Ref.TH.3898-CERN
LAPP-TH-110

GRAVITATIONAL ANOMALIES OF THE ADLER-BARDEEN TYPE

F. Langouche^{*)}, T. Schücker and R. Stora⁺⁾

CERN - Geneva

Gravitational anomalies of the Adler-Bardeen type are defined and studied by means of the usual Slavnov algebra, adapted to the case of an even-dimensional "space-time" manifold.

*) On leave of absence from Kath. Univ. Leuven, Celestijnenlaan 200D, B-3030 Heverlee, Belgium. Onderzoeker IIKW Belgium.

+) Permanent address : LAPP, IN2P3, B.P. 909, F-74019 Annecy-le-Vieux Cedex, France.

The occurrence of gravitational anomalies has been pointed out by L. Alvarez Gaumé and E. Witten¹⁾ in a recent paper concerned with the interaction of chiral fields with an external gravitational field in $2n-2 = 4k-2$ dimensions, represented by a vielbein e and a spin connection ω . Inspection of the explicit results they obtained in a number of concrete cases suggests (footnote 1) that the gravitational anomalies they found are disguised forms of the usual Adler Bardeen chiral anomaly for the $SO(4k-2)$ orthogonal gauge group associated with the Riemannian metric of space time, (in a euclidean version), whose consideration is forced by the inclusion of half integer spins and is convenient anyway. Because of the many equivalent ways in which these anomalies can be exhibited it may be relevant to review some of their properties.

Let $\Gamma(e, \omega)$ be the vacuum functional for the propagation of some chiral field in the external gravitational field (e, ω) . It fulfills possibly anomalous Ward identities related to the violation of invariance under local infinitesimal $SO(4k-2)$ orthogonal gauge transformations parametrized by Ω , and of the invariance under infinitesimal diffeomorphisms represented by vector fields ξ defined on the base manifold M_{2n-2} . Care has to be exercised, if the base manifold has non trivial topology because the action of ξ on e, ω is only defined up to a gauge transformation. We choose a fixed background field $(\tilde{e}, \tilde{\omega})$ and write down the Ward identities

$$W(\Omega) \Gamma(e, \omega) = \int \alpha(\Omega) \quad (1a)$$

$$W(\xi) \Gamma(e, \omega) = \int \alpha(\xi) \quad (1b)$$

where $W(\Omega)$, $W(\xi)$ are induced by

$$W(\Omega) e = -\Omega e \quad (2a)$$

$$W(\Omega) \omega = d\Omega + [\omega, \Omega] =: D\Omega \quad (2b)$$

$$W(\xi) e = -L_\xi e - (i_\xi \tilde{\omega})e =: -\tilde{L}_\xi e =: -(i_\xi \tilde{\omega} + \tilde{\omega} i_\xi) e \quad (2c)$$

$$W(\zeta)\omega = -L_{\zeta}\omega + d i_{\zeta} \tilde{\omega} + [\omega, i_{\zeta} \tilde{\omega}] = -i_{\zeta} R(\omega) - D i_{\zeta} (\omega - \tilde{\omega}) \quad (2d)$$

$$W(\mathcal{N})\tilde{e} = W(\zeta)\tilde{e} = W(\mathcal{N})\tilde{\omega} = W(\zeta)\tilde{\omega} = 0 \quad (2e)$$

where $L_{\xi} = i_{\xi}d + di_{\xi}$ is the Lie derivative and i_{ξ} indicates the evaluation of differential forms on vector field ξ , $\tilde{D} = d + [\tilde{\omega}, \cdot]$, $R \equiv R(\omega) = d\omega + \frac{1}{2} [\omega, \omega]$.

The following commutation rules hold:

$$[W(\mathcal{N}), W(\mathcal{N}')] = W([\mathcal{N}, \mathcal{N}']) \quad (3a)$$

$$[W(\zeta), W(\mathcal{N})] = W(\tilde{\mathcal{L}}_{\zeta} \mathcal{N}) = W(i_{\zeta} \tilde{D} \mathcal{N}) \quad (3b)$$

$$[W(\zeta), W(\zeta')] = W([\zeta, \zeta']) - W(i_{\zeta} i_{\zeta'} \tilde{R}) \quad (3c)$$

where $\tilde{R} = R(\tilde{\omega}) = d\tilde{\omega} + \frac{1}{2} [\tilde{\omega}, \tilde{\omega}]$, and $[\xi, \xi']$ is the Lie bracket of vector fields ξ, ξ' . We have thus constructed a Lie algebra $\mathfrak{E}(\tilde{\omega})$ with an ideal $\{\Omega\}$ such that the quotient $\mathfrak{E}(\tilde{\omega})/\{\Omega\}$ is the Lie algebra $\{\xi\}$ of vector fields on M_{2n-2} .

The Wess Zumino⁷⁾ consistency conditions follow easily and are best expressed by turning (Ω, ξ) into Fadeev Popov ghosts (footnote 2): define s by

$$se = -\mathcal{N}e - \tilde{\mathcal{L}}_{\zeta} e \quad (4a)$$

$$s\omega = -D\mathcal{N} - i_{\zeta} R + D i_{\zeta} (\omega - \tilde{\omega}) \quad (4b)$$

$$s\mathcal{N} = -\frac{1}{2} [\mathcal{N}, \mathcal{N}] - \tilde{\mathcal{L}}_{\zeta} \mathcal{N} - \frac{1}{2} i_{\zeta} i_{\zeta} \tilde{R} \quad (4c)$$

$$s\zeta = -\frac{1}{2} [\zeta, \zeta] \quad (4d)$$

s is the coboundary operator for the cohomology of $\mathcal{E}(\tilde{\omega})$ with values in the space of functionals of e and ω ⁸⁾:

$$s^2 = 0 \quad (5)$$

The Wess Zumino consistency conditions read

$$s\alpha \simeq 0 \quad (6)$$

where α is the pair $[\alpha(\Omega), \alpha(\xi)]$ linear in $(\Omega, \xi) \in \mathcal{E}(\tilde{\omega})$, and $\simeq 0$ means up to d (local form), by the algebraic Poincaré Lemma^{3),4),5)}.

It is convenient to change variables from (Ω, ξ) to $(\tilde{\Omega}, \xi)$ where

$$\tilde{\mathcal{L}} = \mathcal{L} - i_{\xi}(\omega - \tilde{\omega}) \quad (7)$$

(Eqs 4) then read:

$$s e = -\tilde{\mathcal{L}} e - (i_{\xi} D - D i_{\xi}) e \quad (8a)$$

$$s \omega = -D \tilde{\mathcal{L}} - i_{\xi} R \quad (8b)$$

$$s \tilde{\mathcal{L}} = -\frac{1}{2} [\tilde{\mathcal{L}}, \tilde{\mathcal{L}}] + \frac{1}{2} i_{\xi} i_{\xi} R \quad (8c)$$

$$s \xi = -\frac{1}{2} [\xi, \xi] \quad (8d)$$

s now expresses parallel transport⁹⁾. Adler Bardeen anomalies are then defined as follows: let $\mathcal{I}_n$ be a symmetric invariant polynomial of degree n on Lie

$SO(4k-2)$ (non trivial invariants only exist when $n = 2k$, because of the anti-symmetry of the elements of Lie $SO(4k-2)$). One has the identity^{3),4),5)} valid in an arbitrary number of dimensions

$$J_n(R^n) - J_n(\dot{R}^n) = (d+n) Q_{2n-1} \quad (9)$$

where

$$\begin{aligned} R &:= (d+n)(\omega + \tilde{\mathcal{L}}) + \frac{1}{2} [\omega + \tilde{\mathcal{L}}, \omega + \tilde{\mathcal{L}}] = \\ &= R - i_\xi R + \frac{1}{2} i_\xi i_\xi R \end{aligned} \quad (10)$$

$$Q_{2n-1} := n \int_0^1 dt J_n(\omega - \dot{\omega} + \tilde{\mathcal{L}}, R_t^{n-1}) \quad (11)$$

$$\begin{aligned} R_t &:= (d+n) [\dot{\omega} + t(\omega - \dot{\omega} + \tilde{\mathcal{L}})] + \\ &+ \frac{1}{2} [\dot{\omega} + t(\omega - \dot{\omega} + \tilde{\mathcal{L}}), \dot{\omega} + t(\omega - \dot{\omega} + \tilde{\mathcal{L}})] = \\ &= R_t + (t^2 - t) [\tilde{\mathcal{L}}, \omega - \dot{\omega}] + \frac{t^2 - t}{2} [\tilde{\mathcal{L}}, \tilde{\mathcal{L}}] - t i_\xi R + \frac{t}{2} i_\xi i_\xi R. \end{aligned} \quad (12)$$

$$\begin{aligned} R_t &:= d[\dot{\omega} + t(\omega - \dot{\omega})] + \\ &+ \frac{1}{2} [\dot{\omega} + t(\omega - \dot{\omega}), \dot{\omega} + t(\omega - \dot{\omega})] \end{aligned} \quad (13)$$

Up to second order in ξ (and $\tilde{\Omega}$), the left hand side of Eq.(9) can be expanded as

$$J_n(R^n) - J_n(\dot{R}^n) = i_\xi J_n(R^n) + \frac{1}{2} i_\xi i_\xi J_n(R^n) + \dots \quad (14)$$

whereas Q_{2n-1} can be expanded up to first order in ξ and $\tilde{\Omega}$ as

$$Q_{2n-1} = Q_{2n-1}^0 + Q_{2n-2}^1 + \dots \quad (15)$$

Since, in $2n-2$ dimensions

$$i_{\xi} i_{\xi} J_n(R^n) \equiv 0$$

we see that

$$\hookrightarrow Q_{2n-2}^1 \simeq 0 \quad (16)$$

Hence Q_{2n-2}^1 solves the Wess Zumino consistency condition. We shall define Q_{2n-2}^1 as a gravitational anomaly of the Adler Bardeen type.

Explicitly, we have

$$\begin{aligned} Q_{2n-2}^1 &= n \int_0^1 dt J_n(\tilde{\Omega}, R_x^{n-1}) + \\ &+ n(n-1) \int_0^1 dt J_n(\omega - \tilde{\omega}, (x^2 - t^2)/[\tilde{\Omega}, \omega - \tilde{\omega}], R_x^{n-2}) + \\ &+ n(n-1) \int_0^1 dt J_n(\omega - \tilde{\omega}, -t i_{\xi} R, R_x^{n-2}) = \\ &=: \tilde{\sigma}(\tilde{\Omega}) + \tilde{\sigma}(\xi) \end{aligned} \quad (17)$$

Thus

$$\sigma(\Omega) = \tilde{\sigma}(\Omega) \quad (18)$$

$$\begin{aligned}
\alpha(\xi) &= \alpha(\dot{\lambda}_\xi(\omega - \dot{\omega})) + \tilde{\alpha}(\xi) = \\
&= -\dot{\lambda}_\xi Q_{2n-1}^0 + n(n-1) \int_0^1 dt \, J_n(\omega - \dot{\omega}, (1-t)\dot{\lambda}_\xi \dot{R}, R_\star^{n-2}) \quad (19) \\
&= n(n-1) \int_0^1 dt \, J_n(\omega - \dot{\omega}, (1-t)\dot{\lambda}_\xi \dot{R}, R_\star^{n-2})
\end{aligned}$$

because $i_\xi Q_{2n-1}^0$ vanishes in $2n-2$ dimensions.

Most concrete calculations have been performed for parallelizable base space M_{2n-2} , in which case one may choose $\dot{\omega} = 0$. One then finds, as one should

$$\alpha(\xi) = 0 \quad (20)$$

It is worth mentioning several of the disguises that the Adler Bardeen anomalies may assume:

a) first of all they are naturally ambiguous up to trivial anomalies of the form

$$\alpha_{trivial} = \triangleright Q_{2n-2}^0 \quad (21)$$

where Q_{2n-2}^0 is a local functional of $e, \omega, \dot{e}, \dot{\omega}$.

b) in a direct computation where the orthogonal gauge is fixed, e.g. the matrix of the coefficients of e ($e^a = e_\mu^a dx^\mu$) is symmetric, the Ward operator for infinitesimal diffeomorphisms is

$$W(\xi) + W(\Omega(\xi))$$

where $\Omega(\xi)$ is the infinitesimal Wigner-gauge transformation necessary to return to the given gauge after having performed the infinitesimal diffeomorphism ξ . The gravitational anomaly thus found is

$$\alpha(\xi) + \alpha(\Omega(\xi))$$

(in the case $\dot{\omega} = 0$, it is just $\mathcal{Q}_{(\Omega(\xi))}$ where $\mathcal{Q}$ is now the usual Adler Bardeen anomaly).

c) if ω is replaced by any other connection, a local function of ω and e , the new anomaly differs from the previous one by a trivial anomaly⁴⁾, so, one may assume that the anomaly is expressed in terms of $\omega(e)$, the Levi-Civita connection corresponding to the metric

$$g = e^a \eta_{ab} e^b \quad (22)$$

where $\eta_{ab} = \delta_a^b$.

A beautiful remark due to W. Bardeen and B. Zumino¹⁰⁾ shows that $\mathcal{Q}_{(\Omega)}$ can be completely compensated by the Wess-Zumino-Witten action⁷⁾. The Goldstone field we shall use here is the vielbein e now considered as a 0-form with values in the general linear group. The corresponding Wess-Zumino-Witten action reads:

$$\Gamma_{WZW} = \int_{M_{2n-1}^+} [Q_{2n-1}^0(\omega(e), \omega(\dot{e})) - Q_{2n-1}^0(\Gamma, \dot{\Gamma})] \quad (24)$$

$$\Gamma := e^{-1} d e + e^{-1} \omega(e) e \quad (25)$$

$$\dot{\Gamma} := \dot{e}^{-1} d \dot{e} + \dot{e}^{-1} \omega(\dot{e}) \dot{e} \quad (26)$$

We use continuations of invariant polynomials on Lie $SO(2n-2)$ to Lie $GL(2n-2)$. The two arguments of Q_{2n-1}^0 correspond to ω and $\dot{\omega}$ in (11). M_{2n-1}^+ is a $(2n-1)$ dimensional contractible manifold whose boundary is M_{2n-2} ; e and $\dot{e}$ are extended to functions on M_{2n-1}^+ . Note however that the integrand of (24) is closed and therefore exact. Consequently (24) is an integral over M_{2n-2} only.

Adding Γ_{WZW} to the effective action replaces the anomaly $\{\mathcal{O}(\Lambda), \mathcal{O}(\xi)\}$, equations (18), (19) by

$$\overline{\sigma}(\Lambda) = 0 \quad (27)$$

and

$$\overline{\sigma}(\xi) = - \int_{M_{2n-1}^+} Q_{2n-1}^0(\Gamma, \dot{\Gamma}) = \tilde{\sigma}(\tilde{\Lambda}(\xi)) + \tilde{\sigma}(\xi) \quad (28)$$

with
$$\tilde{\Lambda}(\xi) := \left\{ - (D_\mu(r) \xi)^\vee \right\} \quad (29)$$

and all ω 's replaced by Γ 's.

This simple form is a consequence of the usual Slavnov algebra (8b) (8c)

$$\hookrightarrow \Gamma = - D \tilde{\Lambda}(\xi) - i_\xi R(\Gamma) \quad (30a)$$

$$\hookrightarrow \tilde{\Lambda}(\xi) = -\frac{1}{2} [\tilde{\Lambda}(\xi), \tilde{\Lambda}(\xi)] + \frac{1}{2} i_\xi i_\xi R(\Gamma) \quad (30b)$$

$\mathcal{O}(\xi)$ can also be written as

$$\overline{\sigma}(\xi) = \sigma(\Lambda(\xi)) + \sigma(\xi) \quad (31)$$

with
$$\Lambda(\xi) := \left\{ - (D_\mu(\dot{r}) \xi)^\vee \right\} \quad (32)$$

$\Lambda(\xi)$ takes into account the Wigner linear gauge transformation necessary to return to the holonomic frame $\{\partial_\mu\}$ after an infinitesimal diffeomorphism ξ . It is a matter of algebra to verify that the anomaly $\bar{\sigma}(\xi)$ fulfills the consistency condition for purely gravitational anomalies

$$W(\xi) \bar{\sigma}(\xi') - W(\xi') \bar{\sigma}(\xi) \simeq \bar{\sigma}([\xi, \xi']) \quad (33)$$

This procedure is precisely the same as that transforming Wigner type amplitudes into spinorial type amplitudes involved in the theory of representations of the Poincaré group¹¹⁾.

The same game can be played a second time to construct the covariant anomaly¹⁰⁾, i.e. the covariant divergence of the energy momentum tensor which is the analogue for gravity of the Bardeen current in the chiral gauge case: consider both a left and a right metric, $g^{L,R}$, left and right diffeomorphisms $\xi^{L,R}$, left and right chiral fields giving rise to "opposite anomalies" $\alpha(\xi^R, \Gamma^R, \tilde{\Gamma})$, $-\alpha(\xi^L, \Gamma^L, \tilde{\Gamma})$ referred to the holonomic frame $\{\partial_\mu\}$. Using the same method as in 4) (section III Eq.47), one first eliminates the diagonal anomaly ($\xi^L = \xi^R = \xi$) by the local action $\Gamma^{L,R}$

$$\Gamma^{L,R} = \int_{M_{2n-1}}^+ \left[Q_{2n-1}^0(\Gamma^R, \tilde{\Gamma}) - Q_{2n-1}^0(\Gamma^L, \tilde{\Gamma}) - Q_{2n-1}^0(\Gamma^R, \Gamma^L) \right] \quad (34)$$

The non diagonal Ward identity evaluated at $(g^L = g^R = g)$ involves

$$\frac{\delta \Gamma^{LR}}{\delta g^R} - \frac{\delta \Gamma^{LR}}{\delta g^L} \Big|_{g^R = g^L} \quad \text{and the corresponding anomaly now becomes covariant, and}$$

independent of $\tilde{\Gamma}$:

$$\sigma^{cov}(\xi) = n \int_n \left((D_\mu(\Gamma) \xi)^\nu - (D^\nu(\Gamma) \xi)_\mu, R^{\mu\nu} \right) \quad (35)$$

in which one should note the usual factor n , characteristic of the Bardeen current¹⁰⁾.

A few final remarks are in order: although the general solution of the Wess Zumino consistency conditions in the pure gauge case (following from Eq.3a) is not

known except for renormalizable theories in $d = 4$ dimensions¹²⁾, it is believed to be of the Adler-Bardeen type. In the purely gravitational case, the situation is much less under control: does Eq. (33) admit other solutions than those of the Adler-Bardeen type?

Besides the concrete calculations of Ref. 1), we should finally like to quote one more result¹³⁾. In 2 dimensions Lorentz and gravitational anomalies can be computed simultaneously from Feynman diagrams without resorting to any approximation. The result is of the Adler Bardeen type:

With $\omega = 0$

$$\sigma(\mathcal{N}) = -\frac{i}{96\pi} \text{Tr } \mathcal{N} \, d\omega \quad (36a)$$

$$\sigma(1) = 0 \quad (36b)$$

The case of mixed gravitational and internal gauge anomalies can of course be treated similarly.

A detailed version of this work is in preparation.

ACKNOWLEDGEMENTS

One of us (R.S.) wishes to thank I.M. Singer and B. Zumino for communicating their results prior to publication as well as for stimulating correspondence on various aspects of the general problem of anomalies. It is also a pleasure to thank L. Alvarez Gaumé, P. Ginsparg, R. Jackiw, Ed Witten for communicating some remarks and results of their work, and M.F. Atiyah for a comment which suggested the necessity to introduce a background field in order to cope with non trivial topology.

FOOTNOTES

- (f1) It has been known for some time²⁾ that gauge anomalies in $2n-2$ dimensions are related to symmetric invariant polynomials of degree n on Lie G , the Lie algebra of the structure group G , here $SO(4k-2)$. For a review of this phenomenon see for instance references 3), 4), 5), 6) where a bibliography can be found.

This remarkable fact has now been understood on a more fundamental basis, allowing in each specific case to relate the relevant invariant to the index of an elliptic operator in $2n$ dimensions (O. Alvarez, I.M. Singer, B. Zumino, to be published; L. Alvarez Gaumé, P. Ginsparg, to be published).

- (f2) A recent paper by L. Baulieu and J. Thierry-Mieg parallels an older treatment of the Faddeev-Popv algebra which, however, has not so far been adapted to the introduction of a background field.

REFERENCES

- 1) L. Alvarez Gaumé, E. Witten, Nucl. Phys. B234, 269 (1984).
- 2) R. Stora, Lecture notes, Cargèse 1976 in "New developments in quantum field theory and statistical mechanics", M. Levy, P. Mitter Ed. NATO ASI Ser B vol. 26, Plenum Press 1977.
- 3) B. Zumino, Lectures, Les Houches 1983 in Relativity Groups Topology II, B.S. De Witt, R. Stora Ed. North Holland 1984 (to appear).
- 4) R. Stora, Lectures, Cargèse 1983 Progress in gauge field theory, H. Lehmann Ed. (to appear).
- 5) R. Stora, B. Zumino, in preparation.
- 6) R. Jackiw, Lectures, Les Houches 1983 as in 3).
- 7) J. Wess, B. Zumino, Phys. Lett. 37B, 95 (1971);
E. Witten, Nucl. Phys. B223, 422 (1983).
- 8) D. Sullivan, IHES Pub n° 47, p. 269 (1977).
- 9) F.W. Hehl, P. von der Heyde, G.D. Kerlick, Rev. Mod. Phys. 48, 393 (1976),
Eqs (4.32), (4.33).
- 10) W.A. Bardeen, B. Zumino, preprint, LBL17639, UCB-PTH 84-12, Fermilab PUB
84/38-T.
- 11) P. Moussa, R. Stora, in Methods in Subnuclear Physics, Hercegovi Lectures (1966), M. Nikolic Ed. Gordon and Breach.
- 12) C. Becchi, A. Rouet, R. Stora, in Field Theory, Quantization and Statistical Physics, E. Tirapegui Ed. Reidel (1981), pp. 3-32.
- 13) F. Langouche, to be published.