EXPERIMENTALLY FAVOURED POTENTIAL FOR MESONSD. Olivier^{*)}

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ABSTRACT

We show first that a natural potential to be used in a semirelativistic wave equation for the $S = 1$, $J = J_{\max}$ light and heavy mesons is, for $r \gtrsim 1$ Fermi, $V(r) = ar + b/r$ with $b \approx -\pi/2$, for $r \lesssim 1$ Fermi, $V(r) = c \ln(r/r_0)$. Next we show that this particular b value favours in fact a QCD-like potential for $r \lesssim 1$ Fermi so that a logarithmic potential is only an effective one in a small r region.

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Several years ago, C. Quigg and J. Rosner¹⁾, motivated by the apparent equal spacing $M(T') - M(T) = M(\psi') - M(\psi)$ showed that a natural potential to be used in a Schrödinger equation describing heavy quarkonia must be $V(r) = c \ln(r/r_0)$ in a suitable range of r . Here we will somehow extend their analysis by showing that a natural parametrization of a potential to be used in a semirelativistic wave equation (2), which is suitable to describe light and heavy mesons, is

$$\begin{aligned} r \gtrsim 1 \text{ f} \quad V(r) &= ar + b/r \\ r \lesssim 1 \text{ f} \quad V(r) &= c \ln(r/r_0) \end{aligned}$$

where the three constants a , b and c have a direct experimental interpretation, namely a is, as is well known, related to the common slope of the Regge trajectories of the ρ , K^* and ϕ mesons, b is related to the intercept of the ρ trajectory and c is, as recalled above, related to the spacing of radial excitations in the charmonium and bellonium spectra. r_0 , on its side, provides a dimensionless argument for the logarithm and may simply be adjusted to lead a smooth behaved $V(r)$ for r around 1 fermi. Moreover, a and b will both be seen to have a theoretical interpretation which, in turn, will suggest a QCD-like potential at short distance so that the logarithmic behaviour for $r \lesssim 1 \text{ f}$ will finally be considered only as a convenient effective one.

The paper is organized as follows: firstly, we recall the well-known structure of the spectra of light and heavy mesons with $S=1$ and $J=J_{\text{max}}$; secondly, using light quarkonia we derive the potential for large r ; then, using heavy quarkonia, we recover the Quigg and Rosner result in the context of a semi-relativistic equation and finally we discuss our results in the light of a model recently proposed.

I. Structure of the mesonic spectral data

We will concentrate on the mesonic states with spin $S=1$ and total kinetic momentum $J=J_{\text{max}}$, i.e., $J=l+1$ where l is the orbital momentum.

We recall in Fig. 1 the Regge trajectories of the ρ , K^* and ϕ mesons. These trajectories appear to be linear to a good approximation, to share approximately the same slope and to be nearly equally spaced. From Fig. 1 we deduce the value of the common slope to be $\gamma = 1.2 \text{ GeV}^2$, the intercept of the

ρ trajectory to be $\beta_\rho = -0.45$, the spacing between the trajectories to be $\delta = -0.21$. We wrote $E^2 = \gamma(\lambda - \beta - n\delta)$ $n = 1$ or 2 so that $\gamma = 1/\alpha'$ and $\beta = \lambda_\rho(0)$ in the usual notation where $\lambda(E) = \alpha'E^2 + \lambda(0) + n\delta$.

In Fig. 2 we recall the approximate equal spacing between the three first radial excitations of the charmonium ψ and of the bellowium Υ . Note that we have scaled the ψ and Υ spectra to coincide with the first two levels of the non-relativistic Hamiltonian $-(d^2/dx^2) + \lambda \ln|x|$. For that Hamiltonian we have $E_2 - E_1 = 0.703^3$.

These two graphs are all we need to derive the parametrization of $V(r)$ mentioned in the introduction as we shall see now.

II. Behaviour of the potential for large r

The semirelativistic equation for the light mesons we are interested in writes

$$\left[\sqrt{\vec{p}^2 + m_1^2} + \sqrt{-\vec{p}^2 + m_2^2} + V_L(r) \right] \Psi(\vec{r}) = E \Psi(\vec{r}) \quad (1)$$

where the m_i are the constituent quark masses and $\vec{p}$ is the momentum of quark 1 in the centre-of-mass frame.

We will consider first the $m_1 = m_2 = 0$ case. In that case, imposing linear Regge trajectories leads heuristically to $V(r) = ar + b/r$.

Indeed, if we postulate first $V(r) = ar^\nu$ and approximate [thus neglecting terms of order $O(1/\sqrt{\lambda})$]^{*)}

$$2 \sqrt{\vec{p}^2} \approx 2 \sqrt{p_r^2 + \frac{\ell(\ell+1)}{r^2}} \sim 2 \frac{\sqrt{\ell(\ell+1)}}{r}$$

for large enough ℓ , thus neglecting the radial kinetic term $[p_r^2]$ as compared to the centrifugal barrier, we are led to

$$\langle r \rangle = r_m = \left[\frac{2 \sqrt{\ell(\ell+1)}}{a \nu} \right]^{\frac{1}{1+\nu}}$$

^{*)} This was noticed by A. Martin.

where r_m realizes the minimum of the effective potential $V_{\text{eff}} = [2\sqrt{\ell(\ell+1)}]/r + ar^\nu$ and then to

$$E_\ell = \left[2 \sqrt{\ell(\ell+1)} \right]^{\frac{\nu}{1+\nu}} \left[1 + \frac{1}{\nu} \right] \left[\nu a \right]^{\frac{1}{1+\nu}}$$

which in turn gives linear trajectories for $\nu = 1$. [Such an argument to obtain the long-range potential was used by H. Grosse and A. Martin in Ref. 4)]. Explicitly we have $E_\ell^2 \sim 8a(\ell + \frac{1}{2})$ for $V_L(r) = ar$. We expect the slope $8a$ to be in good agreement with the exact slope as the approximations used should be accurate for high enough ℓ , but the particular intercept $-\frac{1}{2}$ which involves low ℓ may be only a crude estimate. Next it is clear that adding a term $+b/r$ to ar just translates the Regge trajectories by $b/2$ leading to the most general linear relation

$$E^2 \simeq 8a \left(\ell + b/2 + \frac{1}{2} \right)$$

in the same approximation. The above heuristic reasoning may be completely justified by explicitly computing the eigenvalue of (1) with $V(r) = ar$ which in turn will provide us with the true value of the intercept.

The results are summarized in Table 1 and Fig. 3 where we have also reported the BKW Regge trajectories⁵⁾.

This Table and this graph correspond to the scaled equation:

$$\left[P + \frac{1}{2\pi} r \right] \psi = E \psi \quad (2)$$

which was numerically solved by the following method:

- 1) we Fourier-transformed Eq. (2);
- 2) we used the Multhopp method⁶⁾ to transform the singular integral equation thus obtained to a set of linear equations;
- 3) we then solved numerically this linear eigenvalue problem.

Many more details are available⁷⁾.

From Table 1 we deduce the following very good approximate formula for the exact eigenvalues of the Hamiltonian (1) with $m_1 = m_2 = 0$ and $V(r) = ar$

$$E_{n,\ell}^2 = 4\pi a \left(n + \lambda(n) \ell/2 + 3/4 + \mu(n) \right) \quad n \geq 0 \quad \ell \geq 0$$

where the first values of $\lambda(n)$ and $\mu(n)$ are listed in Table 2.

It is readily checked that $\lambda(1) \approx 8/2\pi$ thus confirming our heuristic argument for the leading trajectory. We also obtain the intercept of the leading Regge trajectory to be $B = 1.25$.

Now it is easy to see that no more information is contained in the Regge trajectories of the light mesons as their particular pattern follows from the use of the relativistic kinematics associated with first order perturbation theory. Indeed we may write

$$\sqrt{p^2 + m_1^2} = \sqrt{p^2} + \frac{m_1^2}{\sqrt{p^2 + m_1^2} + \sqrt{p^2}}$$

so that (1) becomes

$$\left[2 \sqrt{p^2} + \frac{m_1^2}{\sqrt{p^2 + m_1^2} + \sqrt{p^2}} + \frac{m_2^2}{\sqrt{p^2 + m_2^2} + \sqrt{p^2}} + V_L(r) \right] \psi(\vec{r}) = E \psi(\vec{r})$$

It is then justified to use first order perturbation theory because for a fixed ℓ , the energy differences between the various families (ρ, K^*, ϕ) are small as compared to the energy difference in each family between the fundamental and the first radial excitation and that for all ℓ . Therefore we can approximate

$$E_\ell(m_1, m_2) \approx E_\ell(0, 0) + (m_1^2 + m_2^2) \left\langle \frac{1}{2\sqrt{p^2}} \right\rangle$$

But

$$\left\langle \frac{1}{2\sqrt{p^2}} \right\rangle \sim \frac{1}{2 \langle \sqrt{p^2} \rangle} \sim \frac{2}{E_\ell(0, 0)}$$

for large enough l as can be seen from the Feynmann-Hellmann theorem applied to l . Here it is useful to note that apart from providing a coherent kinetic energy for the light quarks, the relativistic kinematics is also important to understand the parallelism of light meson Regge trajectories.

From the experimental data it follows that

$$a = 8/8 = 0.15 \text{ GeV}^2$$

and

$$b = -2 \left(\beta_p + B + \frac{m_u^2}{a} \right) = -2 \left(0.80 + \frac{m_u^2}{a} \right)$$

where $V_L(r)$ is written as $ar + b/r$.

We also have $m_s^2 - m_u^2 = -2a\delta = 0.062 \text{ GeV}^2$ which is quite reasonable. We expect this parametrization to be adequate for $V(r)$ in the range of r tested by the light mesons families, that is for $r \gtrsim 1\text{f}$.

III. Behaviour of the potential for small r

We start again with Eq. (1) which writes for heavy quarkonia in which we are interested, namely charmonium and bellonium,

$$\left[2 \sqrt{p^2 + m_q^2} + V_H(r) \right] \psi(\vec{r}) = E \psi(\vec{r}) \quad (3)$$

Now it is clear that the only thing we have to show in order to recover the C. Quigg and J. Rosner result is that, in spite of the emerging relativistic behaviour of the charmonium, which is well known [see for instance Refs. 1) and 3)], the characteristic mass-independent excitation spacing of the log potential is not altered, at least for the first levels, by the use of relativistic kinematics in the mass region of the c quark. But this last point immediately follows from first-order perturbation theory applied to Eq. (3).

Indeed we have

$$\sqrt{p^2 + m^2} = m + \frac{p^2}{2m} - \frac{1}{2m} \left(\frac{p^2}{2m} \right)^2 \frac{1}{\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{p^2}{m^2}} \right)^2}$$

which leads for $V_H(r) = \ln(r/r_0)$ to

$$\Delta E_{(\text{rel} - \text{non rel})} \simeq - \frac{c^2}{16 m} \left(\frac{1}{\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{c}{2m}}} \right)^2$$

We have used the virial theorem to obtain $\langle p^2/2m_q \rangle = c/4$ (assuming a distribution of momentum p sharp enough) which thus appears to be level independent.

To evaluate $\Delta E(\text{rel-non-rel})$ we use the c value given by the experimental spacing $E(T') - E(T) \sim 0.5460$ GeV. We have $c = 0.7$ GeV and for $m_c \approx 1.5$ GeV $\Delta E(\text{rel-non-rel}) \approx 0.02$ GeV, which is indeed small as compared to the relevant excitation energies for the first few radially excited levels of the log potential.

Therefore, assuming the approximate equal spacing exhibited in ϕ and T spectra will remain for all heavier $q\bar{q}$ families, we conclude as in Ref. 1) that a natural potential to be used in the r region tested by these mesons is $V_H(r) = \ln(r/r_0)$ with $c = 0.7$ GeV.

In fact, we know from Ref. 8) that without any extra assumption, V_H is determined to be logarithmic only in the range $0.1 f \lesssim r \lesssim 1f$, $r_L = 1f$ being therefore the upper limit of the small r region.

r_0 is adjusted so as to provide the smooth universal shape for $V(r)$ in the one-Fermi region. We found $c \ln r_0 = 0.800$ GeV to be a good value.

IV. Discussion

So far we have shown on experimental grounds only that the potential to be used to reproduce the observed spectra of light and heavy $S = 1$ and $J = J_{\text{max}}$ mesons must be parametrized as

$$V_L(r) = ar + b/r \quad r \gtrsim 1f$$

$$V_H(r) = c \ln(r/r_0) \quad r \lesssim 1f$$

where each relevant parameter (a, b , and c) has a straightforward physical interpretation; but a natural question arises: what could be their theoretical meaning?

Consider the small r behaviour of $V(r)$ first. As stated above, the log parametrization appears, from the current spectroscopy, to be strictly valid in a fairly restricted range of r ; moreover it does not correspond to any common theoretical expectation, so that altogether it sounds much more like a convenient form of an effective potential than anything fundamental. On the contrary, the ar part of the large r behaviour of $V_L(r)$ is well known to reflect the string dynamics of light mesons so that we may wonder if the b/r term could not also have such an interpretation. In fact it does! Indeed, we expect from the string model an additional contribution to the large distance potential coming from the quantum fluctuations of the string, a contribution which has been shown to write κ/r ^{9),10)} where $\kappa = -\pi$ for the Nambu string and $\kappa = -\pi/2$ in the Neveu-Schwartz model. Comparing with our b value we see that for a sensible u constituent quark mass ($m_u \approx 0.1$ GeV), b is quite near the Neveu-Schwartz value, which is noticeable. Moreover, using a preliminary model recently proposed in Ref. 11), we can reach a complete agreement with it. Indeed in that model, based on the coupled dynamics of a string and of a $U(1)$ gauge field, the potential to be used in a semirelativistic equation is shown to be for large r : $V(r) = ar + \kappa/r - (M_0^2/r)$ where (M_0^2/r) comes from the coupling with the $U(1)$ field and extends to short distances as well and where M_0^2 remains to be determined. From the b value we get

$$\begin{aligned} M_0^2 &= K - b = 2 \left(0.80 + \frac{m_u^2}{a} \right) - \pi/2 \\ \Rightarrow M_0^2 &= 0.03 + \frac{2m_u^2}{a} \end{aligned}$$

so for $0.1 < m_u < 0.2$ GeV, $0.16 < M_0^2 < 0.57$ which, remarkably, is seen to include the particular values used in the early calculation of Eichten et al¹²⁾. Such a result may suggest that the true short distances potential is in fact QCD-like^{*)}.

*) We should not worry too much about how to define the potential in the 1 Fermi region. Indeed from Ref. 10) it is clear that the form $ar + \kappa/r$ is not an accurate approximation to the string potential in that region (its exact form is for

$$r \gtrsim \sqrt{\frac{2K}{a}} \quad V(r) = \sqrt{(ar)^2 - 2Ka}$$

so that what we should do if we want to use the potential

$$\begin{aligned} r \gtrsim 1f & \quad V(r) = ar - \frac{K}{r} - \frac{M_0^2}{r} \\ r \lesssim 1f & \quad V(r) = - \frac{M_0^2}{r} \end{aligned}$$

is to join the $r \gtrsim 1f$ curve and the $r \lesssim 1f$ curve with a smooth line.

Let us emphasize finally that by using the transparent b/r trick, there is no need for any negative constant to be added to the linear term. We think that removes a long-lived embarrassment [see for instance Ref. 13)].

ACKNOWLEDGMENTS

We are grateful to J.L. Basdevant for suggesting to look at the meson spectroscopy in the context of a semirelativistic Schrödinger equation and for informing us about the Multhopp method. We also thank him for discussions, and thank A. Martin for a careful reading of the manuscript.

TABLES

N^L	Eigen energies			
	0	1	2	3
1	0.8908 (0.79)	1.1917 (1.42)	1.322 (2.05)	1.639 (2.69)
2	1.3287 (1.77)	1.5393 (2.37)	1.729 (2.99)	1.902 (3.62)
3	1.6615 (2.76)	1.8286 (3.34)	1.988 (3.95)	2.138 (4.57)
4	1.9386 (3.76)	2.0803 (4.33)	2.219 (4.92)	2.352 (5.53)
5	2.181 (4.76)	2.306 (5.32)	2.43 (5.90)	2.55 (6.50)

Table 1

Eigenvalues table of the Hamiltonian $(p+1/2\pi r) \phi(\vec{r}) = E_{N,L} \phi(\vec{r})$

Least Squared Linear Approximation Parameters					
N	1	2	3	4	5
α	0.633	0.617	0.603	0.590	0.58
λ	1.27	1.23	1.21	1.18	
μ	4.1×10^{-2}	9×10^{-3}	9×10^{-4}		

Table 2

$\lambda(N)$ and $\mu(N)$ are obtained from Table 1

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FIGURES CAPTIONS

Fig. 1: Experimental results drawn on this figure are from the Rosenfeld Tables¹⁴⁾. The first daughter trajectory is seen to emerge around 2.7 GeV².

Fig. 2: Experimental results are from (14). We used $H_{\text{Log}} = -(d^2/dx^2) + \ln|x|$ and $\Delta(\mathbb{T}) \times \text{Ech}(\mathbb{T}) = \Delta_{\text{Log}}$.

Fig. 3: Exact and BKW eigenvalues of the Hamiltonian $(p + 1/2\pi r) \psi(\vec{r}) = E_{N,L} \psi(\vec{r})$

- Figure 1 -

- Figure 2 -

- Figure 3 -