
 \delta F_- &= D_- \delta A_R - D_R \delta A_-
 \end{aligned} \tag{17}$$

The action I is invariant under the supersymmetric transformations

$$\delta A_\mu = i \epsilon^\alpha (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}; \quad \delta \lambda_\alpha = F_i (\sigma_i)_\alpha{}^\beta \epsilon_\beta; \quad \delta \bar{\lambda} = 0 \tag{18}$$

It is, however, well known that gauge fixing and accompanying Faddeev-Popov terms break supersymmetry. This is of course a complication with respect to the quantum mechanical case. Also, as suggested by Nicolai, we have three "Gaussian" variables as compared to four A_μ 's⁶⁾. This latter fact suggests taking an axial gauge $n^\mu A_\mu = 0$, which eliminates directly one component of A_μ and does not require an interacting Faddeev-Popov ghost⁷⁾.

In such a class of gauges we can still save part of the supersymmetry (18) if the equation

$$\delta(n \cdot A) = i \epsilon n^\mu \sigma_\mu \bar{\lambda} \tag{19}$$

has solutions other than $\epsilon^\alpha = 0$. This is the case if and only if $\det(n \cdot \sigma) = n^2 = 0$, i.e., in light cone gauges^{*)}. In particular, if we take $n \cdot A = A_0 - A_3 = 0$, Eq. (19) will hold provided $\epsilon^\alpha = \epsilon_s^\alpha = \delta_{\alpha 2} \epsilon$. We shall call δ_s the restricted transformation associated with such an ϵ^α . It is straightforward

*) This might explain the problems met by Nicolai⁶⁾ in the $A_0 = 0$ gauge. Similar difficulties would also occur in covariant gauges.

to write down the restricted transformations for the components of A_μ and λ_α using the definition of the restricted variations and Eqs. (18): they are

$$\begin{aligned} \delta_s A_L = \delta_s A_+ = 0 ; \quad \delta_s A_R = 2i\epsilon \bar{\lambda}_1 ; \quad \delta_s A_- = 2i\epsilon \bar{\lambda}_2 \\ \delta_s \lambda_1 = -F_3 \epsilon ; \quad \delta_s \lambda_2 = -F_+ \epsilon ; \quad \delta_s \bar{\lambda} = \delta_s \omega^\dagger = 0 \end{aligned} \quad (20)$$

Equations (17) take a particularly simple form, to be used later, in the $A_L = 0$ gauge. For the moment we just use it to derive that

$$\delta_s F_+ = 0 ; \quad 2\delta_s F_3 = 2\omega_2^\dagger \epsilon ; \quad \delta_s F_- = -2\omega_1^\dagger \epsilon \quad (21)$$

Invariance of the action (14.b) under (20), (21) can be easily checked.

Within the (standard) assumption that supersymmetry suffers no quantum level anomalies, we can now derive some particularly simple Ward identities that follow from $\delta_s I = 0$. These take the form

$$\begin{aligned} 0 &= \langle \delta_s [\lambda_\alpha(x) F_i(y)] \rangle = \\ &= - \langle \epsilon F_j(x) (\sigma_j)_\alpha{}^i F_i(y) \rangle + \langle \lambda_\alpha(x) \delta_s F_i(y) \rangle \end{aligned} \quad (22)$$

Using (21) we see that $\delta_s F_i$ is either 0 or proportional to some $\omega_\alpha^\dagger = \delta I / \delta \lambda^\alpha$ so that the last term in Eq. (22) is either 0 or a δ -function depending on the values of α and i . Writing out all the constraints following from (22) and solving, we find:

$$\begin{aligned} \langle F_3(x) F_3(y) \rangle &= \frac{1}{2} \langle F_+(x) F_-(y) \rangle = \delta^4(x-y) \\ \langle F_3(x) F_+(y) \rangle &= \langle F_3(x) F_-(y) \rangle = \langle F_+(x) F_+(y) \rangle = 0 \end{aligned} \quad (23)$$

which are our "stochastic identities", since all the correlations in (23) coincide with those of the free theory. In order to get a handle on the single missing correlation function $\langle F_-(x) F_-(y) \rangle$ we use the identity

$$\frac{\delta I}{\delta A_+(x)} \epsilon + \int_S (D_- \lambda_1(x)) = \frac{1}{2} \partial_L F_- \epsilon \quad (24)$$

which can be easily checked from Eqs. (14) and (20). We then write:

$$\begin{aligned} \langle F_+(x) \frac{\epsilon}{2} \partial_L F_-(y) \rangle &= \langle F_+(x) \left[\epsilon \frac{\delta I}{\delta A_+(y)} + \int_S (D_- \lambda_1(y)) \right] \rangle = \\ &= \langle -\epsilon \delta F_+(x) / \delta A_+(y) \rangle - \langle \int_S F_+(x) D_- \lambda_1(y) \rangle \end{aligned} \quad (25)$$

In the case of F_- it is easy to verify that both terms vanish giving

$$\partial_L \langle F_-(x) F_-(y) \rangle = 0 \quad (26)$$

while for F_+ we pick up $\partial_L \delta^4(x-y)$ from $\delta F_+ / \delta A_+$ and recover the derivative of one of Eqs. (23).

Equation (26) is consistent with (and suggests) the conclusion that the missing relation also holds:

$$\langle F_-(x) F_-(y) \rangle = 0 \quad (27)$$

i.e., that the integration "constant" of (26) is zero, as it is for the $\langle F_+ F_+ \rangle$ case. This conclusion may be unwarranted and we could be left with a non-trivial function of $x_L, x_\pm$ for $\langle F_- F_- \rangle$. On the other hand, we should remember that in the $A_L = 0$ gauge we have still the possibility of performing x_R independent gauge transformations and it is quite possible that $\langle F_- F_- \rangle$ can be set to 0 after completely fixing the gauge^{*)}.

*) This is related to fixing an ϵ prescription around $p_L = 0$. As discussed by Mandelstam⁸⁾, one particular prescription ("modified light cone") ensures both good ultra-violet behaviour and a regular $p_L \rightarrow 0$ limit. Thus, Eq. (27) should follow at least in that gauge.

If Eq. (27) and its n -point generalizations [cf., Eq. (22)] hold true, as we believe to be the case, that would mean that the bosonic action is Gaussian in the variables F_i .

There is another way to arrive at the same conclusion: it simply consists of looking at the Nicolai transformation (17) in the light cone gauge ($\delta A_L = 0$) and computing its Jacobian:

$$J = \det \left(\frac{\delta F}{\delta A} \right) = \det(-\partial_L) \det(D_\alpha \alpha) \quad (28)$$

It is easy to see that this Jacobian coincides with the product of the ghost determinant (which is a number) and the fermion determinant. The only subtle point about this approach is that the functional integration over the F_i is over a complex domain (but not the whole complex plane). In order to have real F 's, we would have to work in Euclidean space but, in such a case, we cannot choose the light cone gauge. Again, we are not sure whether these are anything more than purely formal difficulties. The "stochastic identities" method avoids this problem by working all the time in Minkowski space.

4. Our conclusion is that the correspondence between stochastic quantization and supersymmetry can indeed be extended from elementary models to a gauge theory in the physical 3+1 space time. The main result is that the "stochastic identities" can be directly derived from supersymmetry.

The use of the new procedure is crucial in gauge theories where it allows to cope with the new features introduced by gauge fixing. Indeed, the fact that only light cone gauges are invariant under a subgroup of supersymmetric transformations tells us that only in those gauges the stochastic structure of supersymmetric theories becomes clear and apparent.

Work is in progress to apply our treatment to extended supersymmetric gauge theories. The situation looks particularly promising in the $N = 2$ case, where some stochastic identities can indeed be derived. The question of course arises whether also in this case the full Nicolai transformation

can be applied or whether one should be satisfied with a subset of stochastic identities which would not imply the full Gaussian structure of the theory.

The very fact that even in the $N = 1$ case all the stochastic identities are not on the same ground might hint to the possibility that our procedure is not always equivalent to that of Nicolai.

The physical interest of the stochastic identities will be discussed in the future; of course their presence is related to the special cancellation of infinities appearing in supersymmetric gauge theory. This is also implied by the fact that our stochastic identities are valid in the light cone gauge where the cancellations of infinities are particularly transparent⁸⁾.

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R E F E R E N C E S

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