Ref.TH.3914-CERN

CONSERVATION LAWS AND BOSONIZATION

Y. Frishman^{*)}

CERN - Geneva

A B S T R A C T

Simple arguments are given to demonstrate that the bosonization procedure of Witten forces a certain regularization for the fermion currents. This is done for interacting cases as well. It is then argued that when a non-Abelian vector symmetry is ungauged, it can still be used for classification, in spite of possible anomalies in Green's functions.

^{*)} On leave from the Weizmann Institute, Rehovot, Israel.

Ref.TH.3914-CERN

May 1984

A new bosonization scheme was suggested by Witten¹⁾. Its main advantage over the previous scheme^{2),3)} is that non-Abelian symmetries appear explicitly. Moreover, when both colour and flavour degrees of freedom are present, the bosonic Hamiltonian in the previous scheme was found to have non-local momentum dependent interactions⁴⁾.

A term of the form

$$(Const.) \mu^2 N_\mu \sum_{i \neq j} \sum_{\alpha, \beta} K_{\alpha\beta, ij} \sin \left[\int_{-\infty}^x (\pi_{\alpha i} - \pi_{\beta i} + \pi_{\beta j} - \pi_{\alpha j}) dx \right] - (\phi_{\alpha i} + \phi_{\beta i} - \phi_{\alpha j} - \phi_{\beta j}) / \left[\sum_{\alpha} (\phi_{\alpha i} - \phi_{\alpha j}) \right] \quad (1)$$

appears in the bosonic Hamiltonian⁴⁾. We deal here with two dimensions, one space and one time. N_μ is normal ordering with respect to the mass μ , (α, β) are flavour indices and (ij) colour indices, π is the canonical conjugate to ϕ , and K is an ordering operator⁴⁾ involving $(\pm)$ factors. The term in (1) renders applicability difficult. It is thus worth exploring properties of the new bosonization scheme.

In the case of both colour and flavour, the new suggestions¹⁾ may have a difficulty. Suppose we have N colours and n flavours (we anticipate gauging colour later or different gauge groups for colour and flavour). The suggestion is to use $n \times n$ matrices g for the flavour and $N \times N$ matrices h for the colour. But this way the number of degrees of freedom in internal space is $n+N$, while in the fermion sector it is (nN) . Thus when one is interested not only in certain vector and axial vector currents, but also in scalar and pseudoscalar densities, the description in terms of g and h alone is in general not enough. Such will be the case when the latter densities appear in the Hamiltonian. In the following we restrict ourselves to $N = 1$.

It was recently pointed out⁵⁾ that the correspondence between the scheme with bosonic variables and the one with fermionic variables, in the case of free fermions, holds only with a certain regularization for the fermion currents. This is the regularization where the singlet vector current is conserved, and the Green's functions of the (right) $\times$ (left) currents of $SU(n) \times SU(n)$ factorize into a product of a Green function with only right currents and another with only left currents. Thus the vector $SU(n)$ currents are not conserved. These results were obtained by explicitly computing the

generating functional in the presence of external sources coupled to the above currents.

We first demonstrate that these conservation laws are built into the bosonic description and are of a somewhat "geometric" nature. In fact, the singlet vector current must be conserved also with interactions, while the anomaly of the vector SU(N) in Green's functions follows from any bosonic Lagrangian that is globally right×left symmetric. However, although the non-Abelian vectors are not conserved in Green's functions, we shall see that one can still, in the case of a vector flavour symmetry, classify states by the appropriate quantum numbers.

The bosonic action in two-dimensional Minkowski spacetime is

$$S(U) = \frac{1}{8\pi} \int d^2x \operatorname{Tr}[(\partial_\mu U)(\partial^\mu U^\dagger)] \\ + \frac{1}{12\pi} \int_Q d^3\xi \epsilon^{ijk} \operatorname{Tr}[(U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U)] + S_I(U) \quad (2)$$

Q is a three-dimensional volume with two-space (compactified) as boundary. U are n×n unitary matrices of U(n). $S_I(U)$ is a possible interaction term. When $S_I = 0$ this is equivalent to n free massless fermions with the above-mentioned regularizations^{1),5)}. The currents are

$$J_+ = \frac{i}{4\pi} U^{-1} \partial_+ U \quad (3a)$$

$$J_- = \frac{i}{4\pi} U(\partial_- U^{-1}) = -\frac{i}{4\pi} (\partial_- U) U^{-1} \quad (3b)$$

Let us start with the singlet current,

$$J_+^S = \frac{i}{4\pi} \operatorname{Tr}[U^{-1}(\partial_+ U)] \quad (4a)$$

$$J_-^S = -\frac{i}{4\pi} \operatorname{Tr}[(\partial_- U) U^{-1}] = -\frac{i}{4\pi} \operatorname{Tr}[U^{-1}(\partial_- U)] \quad (4b)$$

When computing Green's functions involving the singlet current and operators $O_1 \dots O_k$ we have

$$\begin{aligned} \langle T J_\mu^S(x) O_1 \dots O_k \rangle &= \\ &= \frac{\int D\psi J_\mu^S(x) O_1 \dots O_k \exp[iS(\psi)]}{\int D\psi \exp[iS(\psi)]} \end{aligned} \quad (5)$$

The change of orders in (4b) follows from the cyclicity of the trace and the fact that under the functional integral the U_{ij} are numbers. Now

$$J_\mu^S = \frac{i}{4\pi} \epsilon_{\mu\nu} \text{Tr}[\psi^{-1} \partial^\nu \psi] \quad (6)$$

Notice that $\partial^\mu J_\mu^S = 0$ for any U , and not just the ones satisfying the equations of motion. J_μ^S will then obey conservation also under the functional integral. If we define our generalized time ordering in such a way that the Green's functions are given by the functional integral, the singlet vector will be conserved. In the fermionic language this means the following. First, any interaction that does not obey vector conservation cannot be bosonized with the correspondence of the fermionic vector $\bar{\psi} \gamma_\mu \psi$ with the expression in Eq. (6) (ψ_α are the Fermi fields). Second, if such a correspondence can be made through the equations of motion, one has to regulate the Green's functions in such a way that the singlet vector is conserved. This in general implies extra terms with derivatives of δ functions (in differences of the co-ordinates of the various local operators), added to the naive time ordering.

Let us now consider the $SU(n) \times SU(n)$ currents of right $\times$ left,

$$J_+^a = \frac{i}{4\pi} \text{Tr} [T^a \psi^{-1} (\partial_+ \psi)] \quad (7a)$$

$$J_-^a = -\frac{i}{4\pi} \text{Tr} [T^a (\partial_- \psi) \psi^{-1}] \quad (7b)$$

T^a are $n \times n$ matrices of the fundamental representation of $SU(n)$. Consider the two-point function

$$\langle T J_+^a(x) J_-^b(0) \rangle = \frac{\int D U J_+^a(x) J_-^b(0) \exp[i S(U)]}{\int D U \exp[i S(U)]} \quad (8)$$

Perform now a change $U(x) \rightarrow AU(x)$, with A a fixed matrix (independent of x). This is a global left transformation. We take $S(U)$ to be invariant under global $[SU(n)]_R \times [SU(n)]_L$. Now, as follows from Eqs. (7a) and (7b), J_+^a is invariant while

$$J_-^b \rightarrow J_-^{(A^T b A)} \equiv -\frac{i}{4\pi} \text{Tr} [(A^T T^b A) (\partial_\mu U) U^{-1}] \quad (9)$$

Thus the right-hand side of Eq. (8) is invariant under rotations of T^b appearing in J_-^b , and together with linearity we deduce that it must be proportional to $(\text{Tr } T^b) = 0$. Hence $\langle T J_+^a J_-^b \rangle = 0$, and thus the two-point function of $SU(n)$ vector currents, in momentum space, is proportional to

$$\langle T V_\mu^a V_\nu^b \rangle \propto (g_{\mu\nu} p^2 - 2 p_\mu p_\nu) \quad (10)$$

which is not conserved. This is an anomaly in the Green's functions, which is there even in the free fermion case, where there is certainly no anomaly in the equations of motion. Thus all $[U(n)]_R \times [U(n)]_L$ currents are conserved as operator equations

$$\partial^\mu J_\mu = 0 \quad (11)$$

meaning

$$\langle F_1 | \partial^\mu J_\mu | F_2 \rangle = 0 \quad (12)$$

for any states $|F_1\rangle, |F_2\rangle$.

Consider now a theory where there is an $SU(n)$ global vector symmetry. We argue that one may still go ahead and bosonize using the procedure of Ref. 1), if one uses the vector symmetry only to classify states. This is so

since if one uses the charges Q_V^a only for classification, only the conservation as in Eq. (12) is needed, and the fact that there is an anomaly in the divergence in Green's functions never shows up.

The simplest example would be two-dimensional QED with n flavours. Here one has an interaction $e(\sum_{\alpha} \bar{\psi}_{\alpha} \gamma_{\mu} \psi_{\alpha}) A^{\mu}$, where A^{μ} is the photon field. Only the singlet fermion vector current appears, which is conserved by the procedure of bosonization. Here one gets an anomaly in the divergence of the axial current as a result of the interaction

$$\partial^{\mu} (\bar{\psi} \gamma_{\mu} \gamma_5 \psi) = \frac{e}{2\pi} n \epsilon^{\mu\nu} \partial_{\mu} A_{\nu} \quad (13)$$

Equation (13) is for the case of massless fermions. For massive fermions an extra $2m\bar{\psi}\gamma_5\psi$ (normal divergence term) is added to the right-hand side. In any case, the vector $SU(n)$ can be used to classify states. [The axial $SU(n)$ is broken even for $m = 0$ since $\bar{\psi}\psi$ gets a dynamical vacuum expectation value.]

In fact, in this case the situation is very simple. One writes $U = e^{i(\phi/\sqrt{n})} V$, with $\det V = 1$. Then A_{μ} couples only to ϕ with strength e/n , and ϕ has a kinetic term like a canonical field. Thus the ϕ, A_{μ} system is like the old $n = 1$ case (but with coupling e/n), and the V have the dynamics of the free case of n flavours but with the constraint $\det V = 1$.

The situation is somewhat more complex in the non-Abelian Thirring model. Consider the Lagrangian

$$\begin{aligned} \mathcal{L} = & \bar{\psi}(i\not{\partial} - m)\psi - g_1 (\bar{\psi} \gamma_{\mu} \psi) (\bar{\psi} \gamma^{\mu} \psi) \\ & - g_2 (\bar{\psi} \gamma_{\mu} T^a \psi) (\bar{\psi} \gamma^{\mu} T^a \psi) \end{aligned} \quad (14)$$

Here the vector $SU(n)$ currents appear in the Lagrangian, and thus will appear in the various Green's functions when computing the S-matrix. [Note that the interactions break $SU(n)_R \times SU(n)_L$, but conserve $SU(n)_V$.] It thus seems that the anomalies in Green's functions may prevent us from applying the bosonization method. What one can do here is first apply a Fierz transformation

$$(T^a)_{ij} (T^a)_{kl} = \frac{1}{2} \delta_{il} \delta_{kj} - \frac{1}{2n} \delta_{ij} \delta_{kl} \quad (15)$$

$$(\gamma_\mu)_{\alpha\beta} (\gamma^\mu)_{\gamma\delta} = \delta_{\alpha\delta} \delta_{\gamma\beta} + (i\gamma_5)_{\alpha\delta} (i\gamma_5)_{\gamma\beta} \quad (16)$$

so that

$$\begin{aligned} & -(\bar{\psi} \gamma_\mu T^a \psi) (\bar{\psi} \gamma^\mu T^a \psi) = \\ & = \frac{1}{2} [(\bar{\psi} \psi)^2 + (i\bar{\psi} \gamma_5 \psi)^2] + \frac{1}{2n} (\bar{\psi} \gamma_\mu \psi) (\bar{\psi} \gamma^\mu \psi) \end{aligned} \quad (17)$$

The new Lagrangian will involve only the singlet vector and scalars and pseudoscalars. One can now apply the bosonization procedure, and the vector SU(n) currents will never appear in any Green's functions. Following Witten¹⁾, we now use

$$\bar{\psi} \psi = \frac{1}{2} M : [T_2 U + T_2 U^\dagger] : \quad (18a)$$

$$\bar{\psi} \gamma_5 \psi = \frac{1}{2} M : [T_2 U - T_2 U^\dagger] : \quad (18b)$$

Setting $m = 0$ and $g_1 = (1/2n)g_2$, we get an interaction

$$\mathcal{L}_I = \frac{1}{2} g_2 M^2 (T_2 U) (T_2 U^\dagger) \quad (19)$$

(The mass term and the singlet×singlet $V_\mu V^\mu$ interactions may also be included.)

The minimum of the potential (for $g_2 > 0$)

$$V = - \mathcal{L}_I = - \frac{1}{2} g_2 M^2 (T_2 U) (T_2 U^\dagger) \quad (20)$$

is at $U_0 = 1$, yielding a vacuum expectation value for $\bar{\psi}\psi$ and spontaneous mass generation for the fermions, as is well known from other methods⁶⁾.

ACKNOWLEDGEMENTS

It is a pleasure to thank D. Amati and D. Gepner for useful discussions.

REFERENCES

- 1) E. Witten - "Non-Abelian Bosonization in Two Dimensions", Princeton University preprint (1983).
- 2) S. Coleman - Phys.Rev. D11 (1975) 2088;
S. Mandelstam - Phys.Rev. D11 (1975) 3026.
- 3) V. Baluni - Phys.Lett. 90B (1980) 407;
P.J. Steinhardt - Nucl.Phys. 106B (1980) 100;
D. Amati and E. Rabinovici - Phys.Lett. 101B (1981) 407.
- 4) E. Cohen, Y. Frishman and D. Gepner - Phys.Lett. 121B (1983) 180.
- 5) P. Di Vecchia and P. Rossi - CERN Preprint TH. 3808 (1984).
- 6) D. Gross and A. Neveu - Phys.Rev. D10 (1974) 3235.