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ON NON-PERTURBATIVE CONTRIBUTIONS TO VACUUM ENERGY
IN SUPERSYMMETRIC QUANTUM MECHANICAL MODELS

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A B S T R A C T

Saturating the functional integral directly with instanton-anti-instanton type fluctuations, the vacuum energies for a supersymmetric quantum mechanical system with (i) a double well and (ii) a triple well potential are studied. In the former, vacuum energy is raised by these fluctuations, indicating spontaneous breakdown of supersymmetry. In the latter, the vacuum energy stays at zero, indicating that supersymmetry is not broken. In addition, the energy of the next level supersymmetric pair of states has been calculated for the triple well case.

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1. Introduction

Supersymmetry ¹⁾ as a possible fundamental symmetry between fermions and bosons has been studied for over a decade now. It appears to cure the notorious gauge hierarchy problem ²⁾ in the grand unified theories ^{3),4)}. However, at ordinary energy scales, this symmetry is not exact in nature. Whereas perturbative quantum effects respect supersymmetry, it would be desirable if non-perturbative fluctuations were to break it. To this effect, it is interesting to study the non-perturbative fluctuations such as instantons and anti-instantons ⁴⁾⁻⁸⁾.

In non-supersymmetric theories in $0 + 1$ dimensions, the instanton contributions to the vacuum energy have been discussed in Refs. 9) to 11). For supersymmetric models in $0 + 1$ dimensions, the role of instantons has been studied in Ref. 6).

In this paper, we shall discuss instanton-type quantum fluctuations in supersymmetric quantum mechanical systems with (i) a double well and (ii) a triple well potential. In contrast to Ref. 6), where single instanton induced vacuum expectation values of supersymmetric generators were obtained, we shall saturate the functional integral directly with instanton type fluctuations to obtain the vacuum energy. As is well known, single instantons or anti-instantons do not contribute to the vacuum functional integral because of the zero modes of the relevant fermion determinant obtained by integrating over fermionic degrees of freedom. However, in the background of an instanton-anti-instanton, this fermion determinant does not have any exact zero modes. Hence, such fluctuations may in general contribute to the vacuum energy. Here we develop a formalism to calculate these contributions. This can be easily generalized to study possible supersymmetry breaking in field theoretic models in higher dimensions.

The paper is organised as follows. In Section 2, we present the supersymmetric quantum mechanical models in general. In Section 3, we discuss the double well potential. We find that the vacuum energy is shifted upwards due to instanton-anti-instanton fluctuations, thereby breaking supersymmetry spontaneously. The case of triple well potential is discussed in Section 4. Here, in addition to instanton-anti-instanton effects, two-instantons and two-

anti-instantons also contribute. However, the Hamiltonian matrix for the three lowest lying states (which are independent linear combinations of the three classical ground states) has a zero and two equal positive eigenvalues. This implies that supersymmetry is not broken in this case. Finally, Section 5 contains some concluding remarks.

2. Supersymmetric quantum mechanics

The Minkowskian action for a supersymmetric classical particle with anti-commuting degrees of freedom may be written as :

$$A_{Mink} = \frac{1}{2} \int dt [\dot{x}^2 - S^2(x) + i \psi^T \dot{\psi} - S'(x) \psi^T \sigma_2 \psi] \quad (2.1)$$

where ψ is a two component anticommuting variable :

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \{\psi_i, \psi_j\} = 0 \quad (2.2)$$

and σ_2 is the Pauli matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

This action is invariant under the supersymmetric transformations,

$$\begin{aligned} \delta_\epsilon x &= \epsilon^T \sigma_2 \psi, \quad \delta_\epsilon \psi = (-i \sigma_2 \dot{x} - S(x)) \epsilon, \\ \delta_\epsilon \psi^T &= \epsilon^T (i \sigma_2 \dot{x} - S(x)). \end{aligned}$$

We shall be studying the following two specific cases of this action :

(i) double well potential

$$V \equiv \frac{1}{2} S^2 = \frac{\lambda}{4} \left(x^2 - \frac{m^2}{\lambda} \right)^2, \quad (2.3)$$

(ii) triple well potential

$$V \equiv \frac{1}{2} S^2 = \frac{\lambda^2 x^2}{2 m^2} \left(\frac{m^2}{\lambda} - x^2 \right)^2. \quad (2.4)$$

In the former case, we have two classical ground states, denoted by $|\pm\rangle$, corresponding to $x = \pm m/\sqrt{\lambda}$ and $\phi = 0$. In the latter case, we have three classical ground states, denoted by $|\pm\rangle$, 0 , corresponding to $x = \pm m/\sqrt{\lambda}$, 0 and $\phi = 0$. The classical vacuum energy for all these ground states is zero as dictated by supersymmetry. Perturbative quantum fluctuations around any one of these ground states do not change its energy. However, non-perturbative vacuum fluctuations which induce quantum mechanical tunnelling between various ground states may contribute to the vacuum energy.

3. Double well potential

Let us study the effect of non-perturbative vacuum fluctuations such as instantons and anti-instantons in the case of double well potential. Instantons and anti-instantons are the solutions of the Euclidean equations of motion implied by the Euclidean action :

$$A_E = \frac{1}{2} \int dt \left[\dot{x}^2 + S^2(x) + \psi^T \dot{\psi} + S'(x) \psi^T \sigma_2 \psi \right] \quad (3.1)$$

and are given for the double well potential (2.3) as :

$$\begin{aligned} x_I(t-t_1) &= \frac{m}{\sqrt{\lambda}} \tanh \frac{m(t-t_1)}{\sqrt{2}} , \quad \psi_i = 0 ; \\ \text{and } x_{\bar{I}}(t-t_2) &= -\frac{m}{\sqrt{\lambda}} \tanh \frac{m(t-t_2)}{\sqrt{2}} , \quad \psi_i = 0 ; \end{aligned} \quad (3.2)$$

respectively ^{10),11)}. These satisfy the linearized equations of motion $\dot{x}_I = -S(x_I)$, $\dot{x}_{\bar{I}} = +S(x_{\bar{I}})$, respectively. The classical action for both of them is the same :

$$A_0 = \frac{2\sqrt{2} m^3}{3\lambda} . \quad (3.3)$$

To see the effect of these quantum fluctuations, we shall evaluate the following tunnelling amplitudes :

$$\langle \pm | e^{-HT/\hbar} | \mp \rangle = \int_{\substack{x(T_2) = \pm \frac{m}{\sqrt{\lambda}} \\ x(-T_2) = \mp \frac{m}{\sqrt{\lambda}}}} d x(t) d \psi_1(t) d \psi_2(t) e^{-A_E/\hbar} \quad (3.4)$$

where T is the length of the large time-box.

3.1 Single instanton (or single anti-instanton) fluctuations

An instanton contribution to the functional integral (3.4) can be obtained by expanding around the instanton :

$$x(t) = x_I(t) + y(t) \quad (3.5)$$

which yields

$$\langle + | e^{-HT/\hbar} | - \rangle_{x_I} = e^{-A_0/\hbar} \int_{y(-I/2)=0}^{y(I/2)=0} dy(t) d\psi_1(t) d\psi_2(t) \exp \left[-\frac{1}{2\hbar} y D_B y - \frac{1}{2\hbar} \psi^T D_F \psi \right] \quad (3.6)$$

where

$$D_B[x_I] \equiv - \frac{d^2}{dt^2} + (S'^2 + S''S)_{x_I} = - \frac{d^2}{dt^2} + m^2 \left(3 \tanh^2 \frac{m(t-t_1)}{\sqrt{2}} - 1 \right)$$

(3.7)

$$D_F[x_I] \equiv \frac{d}{dt} + \sigma_2 S'(x_I) = \frac{d}{dt} + \sigma_2 \sqrt{2} m \tanh \frac{m(t-t_1)}{\sqrt{2}}.$$

Naïve integration over the Bose fluctuations $y(t)$ yields $\det^{-\frac{1}{2}} D_B$. However, this operator D_B has a zero eigenvalue corresponding to the invariance of the Euclidean action under the translation of the instanton location t_1 . This has to be treated by the collective co-ordinate method, which yields an integration over the instanton location t_1 multiplied by the corresponding Jacobian factor $(A_0/2\pi\hbar)^{\frac{1}{2}}$.

Integration over the fermionic fluctuations yields $\det^{\frac{1}{2}} D_F$ in (3.6). This fermionic operator $D_F[x_I]$ also has a zero mode :

$$\begin{aligned} \psi_0^{(+)}(t-t_1) &= \sqrt{\frac{\hbar}{A_0}} \dot{x}_I(t-t_1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \epsilon_0^{(+)} = \sqrt{\frac{\hbar}{A_0}} \frac{m^2}{\sqrt{2}\lambda} \operatorname{sech}^2 \frac{m(t-t_1)}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \epsilon_0^{(+)} \\ &\equiv \sqrt{\hbar} \chi^{(+)}(t-t_1) \epsilon_0^{(+)} \end{aligned} \quad (3.8)$$

and the corresponding fermionic zero mode for the anti-instanton operator $D_F[x_I^-]$ is

$$\begin{aligned} \psi_0^{(-)}(t-t_2) &= \sqrt{\frac{\hbar}{A_0}} \dot{x}_I^-(t-t_2) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \epsilon_0^{(-)} = \sqrt{\frac{\hbar}{A_0}} \frac{m^2}{\sqrt{2}\lambda} \operatorname{sech}^2 \frac{m(t-t_2)}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \epsilon_0^{(-)} \\ &\equiv \sqrt{\hbar} \chi^{(-)}(t-t_2) \epsilon_0^{(-)} \end{aligned} \quad (3.9)$$

Here $\varepsilon_0^{(\pm)}$ are the anticommuting collective co-ordinates corresponding to these zero modes.

The functional integral (3.6) can now be represented as

$$\langle + | e^{-HT/\hbar} | - \rangle_{x_I} = e^{-A_0/\hbar} \int dt_1 \left(\frac{A_0}{2\pi\hbar} \right)^{1/2} K_{x_I} \int d\varepsilon_0^{(+)} \quad (3.10)$$

with

$$K_{x_I} \equiv \left[\frac{\det' D_F[x_I]}{\det' D_B[x_I]} \right]^{1/2} \quad (3.11)$$

where primes denote that the zero modes have been factored out. Writing

$$\det' D_F[x_I] = \left\{ \det' \left[-\frac{d^2}{dt^2} + (S'^2 + S''S)_{x_I} \right] \det \left[-\frac{d^2}{dt^2} + (S'^2 - S''S)_{x_I} \right] \right\}^{1/2}$$

we have

$$K_{x_I} = \left[\frac{\det \tilde{D}_F}{\det' D_B} \right]^{1/4} \equiv \left\{ \frac{\det \left[-\frac{d^2}{dt^2} + (S'^2 - S''S)_{x_I} \right]}{\det' \left[-\frac{d^2}{dt^2} + (S'^2 + S''S)_{x_I} \right]} \right\}^{1/4} \quad (3.12)$$

where the operator in the numerator does not have a zero mode and hence is not primed. This determinantal factor can be evaluated by using the technique presented in the review of Gelfand and Yaglom¹⁰⁾⁻¹²⁾ in a large box of length T :

$$K_{x_I} = E_0^{1/2} \left[\frac{\overline{M}_I(T/2) M_I(-T/2)}{\overline{N}_I(T/2) N_I(-T/2)} \right]^{1/4} \quad (3.13)$$

where

$$D_B[x_I] N_I = 0, \quad \tilde{D}_F[x_I] M_I = 0 \quad (3.14)$$

and

$$\bar{N}_I(t) = N_I(t) \int_{-\pi/2}^t \frac{dt}{N_I^2(t)}, \quad \bar{M}_I(t) = M_I(t) \int_{-\pi/2}^t \frac{dt}{M_I^2(t)} \quad (3.15)$$

In Eq. (3.13), the expression in the parentheses represents $\det \tilde{D}_F / \det D_B$, where $\det D_B$ contains E_0^2 which is the would-be zero mode when $T \rightarrow \infty$. This has been divided out to get the primed determinant. This would-be zero mode (exact in the limit $T \rightarrow \infty$) can be calculated as in Ref. 10) as

$$E_0^2 = \frac{\bar{N}_I(\pi/2)}{\bar{N}_I(\pi/2) \int_{-\pi/2}^{\pi/2} dt N_I(t) \bar{N}_I(t) - N_I(\pi/2) \int_{-\pi/2}^{\pi/2} dt \bar{N}_I^2(t)} \quad (3.16)$$

Taking

$$N_I(t-t_1) = \text{sech}^2 \frac{m(t-t_1)}{\sqrt{2}}, \quad M_I(t-t_1) = \cosh^2 \frac{m(t-t_1)}{\sqrt{2}} \quad (3.17)$$

and limit $T \rightarrow \infty$, Eqs. (3.13) and (3.16) yield the determinantal factor to be

$$K_{\chi_I} = (2\sqrt{2} m)^{1/2} \quad (3.18)$$

It is interesting to notice that we do not have an exact matching of Bose and Fermi non-zero eigenmodes here ¹³⁾. This is unlike the case of supersymmetric Yang-Mills theory in (3 + 1) dimensions where this ratio would be one due to exact Bose-Fermi cancellations ¹⁴⁾.

Inserting (3.18) in (3.10), we finally can write the single instanton contribution to the vacuum functional integral as

$$\langle + | e^{-HT/k} | - \rangle_{\chi_I} = e^{-A_0/k} \left(\frac{A_0}{2\pi k} \right)^{1/2} (2\sqrt{2} m)^{1/2} \int dt_1 \int d\epsilon_0^{(4)} \quad (3.19)$$

Similarly, the single anti-instanton contribution can be written as

$$\langle - | e^{-HT/\hbar} | + \rangle_{x_{\bar{I}}} = e^{-A_0/\hbar} \left(\frac{A_0}{2\pi\hbar} \right)^{1/2} (2\sqrt{2} m)^{1/2} \int dt_2 \int d\epsilon_0^{(-)} \quad (3.20)$$

Note that $K_{x_I} = K_{x_{\bar{I}}} \cdot \epsilon_0^{(-)}$ is the collective co-ordinate corresponding to the fermionic zero mode in the anti-instanton background.

As is obvious, because of the fermionic integration [over $\epsilon_0^{(+)}$ and $\epsilon_0^{(-)}$] both (3.19) and (3.20) are exactly zero. Hence, single instantons or anti-instantons do not induce any quantum tunnellings. In fact, in general, any number of instantons and anti-instantons does not affect quantum tunnellings as long as there is an excess of one instanton or anti-instanton (kink number = ± 1). However, topologically trivial (kink number = 0) configurations containing an equal number of instantons and anti-instantons do contribute to the vacuum functional integral. This is so because there are no exact fermionic zero modes for such configurations. A pair of instanton and anti-instanton would give the lowest order effect from such vacuum fluctuations.

3.2 Well separated instanton-anti-instanton

An instanton-anti-instanton configuration is depicted in Fig. 1, with the instanton located at t_1 and the anti-instanton located at t_2 :

$$x_{I\bar{I}}(t; t_1, t_2) = \begin{cases} x_I(t-t_1) & -T/2 < t \leq \alpha \\ x_{\bar{I}}(t-t_2) & \alpha \leq t < T/2 \end{cases} \quad (3.21)$$

where $\alpha = (t_1 + t_2)/2$. This configuration is not a stationary point of the Euclidean action, but for infinite separation, $t_2 - t_1 \equiv \beta \rightarrow T \rightarrow \infty$, it approaches a stationary point. Under local translations of the instanton and anti-instanton, t_1, t_2 , the action changes by an exponentially small amount and hence there are two approximate translational symmetries.

The action for this instanton-anti-instanton configuration can be written as

$$\begin{aligned} A_E[x_{I\bar{I}}] &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} dt [\dot{x}_{I\bar{I}}^2 + S^2(x_{I\bar{I}})] \\ &= \int_{-\pi/2}^{\alpha} dt \dot{x}_I^2(t-t_1) + \int_{\alpha}^{-\pi/2} dt \dot{x}_{\bar{I}}^2(t-t_2) \end{aligned} \quad (3.22)$$

It is convenient to shift the integration variable from $t \rightarrow t + t_1$ and $t \rightarrow t + t_2$ in these two terms respectively. Then, in the approximation of large separation, $\beta = t_2 - t_1 \gg \sqrt{2/m}$, (3.22) can be written for $T \rightarrow \infty$ as :

$$\begin{aligned} A_E[x_{I\bar{I}}] &= \int_{-\pi/2}^{\beta/2} dt \dot{x}_I^2(t) + \int_{-\beta/2}^{\pi/2} dt \dot{x}_{\bar{I}}^2(t) \\ &\simeq 2A_0 + A_{int}(\beta) \end{aligned} \quad (3.23)$$

where

$$A_{int}(\beta) = \frac{3}{2} A_0 \left[\tanh \frac{m\beta}{2\sqrt{2}} - \frac{1}{3} \tanh^3 \frac{m\beta}{2\sqrt{2}} - \frac{2}{3} \right]$$

Here A_0 is the single instanton action (3.3). As expected, for infinite separation, $\beta \rightarrow T \rightarrow \infty$, the interaction action $A_{int}(\beta)$ goes to zero.

In order to obtain the contribution of this fluctuation $x_{I\bar{I}}(t; t_1, t_2)$ to the functional integral for $\langle -|e^{-HT/\hbar}| - \rangle$, we expand the action about this configuration. Corresponding to the two approximate translational zero-modes, we introduce the integration over the two collective co-ordinates t_1 and t_2 , with a Jacobian factor $(A_0/2\pi\hbar)^{1/2}$ for each one of them. We shall also evaluate the fermion determinant $K_0(t_2 - t_1)$ in the subspace of the fermionic zero modes (3.8) and (3.9) separately. The rest of the functional integral will be denoted by $K(\beta)$:

$$\begin{aligned} \langle -|e^{-HT/\hbar}| - \rangle_{x_{I\bar{I}}} &= \int_{x(\pi/2)=-\psi/2}^{x(\pi/2)=\psi/2} dx(t) d\psi_1(t) d\psi_2(t) e^{-A_E/\hbar} \\ &\simeq \left(\frac{A_0}{2\pi\hbar} \right) \int_{-\pi/2}^{\pi/2} dt_2 \int_{-\pi/2}^{t_2-\beta_0} dt_1 K(\beta) K_0(\beta) \exp \left[-\frac{2A_0}{\hbar} - \frac{A_{int}(\beta)}{\hbar} \right] \end{aligned} \quad (3.24)$$

for large separations $\beta \gg \sqrt{2}/m$. Here, $\beta_0 \gg \sqrt{2}/m$ is the minimum separation between instanton and anti-instanton up to which our approximation would be valid. For infinite separation, $\beta \rightarrow T \rightarrow \infty$, when the instanton-anti-instanton configuration becomes a stationary point of the action, the factor $K(\beta)$ is simply the product of the instanton and anti-instanton non-zero mode determinantal factors of Eq. (3.18), $K(T) = K_{x_I} K_{x_{\bar{I}}} = 2\sqrt{2} m$. On the other hand, for finite separations, $\beta \ll T$, configuration (3.21) is not a stationary point of the action. This would mean that when the Bose action is expanded around this configuration, the linear term will be dominant and will strongly suppress the functional integral. A straightforward calculation verifies this expectation, $K(\beta \ll T)_{T \rightarrow \infty} \rightarrow 0$.

Next, let us evaluate the fermions determinant in the subspace of zero modes given in Eqs. (3.8) and (3.9) :

$$\begin{aligned}
 K_0(\beta) &= \int d\epsilon_0^{(+)} d\epsilon_0^{(-)} \exp \left\{ -\frac{1}{2} \int dt \left[\chi^{(+)\dagger} \left(\frac{d}{dt} + S'(\chi_{I\bar{I}}) \right) \chi^{(+)} \epsilon_0^{(-)} \epsilon_0^{(+)} \right. \right. \\
 &\quad \left. \left. + \chi^{(+)\dagger} \left(\frac{d}{dt} - S'(\chi_{I\bar{I}}) \right) \chi^{(-)} \epsilon_0^{(+)} \epsilon_0^{(-)} \right] \right\} \\
 &= -\frac{1}{2} \int dt \left[\chi^{(+)\dagger} \left(\frac{d}{dt} + S'(\chi_{I\bar{I}}) \right) \chi^{(+)} - \chi^{(+)\dagger} \left(\frac{d}{dt} - S'(\chi_{I\bar{I}}) \right) \chi^{(-)} \right]
 \end{aligned} \tag{3.25}$$

We break the time integral from $-T/2$ to α and α to $T/2$. Integration by parts and use of the equations satisfied by $\chi^{(+)}$ and $\chi^{(-)}$,

$$\begin{aligned}
 \left[\frac{d}{dt} + S'(\chi_{I\bar{I}}(t-t_1)) \right] \chi^{(+)}(t-t_1) &= 0 & t < \alpha \\
 \left[\frac{d}{dt} - S'(\chi_{I\bar{I}}(t-t_2)) \right] \chi^{(-)}(t-t_2) &= 0 & t > \alpha
 \end{aligned} \tag{3.26}$$

yields for large separation and large T :

$$K_0(\beta) \simeq \frac{3m}{4\sqrt{2}} \operatorname{sech}^4 \frac{m\beta}{2\sqrt{2}} = \frac{1}{A_0} \frac{d}{d\beta} A_{\text{int}}(\beta) \quad (3.27)$$

Notice that for $\beta \rightarrow T \rightarrow \infty$, $K_0 \rightarrow 0$, as expected because in this limit, the fermion zero modes become exact.

Inserting (3.27) into (3.24), we have

$$\begin{aligned} \langle -|e^{-HT/\hbar}|- \rangle_{\chi_{I\bar{I}}} &\simeq -\left(\frac{A_0}{2\pi\hbar}\right) e^{-2A_0/\hbar} \frac{T}{2} \left[\frac{\hbar}{A_0} K(\beta) e^{-A_{\text{int}}(\beta)/\hbar} \right]_{\beta_0}^T \\ &\quad - \frac{\hbar}{A_0} \int_{\beta_0}^T d\beta \frac{dK(\beta)}{d\beta} e^{-A_{\text{int}}(\beta)/\hbar} \end{aligned} \quad (3.28)$$

Using the fact that $K(T) \xrightarrow{T \rightarrow \infty} (2/2m)$ and $K(\beta \ll T) \rightarrow 0$, as argued above, only the upper limit contributes in the first term. The second term can be shown to be higher order in $\hbar$ by successive partial integrations and hence can be dropped. Finally, we can write the instanton-anti-instanton contribution to the functional integral as :

$$\langle -|e^{-HT/\hbar}|- \rangle_{\chi_{I\bar{I}}} \simeq -\frac{mT}{\pi\sqrt{2}} e^{-2A_0/\hbar} \left[1 + O(\hbar/A_0) \right] \quad (3.29)$$

Similar calculations could be done for the contribution of the anti-instanton-instanton contribution to $\langle +|e^{-HT/\hbar}|+ \rangle$ with the same result as given on the right-hand side of (3.29).

From these, we notice that we have two ground states in this model with equal energies :

$$\langle +|H|+ \rangle = \langle -|H|- \rangle \simeq \frac{m\hbar}{\pi\sqrt{2}} e^{-2A_0/\hbar} \quad (3.30)$$

up to the lowest order and supersymmetry is spontaneously broken. This result is in accordance with that of Ref. 6) where the ground state energies were obtained by a different method.

4. Triple well potential

In this case, there is a general argument due to Witten ⁴⁾ that supersymmetry is not broken by quantum effects. He has argued the existence of a normalizable zero energy ground state from general principles. In the following, we shall demonstrate that, indeed, instanton-anti-instanton, two-instanton, and two-anti-instanton effects conspire to leave a zero energy ground state. We shall also calculate the energy of the next level excited states.

The potential is given by the expression in (2.4). This model admits two types of instantons ¹⁵⁾ :

$$(i) \quad \dot{\chi} = -S(\chi) \quad : \quad \chi = \chi_I(t-t_0) \equiv \frac{m}{\sqrt{\lambda}} \left[\frac{1 + \tanh m(t-t_0)}{2} \right]^{\frac{1}{2}}$$

$$(ii) \quad \dot{\chi} = S(\chi) \quad : \quad \chi = \chi_I(t-t_0) \equiv -\frac{m}{\sqrt{\lambda}} \left[\frac{1 - \tanh m(t-t_0)}{2} \right]^{\frac{1}{2}} \quad (4.1)$$

which interpolate between the classical ground states $x = 0$, $x = m/\sqrt{\lambda}$ and $x = -m/\sqrt{\lambda}$, $x = 0$ as $t \rightarrow -\infty$ to $t \rightarrow +\infty$, respectively. There are also two types of anti-instantons ¹⁵⁾ :

$$(iii) \quad \dot{\chi} = S(\chi) \quad : \quad \chi = \chi_{\bar{I}}(t-t_0) \equiv \frac{m}{\sqrt{\lambda}} \left[\frac{1 - \tanh m(t-t_0)}{2} \right]^{\frac{1}{2}}$$

$$(iv) \quad \dot{\chi} = -S(\chi) \quad : \quad \chi = \chi_{\bar{I}}(t-t_0) \equiv -\frac{m}{\sqrt{\lambda}} \left[\frac{1 + \tanh m(t-t_0)}{2} \right]^{\frac{1}{2}} \quad (4.2)$$

which interpolate between the classical ground states $x = m/\sqrt{\lambda}$, $x = 0$ and $x = 0$, $x = -m/\sqrt{\lambda}$ as $t \rightarrow -\infty$ to $t \rightarrow +\infty$, respectively.

The action for all these instantons and anti-instantons is the same :

$$A_0 = \frac{m^3}{4\lambda} \quad (4.3)$$

4.1 Single instanton (anti-instanton) contribution

As in the double well case, single instanton (or anti-instanton) contribution to the vacuum functional integral is completely suppressed due to the fermionic zero modes. These zero modes are the normalizable solutions of

$$\left[\frac{d}{dt} + \sigma_2 S'(x) \right] \psi = 0 \quad (4.4)$$

where x is any one of the configurations x_I , $x_{\bar{I}}$, y_I , $y_{\bar{I}}$ listed in (4.1) and (4.2) and $S(x)$ is as defined in (2.4). These zero modes can be written as :

$$\begin{aligned} \text{(i) } x_I : \quad \psi_0^{(+)} &= \sqrt{\frac{1}{A_0}} \frac{m^2}{2\sqrt{2\lambda}} e^{-m(t-t_0)/2} \kappa e^{\frac{3}{2} m(t-t_0)} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \epsilon_0^{(+)} \\ \text{(ii) } y_I : \quad \psi_0^{(-)} &= \sqrt{\frac{1}{A_0}} \frac{m^2}{2\sqrt{2\lambda}} e^{m(t-t_0)/2} \kappa e^{\frac{3}{2} m(t-t_0)} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \eta_0^{(-)} \\ \text{(iii) } x_{\bar{I}} : \quad \psi_0^{(-)} &= \sqrt{\frac{1}{A_0}} \frac{m^2}{2\sqrt{2\lambda}} e^{m(t-t_0)/2} \kappa e^{\frac{3}{2} m(t-t_0)} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \epsilon_0^{(-)} \quad (4.5) \\ \text{(iv) } y_{\bar{I}} : \quad \psi_0^{(+)} &= \sqrt{\frac{1}{A_0}} \frac{m^2}{2\sqrt{2\lambda}} e^{-m(t-t_0)/2} \kappa e^{\frac{3}{2} m(t-t_0)} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \eta_0^{(+)} \end{aligned}$$

Similar to Eq. (3.10), the various tunnelling amplitudes ($|-\rangle \rightarrow |0\rangle$, $|0\rangle \rightarrow |+\rangle$, etc.) can be expressed in terms of integration over the collective coordinates representing the fermionic and bosonic zero modes and the non-zero mode determinantal factor of the form (3.13) - (3.16). Here we have

$$K_{y_I} = K_{x_{\bar{I}}} = \left[\frac{\det \tilde{D}_F}{\det' D_B} \right]_{y_I, x_{\bar{I}}}^{1/4} = \left[\frac{\det \left[-\frac{d^2}{d\tau^2} + \frac{m^2}{4} (15 \tanh^2 m\tau - 6 \tanh m\tau - 5) \right]}{\det' \left[-\frac{d}{d\tau} + \frac{m^2}{4} (3 \tanh^2 m\tau - 6 \tanh m\tau + 7) \right]} \right]_{y_I, x_{\bar{I}}}^{1/4} \quad (4.6)$$

and $K_{x_I} = K_{y_{\bar{I}}}$ is given by the same expression with $\tau \equiv t - t_0$ replaced by $-\tau \equiv -(t - t_0)$. In the formulae (3.13) - (3.16), the N and M functions are for the present case defined as follows :

$$\begin{aligned} M_{y_I}(\tau) &= M_{x_{\bar{I}}}(\tau) = e^{-m\tau/2} \cosh^{3/2} m\tau \\ N_{y_I}(\tau) &= N_{x_{\bar{I}}}(\tau) = e^{m\tau/2} \operatorname{sech}^{3/2} m\tau \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} M_{x_I}(\tau) &= M_{y_{\bar{I}}}(\tau) = e^{m\tau/2} \cosh^{3/2} m\tau \\ N_{x_I}(\tau) &= N_{y_{\bar{I}}}(\tau) = e^{-m\tau/2} \operatorname{sech}^{3/2} m\tau. \end{aligned} \quad (4.8)$$

Using these, we obtain from (3.13) - (3.16) by a direct computation after taking the limit $T \rightarrow \infty$:

$$K_{x_I} = K_{x_{\bar{I}}} = K_{y_I} = K_{y_{\bar{I}}} = (2\sqrt{2} m)^{1/2} \quad (4.9)$$

Since the single instanton (or anti-instanton) effects are completely suppressed, the possible next order contribution to the vacuum functional integral will again come from instanton-anti-instanton configurations. In this case, we also have to include two-instanton and two-anti-instanton contributions representing quantum tunnellings $|-\rangle$ to $|+\rangle$ and $|+\rangle$ to $|-\rangle$. All these we consider in the next subsection.

4.2 Instanton-anti-instanton, two-instanton and two-anti-instanton contributions

Now we shall evaluate the contribution of the fluctuations of the type depicted in Fig. 2. Figures 2a and 2b contribute to the matrix element $\langle 0 | e^{-HT/\hbar} | 0 \rangle$, Fig. 2c contributes to $\langle - | e^{-HT/\hbar} | - \rangle$ and Fig. 2d to $\langle + | e^{-HT/\hbar} | + \rangle$. The instanton-instanton configuration of Fig. 2e contributes to the tunnelling amplitude $\langle + | e^{-HT/\hbar} | - \rangle$ and the anti-instanton-anti-instanton fluctuation of Fig. 2f contributes to the tunnelling amplitude $\langle - | e^{-HT/\hbar} | + \rangle$. In particular, the contribution from instanton-anti-instanton of Fig. 2a can be written as

$$\langle 0 | e^{-HT/\hbar} | 0 \rangle_{x_I, x_{\bar{I}}} = \left(\frac{A_0}{2\pi\hbar} \right)^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_2 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 K(\beta) K_0^{(1)}(\beta) e^{-A[x_I, x_{\bar{I}}]/\hbar} \quad (4.10)$$

where the action for this configuration can be written as

$$\begin{aligned} A[x_I, x_{\bar{I}}] &= \frac{1}{2} \left[\int_{-\frac{T}{2}}^{\alpha} + \int_{\alpha}^{\frac{T}{2}} \right] dt (\dot{x}^2 + s^2) \\ &= \int_{-\frac{T}{2}}^{\beta/2} dt \dot{x}_I^2(t) + \int_{-\beta/2}^{\frac{T}{2}} dt \dot{x}_{\bar{I}}^2(t) \\ &\simeq 2A_0 + A_{int}^{(1)}(\beta) \\ A_{int}(\beta) &= A_0 \left[\tanh \frac{m\beta}{2} + \frac{1}{2} \kappa \hbar^2 \frac{m\beta}{2} - 1 \right] \end{aligned} \quad (4.11)$$

for large separations $\beta \quad t_2 - t_1 \gg 2/m$. As earlier, $K_0^{(1)}(\beta)$ is the fermionic determinant evaluated in the subspace of the relevant fermionic zero modes listed in (4.5). As in the previous section [Eqs. (3.25)-(3.27)], we can approximate this as follows :

$$K_0^{(1)}(\beta) \simeq \frac{m}{2} e^{-m\beta/2} \kappa e h^2 \frac{m\beta}{2} = \frac{1}{A_0} \frac{d}{d\beta} A_{int}^{(1)}(\beta) . \quad (4.12)$$

Again, for large separation, $\beta \rightarrow T \rightarrow \infty$, when the instanton-anti-instanton configuration becomes a stationary point of the action, the function $K(\beta)$ in (4.10) reduces to the product of the instanton and anti-instanton non-zero mode determinantal factors given in Eq. (4.9), $K(T)_{T \rightarrow \infty} K_{x_I} K_{x_{\bar{I}}} = 2/\sqrt{2}m$. On the other hand, for a finite separation, $\beta \ll T$, when instanton-anti-instanton configuration is not a stationary point of the action, the contribution to the functional integral is strongly suppressed, $K(\beta \ll T)_{T \rightarrow \infty} \rightarrow 0$.

Inserting (4.11) and (4.12) into (4.10), and doing integration by parts, we have

$$\begin{aligned} \langle 0 | e^{-HT/\hbar} | 0 \rangle_{x_I x_{\bar{I}}} &\simeq - \left(\frac{A_0}{2\pi\hbar} \right) e^{-2A_0/\hbar} \frac{T}{2} \left[\frac{\hbar}{A_0} K(\beta) e^{-A_{int}(\beta)/\hbar} \right]_{\beta_0}^T \\ &\quad - \frac{\hbar}{A_0} \int_{\beta_0}^T d\beta \frac{dK(\beta)}{d\beta} e^{-A_{int}(\beta)/\hbar} \end{aligned} \quad (4.13)$$

where only the upper limit contributes in the first term for the reasons given above and the second term can be shown to be higher order in $\hbar$ by successive partial integration. Thus

$$\langle 0 | e^{-HT/\hbar} | 0 \rangle_{x_I x_{\bar{I}}} \simeq - \frac{mT}{\sqrt{2}\pi} e^{-2A_0/\hbar} \quad (4.14)$$

By symmetry, the fluctuation depicted in Fig. 2b contributes the same amount

$$\langle 0 | e^{-HT/\hbar} | 0 \rangle_{y_{\bar{I}} y_I} \simeq - \frac{mT}{\sqrt{2}\pi} e^{-2A_0/\hbar} \quad (4.15)$$

so that adding (4.14) and (4.15) yields energy for this state to be

$$\langle 0 | H | 0 \rangle_{x_{\bar{I}} x_I + y_{\bar{I}} y_I} \simeq \frac{\sqrt{2} m \hbar}{\pi} e^{-2A_0/\hbar} \quad (4.16)$$

With regard to the fluctuations depicted in Figs. 2c to 2f, the calculations proceed exactly as above. The action for all these configurations is equal

$$A[y_I, y_{\bar{I}}] = A[x_{\bar{I}}, x_I] = A[y_I, x_{\bar{I}}] = A[x_{\bar{I}}, y_I] \simeq 2A_0 + A_{\text{int}}^{(2)}(\beta)$$

with

$$A_{\text{int}}^{(2)}(\beta) = A_0 \left[\tanh \frac{m\beta}{2} + \frac{1}{2} \tanh^2 \frac{m\beta}{2} - \frac{3}{2} \right] \quad (4.17)$$

for large separations, $\beta \gg 2/m$. The fermionic determinants in the subspace of the relevant zero modes (4.5), in all these cases, are also equal and can be approximated by

$$K_0^{(2)}(\beta) \simeq \frac{m}{2} e^{m\beta/2} \tanh \frac{m\beta}{2} = \frac{1}{A_0} \frac{d}{d\beta} A_{\text{int}}^{(2)}(\beta) \quad (4.18)$$

for large separations. Now a rerun of the arguments presented above yields :

$$\begin{aligned} \langle - | H | - \rangle_{y_I y_{\bar{I}}} &= \langle + | H | + \rangle_{x_{\bar{I}} x_I} = \langle + | H | - \rangle_{y_I x_{\bar{I}}} = \langle - | H | + \rangle_{x_{\bar{I}} y_I} \\ &\simeq \frac{m \hbar}{\sqrt{2} \pi} e^{-2A_0/\hbar} \end{aligned} \quad (4.19)$$

in the lowest order.

Finally, the Hamiltonian matrix for the low-lying states can be written as

$$H = \begin{pmatrix} X & 0 & X \\ 0 & 2X & 0 \\ X & 0 & X \end{pmatrix}, \quad X \approx \frac{m\hbar}{\sqrt{2}\pi} e^{-2A_0/\hbar} \quad (4.20)$$

in the lowest order. This matrix has a zero eigenvalue and two equal non-zero eigenvalues, $(\sqrt{2} m\hbar/\pi)e^{-2A_0/\hbar}$. Hence supersymmetry is not broken in this case in contrast to that of the double well potential. This is in accordance with the arguments of Ref. 4) and 6). Further, the appearance of the next lying states of equal energy is also as dictated by supersymmetry.

5. Concluding remarks

By saturating the functional integral directly with instanton-anti-instanton, two-instanton and two-anti-instanton fluctuations, we have calculated the vacuum energy of a supersymmetric quantum mechanical particle moving in a double well and a triple well potential. The vacuum energy in the former case does get raised from zero due to these quantum fluctuations, implying a spontaneous breakdown of supersymmetry. This vacuum energy has been calculated to the lowest order and the result is in agreement with that obtained in Ref. 6). In the case of a triple well potential, we find that there does survive a zero energy ground state even when these types of fluctuations are included and therefore supersymmetry is not broken. We have also obtained the energy of the supersymmetric pair of the next level states to the lowest order. They have the same energy in accordance with supersymmetry.

The method presented is fairly general and can be used to explore the possible supersymmetry breaking in other models which admit an instanton type of vacuum fluctuations.

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FIGURE CAPTIONS

Fig. 1 An instanton-anti-instanton configuration for the double well case.

Fig. 2 Instanton-anti-instanton, two-instanton and two-anti-instanton configurations for the triple well case.

Fig.1

Fig.2