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THE EFFECTIVE ACTION FOR PHOTONS IN (2+1) DIMENSIONS

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A B S T R A C T

The topological mass term in the Euler-Heisenberg effective action for (2+1) dimensional spinor QED is studied by functional and canonical methods. The mass term is associated with a quantum Hall effect in the vacuum, and with a vacuum charge in the special case of a pure magnetic field. This charge is shown to become depleted at non-zero temperature.

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1. INTRODUCTION

In (2+1) space-time dimensions, gauge fields may possess a "topological" mass¹⁾. For Abelian gauge fields the Lagrangian density in such circumstances is,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\mu}{4} \epsilon_{\mu\nu\alpha} F^{\mu\nu} A^\alpha \quad (1.1)$$

where,

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1.2)$$

(We use the co-ordinate and metric conventions

$$x^\mu \equiv (t, x, y) \quad (1.3)$$

and,

$$g^{\mu\nu} = \text{diag}(1, -1, -1) \quad (1.4)$$

respectively.)

From (1.1) one may derive the equation,

$$(\square + \mu^2)^* F^\mu = 0 \quad (1.5)$$

where

$$*F^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha} F_{\nu\alpha} \quad (1.6)$$

indicating the massive nature of the gauge field.

Recently, it has been shown^{2),3)} that such a topological mass term is generated by fermionic radiative corrections to the action of a background gauge field [see also Ref. 4)]. These authors calculated the vacuum current density in the presence of a uniform magnetic field by the Schwinger proper-time method⁵⁾, and then functionally integrated this quantity to obtain the effective action. In the first part of this letter we derive the Euler-Heisenberg effective action^{6),7)} for this problem by direct evaluation of the appropriate functional integral.

A consideration of the effective action shows that the fermionic vacuum is charged in the presence of a magnetic field, and exhibits a (fractional) quantum Hall effect in combined magnetic and electric fields. The vacuum charge which we find is in agreement with Ref. 2). The authors of Refs. 3) and 4) have already shown that the vacuum charge arises through the phenomenon of fermion fractionization⁸⁾. In the second part of this letter, we study this charge further by canonical methods, and show that it becomes depleted at non-zero temperatures.

Finally, we comment on the connection between the vacuum charge and the (1+1)-dimensional chiral anomaly.

2. FUNCTIONAL EVALUATION OF THE EULER-HEISENBERG EFFECTIVE ACTION

We consider a Dirac field, ψ , in (2+1) space-time dimensions in the presence of the external gauge potentials,

$$A_0 = A \tag{2.1a}$$

$$A_1 = Bx_2 \tag{2.1b}$$

$$A_2 = 0 \tag{2.1c}$$

where A and B are constants. Although A_0 is a "pure gauge" its presence will serve to reveal the topological mass term in the effective action, and the associated vacuum charge.

The two-component field ψ obeys the Dirac equation,

$$(i \not{D} - m) \psi = 0 \tag{2.2}$$

where,

$$D^\mu \equiv \partial^\mu + ieA^\mu \quad (2.3a)$$

and we choose the representation of the Dirac matrices where,

$$\gamma^\mu \equiv \{\sigma_3, i\sigma_1, i\sigma_2\} \quad (2.3b)$$

We note that for a massive field, ϕ , the theory is neither P nor T invariant¹⁾. The Euler-Heisenberg effective action, S_{eff} , for this problem is defined by,

$$e^{iS_{\text{eff}}} \equiv \langle O_{\text{out}} | O_{\text{in}} \rangle \quad (2.4)$$

in Minkowski space.

After Wick continuation to Euclidean three-space,

$$x_0 \rightarrow ix_3 \quad (2.5a)$$

$$A_0 \rightarrow iA_3 \quad (2.5b)$$

$$S_{\text{eff}} \rightarrow iS_{\text{eff}}^{(E)} \quad (2.5c)$$

the effective action may be represented as the functional integral,

$$e^{-S_{\text{eff}}^{(E)}} = \int d[\psi] d[\bar{\psi}] e^{-\int d^3x \bar{\psi} (i\mathcal{D}^{(E)} - m) \psi} \quad (2.6)$$

This quantity may be evaluated by direct integration after expansion of the Grassmann field variables $\psi, \bar{\psi}$, over the complete set of c-number eigenspinor solutions of,

$$\mathcal{D}^{(E)} \psi_\lambda = \lambda \psi_\lambda \quad (2.7a)$$

where

$$\not{D}^{(E)} \equiv \gamma^0 \left(-i \frac{\partial}{\partial x_3} - eA \right) + \gamma^1 \left(\frac{\partial}{\partial x_1} - ieBx_2 \right) + \gamma^2 \frac{\partial}{\partial x_2} \quad (2.7b)$$

is the Euclidean Dirac operator.

The normalizable eigenspinors and allowed eigenvalues may be determined from the Landau levels of a Schrödinger particle in a uniform magnetic field in three space dimensions⁹⁾. In the gauge chosen these functions have a harmonic oscillator form in the x_2 direction, and are plane waves in the x_1 and x_3 directions. The eigenvalues are thus dependent upon an integer index, $n = 0, 1, 2, \dots$, and two continuous "momentum" variables k_1, k_3 . The eigenspinors are given by,

$$\psi_{n,k_1,k_3,\pm}(x) = e^{i(k_1 x_1 + k_3 x_3)} \begin{pmatrix} \alpha_{n,k_3,\pm} \phi_n[(eB)^{1/2}(x_2 - k_1/eB)] \\ \beta_{n,k_3,\pm} \phi_{n-1}[(eB)^{1/2}(x_2 - k_1/eB)] \end{pmatrix}, \quad n > 0 \quad (2.8a)$$

with eigenvalues

$$\lambda(n, k_1, k_3, \pm) = \pm [2neB + (k_3 - eA)^2]^{1/2}, \quad n > 0 \quad (2.8b)$$

and,

$$\psi_{0,k_1,k_3,-}(x) = e^{i(k_1 x_1 + k_3 x_3)} \begin{pmatrix} \phi_0[(eB)^{1/2}(x_2 - k_1/eB)] \\ 0 \end{pmatrix} \quad (2.8c)$$

with eigenvalues,

$$\lambda(0, k_1, k_3, -) = -(k_3 - eA) \quad (2.8d)$$

and we take $eB > 0$.

There is no normalizable spinor with eigenvalue $+(k_3 - eA)$, and we note the degeneracy of the eigenvalues which is parametrized by k_1 . In (2.8a,c) we have,

$$\phi_n(y) = [\sqrt{\pi} n! 2^n]^{-1/2} e^{-1/2 y^2} H_n(y) \quad (2.9a)$$

where H_n are Hermite polynomials, and the coefficients α, β are,

$$\alpha_{n,k_3,\pm} = \frac{(neB)^{1/2}}{[2neB + (k_3 - eA)^2]^{1/2} [\{2neB + (k_3 - eA)^2\}^{1/2} \pm (k_3 - eA)]^{1/2}} \quad (2.9b)$$

$$\beta_{n,k_3,\pm} = \pm \frac{[\{2neB + (k_3 - eA)^2\}^{1/2} \pm (k_3 - eA)]^{1/2}}{2^{1/2} [2neB + (k_3 - eA)^2]^{1/4}} \quad (2.9c)$$

The ψ_λ constitute an orthonormal set of spinors, so that,

$$S_{\text{eff}}^{(E)} = \sum_{\lambda} \ln(-i\lambda + m)$$

where we see that we need a density of eigenvalues factor in order to facilitate the (generalized) summation. For this we refer to Ref. 9), and introduce large (symbolic) ranges, L_1, L_2, T for the co-ordinates x_1, x_2, x_3 , which have the rôle of infra-red cut-offs. We then find,

$$0 < k_1 < eBL_2 \quad (2.10)$$

in order that the centre of each harmonic oscillator lie within the allowed range of x_2 values. Hence,

$$S_{\text{eff}}^{(E)} = \frac{L_1 T}{4\pi^2} \int_0^{eBL_2} dk_1 \int_{-\infty}^{\infty} dk_3 \left\{ \sum_{n=1}^{\infty} \ln [2neB + (k_3 - eA)^2 + m^2] + \ln [i(k_3 - eA) + m] \right\} \quad (2.11a)$$

so that after introducing the "proper-time" representation of the logarithm, we have,

$$S_{\text{eff}}^{(E)} = \frac{eB L_1 L_2 T}{4\pi^2} \left\{ \frac{1}{2\pi^{1/2}} \int_0^{\infty} \frac{ds}{s^{3/2}} e^{-m^2 s} \coth(eBs) - ieA\pi \frac{m}{|m|} \right\} \quad (2.11b)$$

The imaginary part of (2.11b) arises after the introduction of a gauge-invariant definition of the k_3 -integral of the last term of (2.11a)

The effective Lagrangian density in Minkowski space is then,

$$\mathcal{L}_{\text{eff}} = -\frac{e^2}{4\pi} AB \frac{m}{|m|} + \frac{eB}{8\pi^{3/2}} \int_0^\infty \frac{ds}{s^{3/2}} e^{-m^2 s} \coth(eBs) \quad (2.12)$$

For fields A, B which are slowly varying with respect to the electron Compton wavelength, the parity-violating first term of (2.12) may be re-written as a topological mass term,

$$\mathcal{L}_{\text{TM}} = \frac{e^2}{16\pi} \epsilon_{\mu\nu\alpha} A^\mu F^{\nu\alpha} \frac{m}{|m|} \quad (2.13)$$

up to a total divergence. From (2.13) we find a vacuum charge density of

$$\langle j_0 \rangle = \left. \frac{\partial \mathcal{L}_{\text{eff}}}{\partial A} \right|_{A=0} \quad (2.14a)$$

giving,

$$\langle j_0 \rangle = -\frac{e^2}{4\pi} B \frac{m}{|m|} \quad (2.14b)$$

in agreement with Ref. 2). (We have a negative magnetic field for $B > 0$.)

The origins of the vacuum charge density (2.14) will be studied further in the next section from a Hamiltonian viewpoint. However, we note that if we had started with a magnetic field,

$$B_0 = (B^2 - E^2)^{1/2} \quad (2.15a)$$

with,

$$|E| < |B| \quad (2.15b)$$

then in a frame moving with velocity

$$v = -E/B \quad (2.15c)$$

along the x_1 -axis with respect to the original frame, we would have a magnetic field, B , an electric field, E , directed along the x_2 -direction and a vacuum current density of,

$$\langle j_1 \rangle = \frac{e^2}{4\pi} E \frac{m}{|m|} \quad (2.15d)$$

so that the vacuum would exhibit a quantized Hall effect, with a Hall conductance of

$$\rho_H = \frac{1}{2} \left(\frac{e^2}{2\pi} \right) \quad (2.15e)$$

which is one-half of the fundamental unit^{10),11)}.

We note that the occurrence of the topological mass term, (2.13), in the effective action (2.12) is connected with the asymmetry in the spectrum of the Dirac operator, (2.7b), between positive and negative eigenvalues. It constitutes a violation of parity invariance via quantum effects.

The second term in (2.12) has the familiar (parity-conserving) Euler-Heisenberg form, when allowance is made for the dimensionality of space-time. It contains a field-independent divergence, corresponding to the free-fermion vacuum energy, and a term which renormalizes the classical action by a finite amount, from which the charge renormalization factor,

$$Z_{e^2} = 1 - \frac{e^2}{48\pi|m|} \quad (2.16)$$

may be extracted.

3. CANONICAL EVALUATION OF THE VACUUM ENERGY AND CHARGE IN A MAGNETIC FIELD

In the previous section we evaluated the effective action for the gauge fields (2.1) by direct functional integration. However, for the fermion fields this is a quantum mechanically stable field configuration, and so the effective action may also be written as^{6),7),12)}

$$S_{\text{eff}} = -WT \quad (3.1)$$

where W is the fermionic vacuum energy, and T is the (symbolic) duration of time. We now study this vacuum energy in the case $A_0 = 0$.

Energy eigenspinors are solutions of

$$H\psi_\omega = \omega\psi_\omega \quad (3.2a)$$

where the Dirac Hamiltonian is,

$$H = -i\sigma_2 \left(\frac{\partial}{\partial x_1} - ieBx_2 \right) + i\sigma_1 \frac{\partial}{\partial x_2} + m\sigma_3 \quad (3.2b)$$

and again we take $eB > 0$.

The allowed frequencies, ω , for which normalizable spinors, ψ_ω , exist, are,

$$\omega(n, \pm) = \pm [2neB + m^2]^{1/2}, \quad n > 0 \quad (3.3a)$$

and,

$$\omega(0, +) = m \quad (3.3b)$$

There are no normalizable spinors with $\omega = -m$. Each of the levels (3.3) has the familiar Landau level degeneracy⁹⁾, in a plane, of,

$$\frac{eB}{2\pi} L_1 L_2 \quad (3.3c)$$

where L_1, L_2 , are the (symbolic) lengths of the x_1, x_2 axes.

The vacuum energy is therefore,

$$W = \sum_{\substack{\text{negative} \\ \text{energies}}} (+\frac{1}{2})\omega + \sum_{\substack{\text{positive} \\ \text{energies}}} (-\frac{1}{2})\omega \quad (3.4a)$$

giving,

$$W = \frac{eB}{2\pi} L_1 L_2 \left[- \sum_{n=1}^{\infty} [2neB + m^2]^{\frac{1}{2}} - \frac{1}{2}m \right] \quad (3.4b)$$

The summation may be performed in the "proper-time" representation, giving,

$$W = - \frac{eB}{8\pi^{\frac{3}{2}}} L_1 L_2 \int_0^{\infty} \frac{ds}{s^{\frac{3}{2}}} e^{-m^2 s} \coth(leBs) \quad (3.5)$$

in agreement with (2.12). However, we note that the asymmetry between positive and negative energy levels ω will affect the vacuum charge. In the Dirac sea picture when $m < 0$, the levels (3.3b) must be occupied, and conversely, for $m > 0$, they must be left empty. Hence, with the usual definition of the charge operator,

$$Q = \frac{e}{2} \int d^2x [\psi^\dagger, \psi] \quad (3.6)$$

the vacuum charge is,

$$\langle Q \rangle = \frac{e}{2} \cdot (\text{number of occupied levels} - \text{number of unoccupied levels}) \quad (3.7a)$$

which after gauge-invariant regularization gives,

$$\langle Q \rangle = - \frac{e}{2} \frac{m}{|m|} \frac{eB}{2\pi} L_1 L_2 \quad (3.7b)$$

in agreement with (2.14). The vacuum charge arises through the phenomenon of fermion fractionization^{3),4),8)}, which is also responsible for the fractional Hall conductance, (2.15e), of the vacuum. In terms of the now standard explanations for the (integer) quantized Hall effect^{10),11)}, the fermionic vacuum fluctuations of the Landau levels (3.3b) behave like a single filled Landau level, but with fractional weight of ($\frac{1}{2}$).

From (3.7b), it is easy to see how the term (2.13) arises in (2.12) since in a pure gauge potential A_0 , we expect a contribution to the effective action of

$$\int d^3x A_0 \langle j_0 \rangle \quad (3.8)$$

Because of its origins in fermion fractionization one may expect the charge (3.7b) to become depleted at non-zero temperature, as happens with fractional fermion number in (1+1) dimensions¹³⁾.

In order to study this we now calculate the free energy of the fermionic vacuum in the gauge fields (2.1). We have,

$$F = \frac{eB L_1 L_2}{2\pi} \left\{ \left(-\frac{1}{2}\right) \sum_{n=1}^{\infty} \left\{ [2neB + m^2]^{\frac{1}{2}} - eA \right\} - \frac{1}{2} \sum_{n=0}^{\infty} \left\{ [2neB + m^2]^{\frac{1}{2}} + eA \right\} \right. \\ \left. - \beta^{-1} \sum_{n=1}^{\infty} \ln \left[1 + e^{-\beta \left\{ [2neB + m^2]^{\frac{1}{2}} - eA \right\}} \right] - \beta^{-1} \sum_{n=0}^{\infty} \ln \left[1 + e^{-\beta \left\{ [2neB + m^2]^{\frac{1}{2}} + eA \right\}} \right] \right\} \quad (3.9)$$

where we have assumed $m > 0$.

The vacuum charge is now given by,

$$\langle Q \rangle = \beta^{-1} \frac{\partial F}{\partial A} \Big|_{A=0} \quad (3.10a)$$

which, after gauge-invariant regularization gives,

$$\langle Q \rangle = \frac{eB}{2\pi} L_1 L_2 \left\{ \frac{e}{2} - e(1 + e^{\beta m})^{-1} \right\} \quad (3.10b)$$

We see that at zero temperature, ($\beta \rightarrow \infty$), we recover the result (3.7b), while at high temperatures ($\beta \rightarrow 0$) we have,

$$\lim_{\beta \rightarrow 0} \langle Q \rangle = 0 \quad (3.10c)$$

and similarly in the case $m < 0$.

It is easy to see how the result (3.10c) arises through thermal averaging in the case of large magnetic fields, $eB \gg \beta^{-2} \gg m^2$. In this case, thermal fluctuations will dominantly populate the energy level $\omega = |m|$. The state obtained in this way has opposite charge to the vacuum, leading to a cancellation of the total charge in the thermal average.

4. DISCUSSION

We have shown how the asymmetry of the eigenvalue spectrum of Dirac operators in (2+1) dimensions in a magnetic field is implicated in the electric charge of the vacuum state, through the process of fermion fractionization. Lorentz covariance then implies that the vacuum also exhibits a (fractional) quantum Hall effect.

The authors of Ref. 3) have related the fermion fractionization to the chiral anomaly in (1+1) dimensions. The number of modes (3.3b) which give rise to fermion fractionization in our problem is given by the number of zero energy modes of the operator (3.2b) in the massless limit, $H|_{m=0}$. Moreover, $H|_{m=0}$ possesses a fermion conjugation symmetry under σ_3 , and the zero modes in question are all conjugation eigenstates, of eigenvalue +1. However, $H|_{m=0}$ is identical with the Euclideanized Dirac operator of spinor fields in (1+1) dimensions, in the presence of a uniform electric field of strength B , and σ_3 then plays the rôle of the chirality operator, γ_5 . This last theory possesses an anomaly¹⁴⁾, which is determined by the index of the Dirac operator in question. The same mathematical structure obviously underlie the fermion fractionization in (2+1) dimensions.

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