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LOW ENERGY EXPANSION OF MESON FORM FACTORS^{*)}

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A B S T R A C T

We calculate the corrections to various low energy theorems concerning the behaviour of the pseudoscalar meson form factors near $t=0$. In particular we discuss (i) the Ademollo-Gatto theorem, (ii) Sirlin's relation between the K_0 form factor $f_+^{K\pi}(t)$ and the electromagnetic³ form factors, (iii) the Callan-Treiman relation and (iv) the Dashen-Weinstein relation, which connects the slope λ_0 of $f_0^{K\pi}(t)$ with the ratio F_K/F_π . Furthermore, we point out a remarkable isospin breaking effect which is clearly visible in the experimental rates of the decays $K^+ \rightarrow \pi^0 e^+ \nu$, $K^0 \rightarrow \pi^- e^+ \nu$.

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With the technology developed in a preceding paper (Gasser and Leutwyler 1983) it is straightforward to calculate the form factors of the pseudoscalar mesons to first nonleading order in the low energy expansion. We do not repeat the ingredients of the calculation here, but simply quote the results.

In the first part of the paper we discuss the properties of the vector current matrix elements $\langle M | \bar{q}_Y \gamma^\mu q | M \rangle$. We calculate the invariant form factors up to and including terms of order $t = (p' - p)^2$ and of order M where M is the mass matrix of the light quarks. The explicit analytic expressions for the form factors may be found in section 1, where we restrict ourselves to the isospin symmetry limit $m_u = m_d$. Section 2 contains a discussion of isospin breaking effects in the form factor $f_+^{K^0}(t)$.

In sections 3 - 5 we discuss several low energy theorems concerning the matrix elements of the charge operators and the charge radii. The low energy theorems specify the value of the form factors at special values of the momentum transfer t such as $t = 0$, M_K^2 , $M_K^2 - M_\pi^2$ etc. Since our general low energy representation displays the t -dependence of the form factors explicitly, the low energy theorems are contained in this representation as special cases. [Furthermore, the low energy theorems strictly hold only in the chiral limit $M = 0$. The low energy representation on the other hand accounts for all contributions of order M and hence allows us to correct these theorems for the effects generated by the quark mass matrix.] In section 3 we examine the Ademollo-Gatto theorem in this perspective and section 4 contains a discussion of the momentum dependence of the vector form factors (charge radii, Sirlin's theorem). The t -dependence of the scalar form factors $f_0^{K^0}(t) = f_+^{K^0}(t) + t f_-^{K^0}(t)/(M_K^2 - M_\pi^2)$ is compared with the Dashen-Weinstein and Callan-Treiman relations in section 5. Finally, in section 6, we analyze the low energy properties of the scalar matrix element $\langle \pi | \bar{u}u + \bar{d}d | \pi \rangle$. The slope of this form factor is of interest, because it determines the amount by which the pion decay constant changes as one approaches the $SU(2) \times SU(2)$ limit. A summary and some conclusions are given in section 7.

Before we proceed to describe our own work we briefly comment on the abundant literature on the subject of meson form factors. We have made a serious effort to quote papers which have an immediate bearing on this work. We did not attempt, however, to compile a list of the many references which combine the general low energy properties ensuing from chiral symmetry with specific model assumptions which go beyond QCD (vector meson dominance, σ -models etc.). Also many authors

have tried to calculate corrections to the low energy theorems of current algebra and PCAC by reanalyzing the derivation of these theorems and evaluating the corrections induced by the quark mass matrix in the various terms occurring in the derivation. Almost invariably, this type of analysis misses part of the correction: it is extremely difficult to calculate the corrections with methods that are not based on a Lagrangian, but instead only make use of commutation relations, pole dominance etc. It is here that our general technology visibly pays - the relevant contributions are easily identified as classes of graphs of an effective Lagrangian; the evaluation of these graphs is straightforward.

To complete our bibliography we quote the review papers by Chounet, Gaillard and Gaillard (1972), Gourdin (1974), Zovko (1975), Boal and Graham (1977), Gaillard, Gaillard and Vannucci (1977), Perez-y-Jorba and Renard (1977), Geddes and Graham (1980), Heyn and Lang (1981). These articles contain an extensive list of references to earlier work.

1. Form factors of the vector currents

In the limit $m_u = m_d$, isospin symmetry and charge conjugation invariance reduce the matrix elements of the vector current octet between the eight lightest pseudoscalar mesons to seven independent form factors. It is convenient to identify the independent matrix elements with the electromagnetic form factors, with the $K_{\ell 3}$ form factors and with the analogous $K\pi$ transition matrix elements:

$$\begin{aligned} \langle \pi^+ | j_\mu | \pi^+ \rangle &= (\rho'_\mu + \rho_\mu) F_V^\pi(t) \\ \langle K^+ | j_\mu | K^+ \rangle &= (\rho'_\mu + \rho_\mu) F_V^{K^+}(t) \\ \langle K^0 | j_\mu | K^0 \rangle &= (\rho'_\mu + \rho_\mu) F_V^{K^0}(t) \\ \langle K^+ | \bar{u} \gamma_\mu s | \pi^0 \rangle &= \frac{1}{\sqrt{2}} \{ (\rho'_\mu + \rho_\mu) f_+^{K^+}(t) + (\rho'_\mu - \rho_\mu) f_-^{K^+}(t) \} \\ \langle K^+ | \bar{u} \gamma_\mu s | \eta \rangle &= \sqrt{\frac{2}{3}} \{ (\rho'_\mu + \rho_\mu) f_+^{K^+}(t) + (\rho'_\mu - \rho_\mu) f_-^{K^+}(t) \} \end{aligned} \quad (1.1)$$

The low energy representation of the generating functional to order p^4 given in III determines these form factors up to and including contributions of order $t = (p' - p)^2$ and of order m_{quark} .

The operator j_μ stands for the electromagnetic current carried by the light quarks

$$\begin{aligned} j_\mu &= \frac{2}{3} \bar{\psi} \gamma_\mu \psi - \frac{1}{3} \bar{\psi} \gamma_\mu \psi - \frac{1}{3} \bar{\psi} \gamma_\mu \psi \\ &= \frac{2}{3} \bar{\psi} \gamma_\mu \psi + \frac{1}{3} \bar{\psi} \gamma_\mu \psi \end{aligned} \quad (1.2)$$

The electromagnetic current also receives a contribution $j_\mu^h = \frac{2}{3} \bar{\psi} \gamma_\mu \psi + \dots$ from heavy quarks. The meson matrix elements of j_μ^h are however of order p^5 . At the accuracy of our low energy expansion the electromagnetic form factors therefore coincide with the form factors of the current $j_\mu^3 + \sqrt{3} j_\mu^8 / \sqrt{3}$. [To verify that at low energies the current j_μ^h is suppressed, write the matrix elements $\langle M^h | j_\mu^h | M^p \rangle$ in the form $\langle p' + p | h_{ab}^+ + (p' - p) h_{ab}^- \rangle$. As can be seen, e.g., from the Ward identity for $\langle 0 | T j_\mu^h A_\mu | 0 \rangle$, the meson expectation values of the conserved charge $\int d^3x j_0^h$ vanish: $h_{ab}^+(0) = 0$. The low energy expansion of $h_{ab}^+(t)$ therefore starts with $c^{ab} t + \dots$. Expand c^{ab} in powers of the light quark masses. The leading term is flavour symmetric $c^{ab} = \delta^{ab} c + O(m)$. Bose symmetry however requires the π^0 current matrix elements to be even under $p' \leftrightarrow -p$. Hence h_{ab}^+ vanishes - the constant c is zero. The low energy expansion of the form factor h_{ab}^+ therefore starts with a contribution of order p^4 . Finally, current conservation implies that h_{ab}^- is of the same order as h_{ab}^+ . This completes the proof of the above claim.]

We refer to the quantities $F_V^\pi(t)$, $F_V^K(t)$, $F_V^{K^0}(t)$, $F_+^{K\pi}(t)$, $F_+^{K\eta}(t)$ as vector form factors: they specify the P-wave projection of the crossed channel matrix elements $\langle 0 | \bar{q}_\mu \lambda^a q | M, M' \rangle$. The S-wave projection is described by the scalar form factors

$$\begin{aligned} f_+^{K\pi}(t) &= f_+^{K\pi}(t) + \frac{t}{M_K^2 - M_\pi^2} f_-^{K\pi}(t) \\ f_+^{K\eta}(t) &= f_+^{K\eta}(t) + \frac{t}{M_K^2 - M_\eta^2} f_-^{K\eta}(t) \end{aligned} \quad (1.3)$$

(since the electromagnetic current is conserved, the corresponding form factor $f_-(t)$ vanishes.)

The algebra needed to extract the explicit low energy representation of the form factors from the expression for the generating functional given in III boils down to isospin gymnastics. We have made use of REDUCE (Hearn 1983) to calculate the relevant traces. For the vector form factors the result may be written in the simple form

$$\begin{aligned} F_V^\pi(t) &= 1 + 2 H_{\pi\pi}(t) + H_{KK}(t) \\ F_V^{K^0}(t) &= -H_{\pi\pi}(t) + H_{KK}(t) \\ F_V^{K^+}(t) &= F_V^\pi(t) + F_V^{K^0}(t) \\ f_+^{K\pi}(t) &= f_+^{K\eta}(t) = 1 + \frac{3}{2} H_{K\pi}(t) + \frac{3}{2} H_{K\eta}(t) \end{aligned} \quad (1.4)$$

[The nonanalytic contribution generated by the pion loop in the form factor $F_V^\pi(t)$ was first given by Bég and Zepeda (1972).] The quantity $H(t)$ is related to the invariant functions $M^\pi(t)$, $L(t)$ associated with the one loop integral by

$$H(t) = \frac{1}{F_0^2} \left\{ t M^\pi(t) - L(t) \right\} + \frac{2}{3 F_0^2} L_9^r t \quad (1.5)$$

[See III, section 8 for an explicit expression of these functions. The indices attached to $H(t)$ denote the masses of the mesons running around the loop; to obtain $H_{K\pi}(t)$ e.g., put $M_p = M_K$, $M_Q = M_\pi$].

The constant L_9^r is one of the coupling constants in the effective Lagrangian, renormalized at running scale μ . Since L_9^r is the only unknown occurring in the above representation of the form factors $F_V^\pi(t)$, $F_V^{K^0}(t)$, $F_V^{K^+}(t)$, $f_+^{K\pi}(t)$, $f_+^{K\eta}(t)$, we need experimental information on the slope of one of these form factors to obtain a parameter free low energy representation for all of these quantities. (One easily checks that the scale dependence of the function $M^\pi(t)$ compensates the scale dependence of the constant L_9^r : the function $H(t)$ is independent of μ .)

The analogous low energy representation of the scalar form factors $f_0^{K\pi}(t)$ and $f_0^{K\eta}(t)$ is given by

$$\begin{aligned}
 f_0^{K\pi}(t) = & 1 + \frac{1}{8\pi^2} (5t - 2\sum_{K\pi} - 3\frac{\Delta_{K\pi}^2}{t}) \bar{J}_{K\pi}(t) \\
 & + \frac{1}{24\pi^2} (3t - 2\sum_{K\pi} - \frac{\Delta_{K\pi}^2}{t}) \bar{J}_{K\eta}(t) \\
 & + \frac{t}{4\Delta_{K\pi}} (5\mu_\pi - 2\mu_K - 3\mu_\eta) + \frac{4L_5^r}{\pi^2} t \\
 f_0^{K\eta}(t) = & 1 + \frac{3}{8\pi^2} (3t - 2\sum_{K\pi} - \frac{\Delta_{K\pi}^2}{t}) \bar{J}_{K\pi}(t) \\
 & + \frac{1}{24\pi^2} (-9t + 2M_K^2 + 18M_\eta^2 - \frac{\Delta_{K\pi}^2}{t}) \bar{J}_{K\eta}(t) \\
 & + \frac{9t}{4\Delta_{K\pi}} (\mu_\pi - 2\mu_K + \mu_\eta) + \frac{4L_5^r}{\pi^2} t
 \end{aligned} \quad (1.6)$$

The function $\bar{J}(t)$ is the scalar loop integral, subtracted at $t = 0$: Σ_{pq}, Δ_{pq} and μ_p stand for

$$\begin{aligned}
 \Sigma_{pq} &= M_p^2 + M_q^2 \\
 \Delta_{pq} &= M_p^2 - M_q^2 \\
 \mu_p &= \frac{1}{32\pi^2} \ln \frac{M_p^2}{\pi^2}
 \end{aligned} \quad (1.7)$$

The scale dependence of the low energy constant L_5^r is compensated by the scale dependence of the chiral logarithms μ_π, μ_K and μ_η . Since the constant L_5^r at the same time also determines the ratio F_K/F_π (cf. III.10.12)

$$\frac{F_K}{F_\pi} = 1 + \frac{1}{4} (5\mu_\pi - 2\mu_K - 3\mu_\eta) + \frac{4}{\pi^2} (M_K^2 - M_\pi^2) L_5^r \quad (1.8)$$

we may use the measured value (Leutwyler and Roos 1984) $F_K : F_\pi = 1.22 \pm 0.01$ to obtain a parameter free prediction for the scalar form factors.

2. Isospin breaking in $f_+^{K\pi}(t)$

The values of the electromagnetic form factors at $t = 0$ are fixed by the charges: $F_V^\pi(0) = F_V^{K^+}(0) = 1$, $F_V^0(0) = 0$. In the case of the transition form factors $f_+^{K\pi}(t), f_+^{K\eta}(t)$, the value at the origin is different from 1. The momentum dependence of $f_+^{K\pi}(t)$ can be measured in the decay $K \rightarrow \pi\pi$; the value of the form factor at $t = 0$ multiplied by the element V_{us} of the Kobayashi-Maskawa matrix determines the rate. Since this decay is the most accurate source of information on V_{us} , it is important to have a reliable theoretical prediction for $f_+^{K\pi}(0)$. The representation (1.4) shows that the quark mass expansion of this quantity up to and including terms of order M can be calculated in a parameter free manner, because it does not contain any of the low energy constants. Explicitly, the quantity $H_{pq}(0)$ appearing in (1.4) is given by

$$\begin{aligned}
 H_{pq}(0) &= -\frac{1}{48\pi^2\pi^2} (M_p^2 + M_q^2) h_0\left(\frac{M_p^2}{M_q^2}\right) \\
 h_0(x) &= 1 + \frac{2x}{1-x^2} \ln x
 \end{aligned} \quad (2.1)$$

where M_p and M_q denote the masses of the two mesons running around the loop. The algebraic expression for $f_+^{K\pi}(0)$ which follows from eqs. (1.4) and (2.1) agrees with the result given by Wada (1974). (See also Langacker and Pagels 1973; Bačič and Cornwell 1974.)

Numerically, one finds $f_+^{K\pi}(0) = 0.978$. (Unless stated otherwise, we use the numerical values $M_\pi = 135$ MeV, $M_K = 495$ MeV, $M_\eta = 548.8$ MeV and $F_0 \approx F_\pi = 93.3$ MeV). The representation (1.4) however only holds in the isospin symmetry limit $m_u = m_d$. To include isospin breaking effects (see also Isgur et al. 1979; Leutwyler and Roos 1984) we need to distinguish two different form factors, corresponding to the transition $K^0 \rightarrow \pi^-$ and $K^+ \rightarrow \pi^0$ respectively.

Consider first the form factor $f_+^{K^0\pi^-}(t)$. In this case the mass difference $m_u - m_d$ modifies the representation (1.4) as follows

$$f_+^{K^0\pi^-}(t) = 1 + H_{K^0\pi}(t) + \frac{1}{2} H_{K^+\pi}(t) + \frac{3}{2} H_{K^+\eta}(t) + \varepsilon \sqrt{3} \{ H_{K\pi}(t) - H_{K\eta}(t) \} + O(p^4, (m_u - m_d)^2)$$

$$\varepsilon = \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \bar{m}} = (1.00 \pm 0.05) \cdot 10^{-2} = 0.57^\circ \quad (2.2)$$

[Simultaneous expansion in m_q and in t at fixed ratio m_q/t ; $O(p^4)$ stands for terms of order t^2 , $t m_q$ or m_q^2 . In addition, we have treated $m_u - m_d$ as small in comparison to $m_s - \bar{m}$: Terms proportional to $(m_u - m_d)^2$ are neglected. In particular, since $M_{\pi^+ - \pi^0}^2$ is of order $(m_u - m_d)^2$ we do not distinguish these two masses.]

The analogous expression for $f_+^{K^0\pi^-}(t)$ is more complicated because of $\pi^0\eta$ mixing. The expression for the ratio of the two form factors is given by

$$\frac{f_+^{K^0\pi^-}(t)}{f_+^{K^+\pi^-}(t)} = 1 + \frac{1}{2} \sqrt{3} (\varepsilon_1 + \varepsilon_2) + O(p^4, (m_u - m_d)^2) \quad (2.3)$$

where ε_1 and ε_2 are given in III in terms of the physical masses and decay constants [numerically, $\varepsilon_1 = 1.37 \cdot 10^{-2} = 0.78^\circ$, $\varepsilon_2 = 1.11 \cdot 10^{-2} = 0.63^\circ$]. Evaluating the expressions (2.2) and (2.3) at $t = 0$ we obtain

$$f_+(0) = 0.977, \quad f_+^{K^0\pi^-}(0)/f_+^{K^+\pi^-}(0) = 1.022 \quad (2.4)$$

[We have corrected the value of the charged kaon mass M_{K^+} for electromagnetic effects and have used the values $(M_{K^+})_{QCD} = 492.4$ MeV, $(M_{K^0})_{QCD} = 497.7$ MeV, see (I.12.13, I.12.14).]

The difference $1 - f_+^{K^0\pi^-}(0)$ is practically unaffected by $m_u - m_d$, it is due to $m_s - \bar{m}$. The isospin breaking which manifests itself in $f_+^{K^+\pi^0}(0) \neq f_+^{K^0\pi^-}(0)$ is of the same size as the SU(3) breaking in $f_+^{K^0\pi^-}(0)$: In the following we show that this remarkable phenomenon is due to the fact that on the one hand isospin breaking is enhanced by a small energy denominator, whereas, on the other hand, SU(3) breaking is suppressed by a nonrenormalization theorem.

The enhancement of the perturbation generated by $m_u - m_d$ arises as follows. In the chiral limit the vector current is conserved; the current matrix elements $\langle \alpha | v_\mu^b | c \rangle$ are given by $i(p_\mu^c + p_\mu^a) f_{abc} F_V(t)$, where $F_V(t)$ is a common form factor.

In this limit there is however no physical distinction between π^0 and η . The proper limiting states are selected by the perturbation $\bar{q}Mq$. If $m_u \neq m_d$ the states $|\pi^0\rangle$ and $|\eta\rangle$ do not coincide with the octet states $|3\rangle$ and $|8\rangle$ respectively, but differ from these by a rotation through the angle $\varepsilon = \frac{1}{2} \arctg \sqrt{3} (m_d - m_u)/(2 m_s - m_u - m_d)$. The form factor $f_+^{K^+\pi^0}(t)$ is given by $(\cos \varepsilon + \sqrt{3} \sin \varepsilon) F_V(t)$, while there is no mixing in $f_+^{K^0\pi^-}(t) = F_V(t)$. In the chiral limit the ratio is therefore given by

$$\frac{f_+^{K^+\pi^0}(t)}{f_+^{K^0\pi^-}(t)} = \cos \varepsilon + \sqrt{3} \sin \varepsilon = 1 + \sqrt{3} \varepsilon + O(\varepsilon^2) \quad (2.5)$$

The matrix element $\langle 3 | \bar{q}Mq | 8 \rangle$ of the perturbation is of order $m_u - m_d$. The perturbation is enhanced by an energy denominator $M_\eta^2 - M_\pi^2 \sim m_s - \bar{m}$, which vanishes in the chiral limit. The net result is proportional to the ratio $\varepsilon \sim (m_d - m_u)/(m_s - \bar{m})$.

The representation (2.3) for the ratio of form factors includes the corrections of order $\varepsilon \bar{m}$ and εm_s to the leading order formula (2.5). In particular it accounts for terms of order $m_d - m_u \sim \varepsilon(m_s - \bar{m})$. Remarkably, these contributions can be expressed in a parameter free manner in terms of the physical masses. Note, in particular, that the ratio $f_+^{K^+\pi^0}/f_+^{K^0\pi^-}$ does not receive any contribution from $\pi^0\eta$ mixing at this order of the low energy expansion. The η affects this ratio only indirectly: the $\pi^0\eta$ mixing angles $\varepsilon_1, \varepsilon_2$ are sensitive to the mass of the η , which in turn receives a contribution from $\eta\eta'$ mixing. (In our framework this contribution is described by the low energy constant L_7).

Numerically, the leading order formula predicts a ratio of $1 + \sqrt{3} \varepsilon = 1.017$; the contributions of order $\varepsilon \bar{m}$ increase this to 1.022.

The data on the decays $K^+ \rightarrow \pi^0 e^+ \nu$ and $K^0 \rightarrow \pi^- e^+ \nu$ show clear evidence for the presence of the isospin breaking discussed above. Dividing the rates (Particle Data Group 1982) by the relevant phase space integrals one finds

$$\left| \frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} \right|^2 = 1.057 \pm 0.019$$

to be compared with the calculated value $(1.022)^2 = 1.044$.

Note that the matrix element $\langle \pi^+ | \bar{u} \gamma_\mu d | \pi^0 \rangle$, which determines the decay $\pi^+ \rightarrow \pi^0 \nu$, does not receive isospin breaking contributions of first order in $m_u - m_d$. In this case the value of the form factor at $t = 0$ is protected from asymmetries by an $SU(2) \times SU(2)$ nonrenormalization theorem (see II).

3. Ademollo-Gatto theorem

The reason why $f_+^{K^*}(0)$ remains so close to the symmetry limit was identified long ago by Behrends and Sirlin (1960) and by Ademollo and Gatto (1964): if one expands the quantity $f_+^{K^*}(0)$ in powers of the quark mass differences around the symmetry limit ($m_u = m_d = m_s \neq 0$), the terms linear in these differences vanish. This property is easily established by considering the commutation relation

$$[Q^{\bar{u}s}, Q^{\bar{s}u}] = Q^{\bar{u}u} - \bar{s}s \quad (3.1)$$

of the null plane charges (Leutwyler 1968, 1969)

$$Q^{\bar{u}s} = \int_{x^0=x^3} d\sigma_\mu \bar{u} \gamma^\mu s \quad (3.2)$$

The relation (3.1) implies the sum rule (Furlan, Lannoy, Rossetti and Segrè 1965)

$$\sum_n |\langle K^0 | Q^{\bar{u}s} | n \rangle|^2 - \sum_n |\langle K^0 | Q^{\bar{s}u} | n \rangle|^2 = 1 \quad (3.3)$$

The contribution of the π^- intermediate state to the first term on the left hand side is given by the value of the form factor at the origin

$$\langle K^0 | Q^{\bar{u}s} | \pi^- \rangle = f_+^{K^* \pi^-}(0) \quad (3.4)$$

For the other intermediate states current "conservation"

$$\partial_\mu (\bar{u} \gamma^\mu s) = i(m_u - m_s) \bar{u} s \quad (3.5)$$

implies

$$\langle K^0 | Q^{\bar{u}s} | n \rangle = \frac{m_u - m_s}{M_{K^0}^2 - M_n^2} \langle K^0 | \bar{u} s | n \rangle \quad (3.6)$$

In the sum rule (3.3) states for which $M_{K^0}^2 - M_n^2$ remains finite in the limit $m_u = m_d = m_s \neq 0$ therefore produce a contribution proportional to $(m_s - m_u)^2$. In this limit (not in the chiral limit) the pseudoscalar octet is isolated from other intermediate states such as three pion states, A_1 or π' ; only the π^- becomes degenerate with the K^0 . Hence we obtain $f_+^{K^* \pi^-}(t) = 1 + O[(m_s - m_u)^2]$: the symmetry breaking is suppressed.

The low energy representation (2.2) is in accord with this general property: The function $h_0(x)$ which appears in the explicit expression (2.1) for $H_{pq}(0)$ is proportional to $(x-1)^2 \sim (M_p^2 - M_q^2)^2$ near $x = 1$. [This can easily be seen from the property $h_0(x) = h_0(\frac{1}{x})$, $h_0(1) = 0$.] As it is required by the nonrenormalization theorem, the terms in (2.2) sum up to a contribution of order $(m_s - m_u)^2$.

As was pointed out by Langacker and Pagels (1973), the coefficient of $(m_s - m_u)^2$ is however of order $1/m_s$ - in the limit $m_s \gg m_u$, m_d the quantity $f_+^{K^* \pi^-}(0) - 1$ is of order $(m_s - m_u)^2/m_s \sim m_s$. The behaviour of $h(x)$ near $x = 1$, which is governed by the Ademollo-Gatto theorem, is not relevant for the value of $h(M_\pi^2/M_K^2)$ (Wada 1974). To illustrate, we plot the function $h_0(x)$ in Fig. 1. As it is evident from this figure the oscillating curve $\frac{1}{6}(x-1)^2$ (dashed line) fails to represent $h_0(x)$ near $x = M_\pi^2/M_K^2 \sim 0.07$.

In the language of the sum rule (3.3) the function $H(t)$ is generated by three meson intermediate states. In the chiral limit the corresponding energy denominator $M_{K^0}^2 - M_n^2$ vanishes in a corner of phase space and converts the factor $(m_s - m_u)^2$ into $(m_s - m_u)^2/m_s$. The numerical value (2.4) shows that the contribution of these intermediate states, which algebraically dominate the correction, is small: in the vector form factors, the coefficient of the chiral logarithm happens to be small (compare the corresponding contribution to the charge radius, discussed in section 4).

The fact that the terms of order M are small, does not guarantee that the higher order contributions, which are outside the reach of our low energy representation, are even smaller. Estimates for the size of the second order contributions may be found in (Leutwyler and Roos 1984). These con-

tributions arise from intermediate states which do not become massless in the chiral limit (or from states containing more than three Goldstone bosons which do become massless, but are suppressed by phase space). The value of $f_+^{K^0\pi^-}(t)$ obtained on the basis of these estimates is 0.961 ± 0.008 .

In contrast to $f_+^{K^0\pi^-}(0)$ the quantity $f_+^{K^+\pi^0}(0)$ is not protected from the asymmetries in the quark mass term if $m_u \neq m_d$. The sum rule analogous to (3.3) receives a contribution from two octet intermediate states:

$$\frac{1}{4} |f_+^{K^+\pi^0}(0)|^2 + \frac{3}{4} |f_+^{K^+\eta}(0)|^2 = 1 - (m_s - m_u)^2 D \quad (3.7)$$

$$D = \sum_{n \neq \text{octet}} (M_{K^+}^2 - M_n^2)^{-2} \{ | \langle K^+ | \bar{u}s | n \rangle |^2 - | \langle K^+ | \bar{s}u | n \rangle |^2 \}$$

Consequently only the sum $|f_+^{K^+\pi^0}|^2 + 3 |f_+^{K^+\eta}|^2$ obeys an Ademollo-Gatto theorem. In the chiral limit the second term is given by

$$f_+^{K^+\eta}(0) = \cos \varepsilon - \frac{1}{\sqrt{3}} \sin \varepsilon \quad (3.8)$$

The two contributions indeed add up to one for any value of the mixing angle.

4. Momentum dependence of the vector form factors: Charge radii, Sirlin's theorem

It is instructive to compare the low energy representation (1.4) for the pion form factor with the Omnès formula (Omnès 1958)

$$\begin{aligned} F_V^\pi(t) &= \exp \Delta_1(t) \\ \Delta_1(t) &= \frac{t}{\pi} \int_{4M_\pi^2}^{\infty} ds \frac{\delta_1'(s)}{s(s-t)} \end{aligned} \quad (4.1)$$

where $\delta_1'(s)$ is the $I = 1$ P-wave $\pi\pi$ phase shift. The cut in the function $2 H_{\pi\pi}(t)$ which starts at $t = 4 M_\pi^2$ indeed coincides with the cut in $\Delta_1(t)$ to the extent that the phase shift $\delta_1'(s)$ is correctly described by the current algebra formula

$$\delta_1'(s) = \frac{1}{96\pi^2} (s - 4M_\pi^2) \left\{ 1 - 4M_\pi^2/s \right\}^{1/2} \quad (4.2)$$

which follows from $A^{\pi\pi}(s, t, u) = (s - M_\pi^2)/t^2$ (Weinberg 1966). With the low energy representation (4.2) for the phase shift the integral (4.1) does not converge - the low energy representation for $H_{\pi\pi}(t)$ involves a subtraction with an unknown low energy constant L_9^r .

In addition to the term describing the final state $\pi\pi$ interaction, the unitarity correction to the pion form factor also contains a contribution from $K\bar{K}$ intermediate states in the virtual transition $\gamma \rightarrow K\bar{K} \rightarrow \pi\pi$. The corresponding cut starting at $t = 4 M_K^2$ is determined by the lowest order $K\pi$ scattering amplitude.

The low energy representation (1.4) only accounts for the singularities generated by the Goldstone bosons. The presence of other singularities, such as those associated with the bound states ρ , K^* limits the range of validity of this representation. In the low energy expansion to order p^2 the contribution $\sim (1 - t/M_\rho^2)^{-1}$ from the ρ -meson pole to the pion form factor, e.g., is accounted for only to the extent that this contribution may be approximated by the linear function $1 + t/M_\rho^2$. This is the case (to an accuracy better than 5%) for space-like values of t in the range $\sqrt{-t} < 350$ MeV. In the case of the $K_{\ell 3}$ form factor $f_+^{K\pi}(t)$ the physical region is the timelike interval $M_\ell^2 < t < (M_K - M_\pi)^2$; the relevant scale is the mass of the $K^*(892)$. Since in the physical region $1 + t/M_{K^*}^2$ differs from $(1 - t/M_{K^*}^2)^{-1}$ by less than 3% we see no reason to doubt the validity of the low energy representation over the entire Dalitz plot in this case. [Note, however, that the slope of the function $(1 - t/M_{K^*}^2)^{-1}$ changes substantially as t runs from M_ℓ^2 to $(M_K - M_\pi)^2$. As emphasized by Yamada (1980) this curvature may play a significant role in the experimental determination of the slope of $f_+^{K\pi}(t)$ at $t = 0$.]

In the spacelike interval $\sqrt{-t} < 350$ MeV the low energy representation (1.4) for the electromagnetic form factors is very well approximated by the first two terms in the Taylor series expansion

$$\begin{aligned}
F_V^\pi(t) &= 1 + \frac{1}{6} \langle r^2 \rangle_V^\pi \cdot t + \dots \\
F_V^{K^+}(t) &= 1 + \frac{1}{6} \langle r^2 \rangle_V^{K^+} \cdot t + \dots \\
F_V^{K^0}(t) &= \frac{1}{6} \langle r^2 \rangle_V^{K^0} \cdot t + \dots
\end{aligned}
\quad (4.3)$$

Likewise, the linear approximation

$$F_+^{K^\pi}(t) = F_+^{K^\pi}(0) \left\{ 1 + \frac{1}{6} \langle r^2 \rangle_+^{K^\pi} t + \dots \right\} \quad (4.4)$$

reproduces the low energy representation (1.4) for the form factor $F_+^{K^\pi}(t)$ very well (see Fig. 2), in agreement with the observed Dalitz plot distribution, which is consistent with a form factor linear in t . The charge radii are easily worked out with the formula

$$\begin{aligned}
\frac{dH}{dt} \Big|_{t=0} &= \frac{2}{3\pi^2} \left\{ L_9 - \frac{1}{128\pi^2} \ln \frac{M_{PQ}}{\mu^2} \right\} - \frac{1}{192\pi^2\pi^2} h_1 \left(\frac{M_P^2}{M_Q^2} \right) \\
h_1(x) &= \frac{1}{2} \frac{(x^3 - 3x^2 - 3x + 1)}{(x-1)^3} \ln x + \frac{1}{2} \left(\frac{x+1}{x-1} \right)^2 - \frac{1}{3} \quad (4.5)
\end{aligned}$$

which follows from the explicit expressions for the Taylor expansion of the invariant functions given in the Appendix of ref. III. (Note that $h_1(x) = h_1(1/x)$; $h_1(1) = 1$). We obtain

$$\begin{aligned}
\langle r^2 \rangle_V^\pi &= \frac{12L_9}{\pi^2} - \frac{1}{32\pi^2\pi^2} \left\{ 2 \ln \frac{M_\pi^2}{\mu^2} + \ln \frac{M_K^2}{\mu^2} + 3 \right\} \\
\langle r^2 \rangle_V^{K^0} &= -\frac{1}{16\pi^2\pi^2} \ln \frac{M_K}{M_\pi} \\
\langle r^2 \rangle_V^{K^+} &= \langle r^2 \rangle_V^\pi + \langle r^2 \rangle_V^{K^0} \\
\langle r^2 \rangle_V^{K^\pi} &= \langle r^2 \rangle_V^\pi - \frac{1}{64\pi^2\pi^2} \left\{ 3h_1 \left(\frac{M_\pi^2}{M_K^2} \right) + 3h_1 \left(\frac{M_K^2}{M_\pi^2} \right) + \frac{5}{2} \ln \frac{M_K^2}{M_\pi^2} \right. \\
&\quad \left. + \frac{3}{2} \ln \frac{M_\pi^2}{M_K^2} - 6 \right\} \quad (4.6)
\end{aligned}$$

To evaluate these relations numerically, we use the slope λ_+ of the form factor $F_+^{K^\pi}(t)$ measured in K^0_{e3} decay (Particle Data Group 1982)

$$\lambda_+ = 0.0300 \pm 0.0016 \quad (4.7)$$

as an input. [Alternatively, as it was done in III, one may use the charge radius of the pion as an input and predict the slope λ_+ . In terms of the low energy constant L_9 the input (4.7) implies $L_9 = (7.1 \pm 0.4) \cdot 10^{-3}$, whereas using the value (4.10) for the pion charge radius, we obtain $L_9 = (7.4 \pm 0.7) \cdot 10^{-3}$.] In the notation used here this value is equivalent to

$$\langle r^2 \rangle_V^{K^\pi} = \frac{6\lambda_+}{M_{\pi^+}^2} = 0.36 \pm 0.02 \text{ fm}^2 \quad (4.8)$$

Since the relations (4.6) only hold to leading order in the quark mass expansion, we may identify the constant F_0 with the physical decay constant $F_\pi = 93.3 \text{ MeV}$. We then obtain the following predictions for the electromagnetic charge radii:

$$\begin{aligned}
\langle r^2 \rangle_V^\pi &= 0.42 \pm 0.04 \text{ fm}^2 \\
\langle r^2 \rangle_V^{K^+} &= 0.38 \pm 0.03 \text{ fm}^2 \\
\langle r^2 \rangle_V^{K^0} &= -0.04 \pm 0.03 \text{ fm}^2
\end{aligned}
\quad (4.9)$$

to be compared with the observed values

$$\begin{aligned}
\langle r^2 \rangle_V^\pi &= 0.439 \pm 0.03 \text{ fm}^2 \quad (\text{Dally et al. 1982}) \\
\langle r^2 \rangle_V^{K^+} &= 0.28 \pm 0.07 \text{ fm}^2 \quad (\text{Dally et al. 1980}) \\
\langle r^2 \rangle_V^{K^0} &= -0.054 \pm 0.026 \text{ fm}^2 \quad (\text{Molzon et al. 1978})
\end{aligned}$$

(4.10)

The asymmetries in the charge radii may be understood on the basis of the following simple picture. One assumes that the mean square radius of the strange quark in the K^+ is smaller than the mean square radius of the u quark by a factor $(1 - \epsilon)$. The quantity $\langle r^2 \rangle_V^{K^+}$ should then be smaller than $\langle r^2 \rangle_V^\pi$ by the factor $(1 - \frac{1}{3}\epsilon)$ and $\langle r^2 \rangle_V^{K^0}$ should be negative, of order $-\frac{1}{3}\epsilon \langle r^2 \rangle_V^\pi$. The same crude estimate suggests that the matrix element $\langle r^2 \rangle_V^{K^\pi}$, which may be visualized as the mean square radius weighted with the product of the kaon and pion wave functions, should be smaller than $\langle r^2 \rangle_V^\pi$ by $(1 - \frac{1}{2}\epsilon)$. The asymmetries predicted by the low

energy representation are correctly reproduced by this simple picture with $\epsilon = 0.27$. The low energy representation accounts for the asymmetries in the charge distribution due to the exponential tail of the wave functions $\sim \exp(-M_\pi r)$, $\exp(-M_\rho r)$. In the charge radii these unitarity corrections are of order $\log M$. The representation neglects "ordinary" symmetry breaking effects of order $m_S - \bar{m}$. Since the chiral logarithms turn out to have a small coefficient in this case we cannot exclude that higher order terms in the quark mass expansion generate corrections of comparable magnitude. The error bars given in (4.9) correspond to an uncertainty in these residual corrections of order $\epsilon = \pm 0.2$.

A low energy theorem due to Sirin (1970, 1979) states that up to corrections of second order in the mass difference $m_S - \bar{m}$ (and up to contributions from the heavy quark part of the electromagnetic current) the $K_{\ell 3}$ form factor $f_+^{K\pi}(t)$ is related to the electromagnetic form factors by

$$f_+^{K\pi}(t) = \frac{1}{2} \left\{ \pi_V^\pi(t) + \pi_V^{K^*}(t) \right\} + \pi_V^{K^0}(t) + O[(m_S - \bar{m})^2] \quad (4.11)$$

for all t . [The relation generalizes the Ademollo-Gatto theorem for $f_+^{K\pi}(0)$ to $t \neq 0$]. One easily verifies that the low energy representation (1.4) of the vector form factors indeed satisfies this theorem: in the vicinity of the symmetry limit $m_S = \bar{m}$ the difference Δ of the left- and right-hand sides in (4.11) is quadratic in $m_S - \bar{m}$. The asymmetry is however enhanced by small energy denominators due to the infrared singularities of chiral perturbation theory: the correction not only contains terms of order $(m_S - \bar{m})^2/m_S$, but even contributions of order $t(m_S - \bar{m})^2/m_S^2$. Indeed Δ remains different from zero even in the chiral limit, if one approaches this limit at a fixed ratio $\bar{m} : m_S$.

It is straightforward to calculate the infrared singular piece of the asymmetry. We quote the corresponding improved low energy theorem for the charge radii which we write in the form

$$\langle r^2 \rangle_V^{K^*} = 2 \langle r^2 \rangle_V^{K\pi} - \langle r^2 \rangle_V^{K^0} - 2 \langle r^2 \rangle_V^{K^0} + \delta_1 + O[(m_S - \bar{m})^3/m_S^3] \quad (4.12)$$

$$\delta_1 = \frac{3}{32\pi^2 F_0^2} \left\{ h_1 \left(\frac{M_\pi^2}{M_K^2} \right) + h_1 \left(\frac{M_\eta^2}{M_K^2} \right) + \frac{1}{6} \ln \frac{M_\pi^2}{M_K^2} + \frac{1}{2} \ln \frac{M_\eta^2}{M_K^2} - 2 \right\} \quad (4.12)$$

In accordance with Sirin's theorem, the quantity δ_1 is of order $(m_S - \bar{m})^2/m_S^2$. Numerically, it is small: $\delta_1 = 0.01 \text{ fm}^2$.

Inserting the experimental values (4.8), (4.10) this relation predicts

$$\langle r^2 \rangle_V^{K^*} = 0.40 \pm 0.07 \text{ fm}^2 \quad (4.13)$$

In agreement with other theoretical investigations (Zovko 1975; Heyn and Lang 1981) we conclude that the measured charge radii $\langle r^2 \rangle_V^{K^*}$, $\langle r^2 \rangle_V^{K^0}$, $\langle r^2 \rangle_V^{K\pi}$ are consistent with the low energy predictions. As emphasized previously by Blatnik, Stahov and Lang (1979) the experimental value of the K^+ charge radius is however too small by 1/2 to 2 standard deviations - it is difficult to understand how $\langle r^2 \rangle_V^{K^*}$ could be smaller than $\langle r^2 \rangle_V^{K\pi}$.

5. Momentum dependence of $f_0^{K\pi}(t)$: Dashen-Weinstein and Callan-Treiman relations

In the physical region of $K_{\ell 3}$ decay the low energy representation (1.6) for the scalar form factor is approximated by the linear formula

$$f_0^{K\pi}(t) = f_+^{K\pi}(0) \left\{ 1 + \frac{1}{6} \langle r^2 \rangle_S^{K\pi} t + \dots \right\} \quad (5.1)$$

to within an accuracy of order 1 %. (See Fig. 2). Since the scalar bound states [$K(1350)$] occur at an even higher mass than in the vector channel, the curvature generated by the higher order terms is also expected to be negligible in the physical region of the decay - the form factor should be linear in t to a very good approximation.

The slope is determined by the constant L_5 , which in turn is fixed by F_K/F_π . Evaluating the derivative of the right-hand side in (1.6) at $t = 0$ and using the relation (1.8) we obtain

$$\begin{aligned} \langle r^2 \rangle_S^{K\pi} &= \frac{6}{M_K^2 - M_\pi^2} \left(\frac{F_K}{F_\pi} - 1 \right) + \delta_2 + O(m) \\ \delta_2 &= - \frac{1}{192\pi^2 F_0^2} \left\{ 15 h_2 \left(\frac{M_\pi^2}{M_K^2} \right) + \frac{19 M_K^2 + 3 M_\eta^2}{M_K^2 + M_\eta^2} h_2 \left(\frac{M_\eta^2}{M_K^2} \right) - 18 \right\} \end{aligned} \quad (5.2)$$

$$h_2(x) = \frac{3}{2} \left(\frac{1+x}{1-x} \right)^2 + \frac{3x(1+x)}{(1-x)^3} \ln x$$

$$h_2(x) = h_2\left(\frac{1}{x}\right); \quad h_2(1) = 1 \quad (5.2)$$

This prediction is a modified version of the Dasher-Weinstein relation (Dasher and Weinstein 1969), which results if the

nonanalytic contribution δ_2 is dropped. [Note that up to terms of order M^2 the combination $F_K/F_\pi - F_\pi/F_K$ coincides with $2(F_K/F_\pi - 1)$.] Dasher, Li, Pagels and Weinstein (1972) were the first to point out that the low energy singularities generated by the Goldstone bosons affect this relation. The modified relation is formulated as a prediction for the slope of $f_0(t)$ at the unphysical point $t_1 = M_K^2 + M_\pi^2$. Their expression for this slope however has two shortcomings: (i) It does not account for all corrections of order M . (ii) It is evident from Fig. 3 that the slope at t_1 differs substantially from the slope in the physical region of the decay (Auvil and Pritchett 1975; Kenny 1977).

The $K_{\ell 3}$ form factors neatly illustrate the power of our method as compared to direct calculations based on current algebra and PCAC. In our approach the problem is reduced to the straightforward evaluation of a few one loop integrals. The method guarantees that, at the given order in the low energy expansion, the constraints imposed by chiral symmetry on the low energy structure of the theory are exhausted - it is futile to look for further relations. In contrast, direct attempts at extracting low energy information by studying suitable kinematical configurations, for which chiral symmetry implies a low energy theorem, meet two difficulties: (i) The low energy theorem only holds in the chiral limit. It is a very subtle matter to calculate the corrections generated by the quark masses directly from their effect on the various terms which occur in the derivation of the low energy theorem. Most attempts in this direction have failed, because some of the contributions were missed. (ii) The low energy theorem usually concerns the properties of the on-shell amplitude at an unphysical point. To compare with experimental data one needs to extrapolate outside the physical region, a procedure which is not controlled by the low energy theorem and which often leads to substantial uncertainties. Furthermore, the current algebra approach does not tell whether the information obtained is all there is, or whether the analysis of one of the current commutators at a different kinematical point might lead to

additional information. [In fact various, closely related low energy theorems for $K_{\ell 3}$ form factors were proposed by Mathur and Okubo (1970), Mathur and Yang (1972) and by Gaillard (1969); for a review see Geddes and Graham (1980).]

We now return to the prediction (5.2). Algebraically, the correction δ_2 is of the same order in the low energy expansion as the term involving $F_K/F_\pi - 1$. Numerically, the correction is however small: δ_2 reduces the prediction by 11%. With $F_K/F_\pi = 1.22 \pm 0.01$ the low energy theorem (5.2) implies

$$\langle r^2 \rangle_S^{K\pi} = 0.20 \pm 0.05 \text{ fm}^2 \quad (5.3)$$

where the error is an estimate of the uncertainties due to higher order contributions. Expressed in terms of the slope parameter λ_0 the prediction (5.3) is equivalent to

$$\lambda_0 \equiv \frac{1}{6} \langle M_{\pi^+}^2 \rangle \langle r^2 \rangle_S^{K\pi} = 0.017 \pm 0.004 \quad (5.4)$$

The experimental situation concerning the value of the slope parameter is far from clear. Some years ago a high statistics experiment (Donaldson et al. 1974) confirmed the theoretical expectations with the value $\lambda_0 = 0.019 \pm 0.004$. More recent experiments however disagree with this result [Cho et al. (1980) report $\lambda_0 = 0.046 \pm 0.006$ which differs from (5.4) by more than four standard deviations].

In the formulation of Dasher and Weinstein, the Callan-Treiman relation (Callan and Treiman 1966; Mathur, Okubo and Pandit 1966; Suzuki 1966) states that the scalar form factor evaluated at $t = M_K^2 - M_\pi^2$ differs from F_K/F_π only by terms of order m_u, m_d : the quantity

$$\Delta_{CT} = f_0^{K\pi}(M_K^2 - M_\pi^2) - \frac{F_K}{F_\pi} \quad (5.5)$$

is of order $\hat{m}$. This is borne out by the low energy representation (1.6) which leads to

$$\Delta_{CT} = -\frac{M_\pi^2}{2\pi^2} \left\{ \sqrt{K\pi} (M_K^2 - M_\pi^2) + \frac{1}{3} \sqrt{K\eta} (M_K^2 - M_\pi^2) \right\} + O(\hat{m}m) \quad (5.6)$$

Numerically, $\Delta_{CT} = 3.5 \cdot 10^{-3}$. The Callan-Treiman relation should therefore hold to a very high degree of accuracy. If the form factor is linear from $t = 0$ to $t = M_K^2 - M_\pi^2$ then the slope must be very close to

$$\lambda_0 = \frac{1}{M_K^2 - M_\pi^2} \left(\frac{F_K}{F_\pi} - 1 \right) = 0.019 (M_{\pi^+})^2 \quad (5.7)$$

in agreement with (5.4) and with the experimental result of Donaldson et al., but in disagreement with the value found by Cho et al. We see no way to reconcile the value $\lambda_0 = 0.046$ with chiral symmetry.

It is instructive to interchange the role of the pion and of the kaon and to consider the soft kaon analog of the Callan-Treiman relation (Dehne 1966)

$$f_+^{K\pi} (M_\pi^2 - M_K^2) - f_-^{K\pi} (M_\pi^2 - M_K^2) = \frac{F_\pi}{F_K} + O(M_K^2) \quad (5.8)$$

The difference of the left- and right-hand sides is given by

$$\begin{aligned} \Delta_{CT} &= f_+^{K\pi} (M_\pi^2 - M_K^2) - \frac{F_\pi}{F_K} \\ &= -\frac{M_K^2}{2F_0^2} \left\{ \bar{J}_{K\theta} (M_\pi^2 - M_K^2) + \frac{1}{3} \bar{J}_{K\eta} (M_\pi^2 - M_K^2) \right\} + O(M_S^2) \end{aligned} \quad (5.9)$$

Numerically, $\Delta_{CT} = 0.03$, larger than Δ_{CT} roughly by the factor M_K^2/M_π^2 , but still unusually small for a first order $SU(3) \times SU(3)$ breaking effect.

6. Form factors of the scalar densities; F_π, F, F_0

The matrix elements of the scalar operators $\bar{q}q$ may be extracted from the low energy representation for the generating functional by studying the response to an external scalar field.

Isospin and charge conjugation invariance reduce the matrix elements of the strangeness conserving densities $\bar{u}u, \bar{d}d, \bar{s}s$ to eight independent form factors. The strangeness changing scalar operators $\bar{u}s, \bar{s}u, \bar{d}u$ are propor-

tional to the divergence of the corresponding vector currents; their matrix elements may therefore be expressed in terms of the scalar form factors $f_0^{K\pi}(t)$, $f_0^{Kn}(t)$ given in section 1.

As was shown in II the slope of the scalar form factor $\langle \pi | \bar{u}u + \bar{d}d | \pi \rangle$ is of special interest, because it is related to the difference between the physical value of the pion decay constant and the value of this quantity in the limit $m_u = m_d = 0$. The other form factors do not appear to be of immediate interest and we do therefore not list their explicit low energy representations here.

For the pion, there are two independent form factors, describing the matrix elements of the octet and singlet currents respectively:

$$\begin{aligned} \langle \pi^0 | \bar{u}u + \bar{d}d - 2\bar{s}s | \pi^0 \rangle &= \delta^{iK} F^8(t) \\ \langle \pi^0 | \bar{u}u + \bar{d}d + \bar{s}s | \pi^0 \rangle &= \delta^{iK} F^0(t) \end{aligned} \quad (6.1)$$

The low energy representation of these quantities is

$$F^8(t) = F^8(0) \{ 1 + f(t) + O(p^4) \}$$

$$F^0(t) = F^0(0) \{ 1 + f(t) + \tilde{f}(t) + O(p^4) \}$$

$$\begin{aligned} f(t) &= \frac{1}{2F_0^2} (2t - M_\pi^2) \bar{J}_{\pi\pi}(t) - \frac{t}{4F_0^2} \bar{J}_{K\pi}(t) - \frac{M_\pi^2}{6F_0^2} \bar{J}_{\eta\eta}(t) \\ &\quad + \frac{4t}{F_0^2} \left\{ L_5^r - \frac{1}{256\pi^2} \left(4 \ln \frac{M_\pi^2}{\mu^2} - \ln \frac{M_K^2}{\mu^2} + 3 \right) \right\} \end{aligned}$$

$$\begin{aligned} \tilde{f}(t) &= \frac{3t}{4F_0^2} \bar{J}_{K\pi}(t) + \frac{M_\pi^2}{3F_0^2} \bar{J}_{\eta\eta}(t) \\ &\quad + \frac{12t}{F_0^2} \left\{ L_4^r - \frac{1}{256\pi^2} \left(\ln \frac{M_K^2}{\mu^2} + 1 \right) \right\} \end{aligned} \quad (6.2)$$

We do not quote the explicit expressions for $F^8(0)$, $F^0(0)$, because these quantities do not contain new information: the Feynman-Hellman theorem (Hellman 1937; Feynman 1939; Epstein 1954) states that the forward matrix elements of the operators $\bar{u}u$, $\bar{d}d$ and $\bar{s}s$ are equal to the derivatives of the pion (mass)² with respect to m_u , m_d and m_s respectively, e.g.

$$\bar{F}^8(0) = \left(\frac{\partial}{\partial m_u} + \frac{\partial}{\partial m_d} - 2 \frac{\partial}{\partial m_s} \right) M_\pi^2 \quad (6.3)$$

At leading order in the quark mass expansion we therefore have $F^8(0) = F^0(0) = 2 B_0$. The contributions of order M are easily worked out from the explicit expressions for M_π^2 as a function of the quark masses, given in III. The difference $F^0(0) - F^8(0)$ measures the change in M_π^2 produced by a change in m_s ; in the large N_c limit this difference tends to zero (Zweig rule).

The momentum dependence of the octet form factor is described by the function $f(t)$. Since the low energy constant L_5^r determines the ratio F_K/F_π [see (1.8)] we have a parameter free representation of this function. The corresponding mean square radius

$$\bar{F}^8(t) = \bar{F}^8(0) \left\{ 1 + \frac{1}{6} \langle r^2 \rangle_S^8 t + \dots \right\} \quad (6.4)$$

is given by

$$\langle r^2 \rangle_S^8 = \frac{6}{M_K^2 - M_\pi^2} \left(\frac{F_K}{F_\pi} - 1 \right) + \delta_3$$

$$\delta_3 = \frac{1}{64\pi^2 F_\pi^2} \frac{1}{M_K^2 - M_\pi^2} \left\{ 6(2M_K^2 - M_\pi^2) \ln \frac{M_K^2}{M_\pi^2} + 9M_\pi^2 \ln \frac{M_\eta^2}{M_\pi^2} - 2(M_K^2 - M_\pi^2) \left(10 + \frac{1}{3} \frac{M_\pi^2}{M_\eta^2} \right) \right\} \quad (6.5)$$

With $F_K/F_\pi = 1.22$ we obtain the prediction

$$\langle r^2 \rangle_S^8 = 0.55 \pm 0.10 \text{ fm}^2 \quad (6.6)$$

(the error bar is an estimate of the uncertainties due to higher order contributions). Note that this result is more than twice as large as the analogous prediction (5.2) for $\langle r^2 \rangle_S^{K\eta}$.

$$\langle r^2 \rangle_S^{K\pi} = \frac{6}{M_K^2 - M_\pi^2} \left(\frac{F_K}{F_\pi} - 1 \right) + \delta_2 \quad (6.7)$$

which leads to $\langle r^2 \rangle_S^{K\pi} = 0.20 \pm 0.05 \text{ fm}^2$. The chiral logarithms collected in δ_2 and δ_3 are very different ($\delta_2 = -0.025 \text{ fm}^2$, $\delta_3 = +0.33 \text{ fm}^2$), and lead to violent SU(3) asymmetries in the scalar radii.

It is not difficult to understand the origin of these asymmetries. In the limit $m_u = m_d = m_s \neq 0$ the meson masses are degenerate and the formulae for $\langle r^2 \rangle_S^{K\pi}$ and $\langle r^2 \rangle_S^{K\eta}$ predict the same values (with the physical value of L_5^r and a common meson mass of 300 MeV we get $\langle r^2 \rangle_S^{K\pi} = \langle r^2 \rangle_S^{K\eta} = 0.24 \text{ fm}^2$.) In the real world the radii are very different, because $\langle r^2 \rangle_S^{K\pi}$ picks up a contribution proportional to $\log M_\pi^2$ with a large coefficient: the two pion cut strongly influences the slope of the scalar form factor at the origin. (In the case of the vector form factors the residues of the infrared singularities are much smaller - the vector radii are more stable with respect to the perturbations generated by the quark mass differences.)

To the order we are considering here, the difference in the momentum dependence of the octet and singlet form factors is described by the function $\tilde{f}(t)$. This function also determines the difference between the octet form factor and the matrix element

$$\begin{aligned} \langle \pi^+ | \bar{u}u + \bar{d}d | \pi^+ \rangle &= \delta^{ik} \bar{F}_S^\pi(t) \\ \bar{F}_S^\pi(t) &= \bar{F}_S^\pi(0) \left\{ 1 + f(t) + \frac{2}{3} \tilde{f}(t) + O(\rho^4) \right\} \\ &= \bar{F}_S^\pi(0) \left\{ 1 + \frac{1}{6} \langle r^2 \rangle_S^\pi t + \dots \right\} \quad (6.8) \end{aligned}$$

In the large N_c limit the function $\tilde{f}(t)$ vanishes. The infrared singularities in $\tilde{f}(t)$, of order $1/N_c$, are generated by the $K\bar{K}$ and $\eta\eta$ cuts; the two pion cut does not contribute, because, to leading order, the matrix element $\langle \pi | \bar{s}s | \pi \rangle$ vanishes. For the radii we obtain

$$\begin{aligned}\langle r^2 \rangle_S^0 &= \langle r^2 \rangle_S^8 + \langle \delta r^2 \rangle_S \\ \langle r^2 \rangle_S^\pi &= \langle r^2 \rangle_S^8 + \frac{2}{3} \langle \delta r^2 \rangle_S\end{aligned}\quad (6.9)$$

where the Zweig rule violating contribution $\langle \delta r^2 \rangle$ is given by

$$\langle \delta r^2 \rangle_S = \frac{6}{F_0^2} \left\{ 12 L_4^r - \frac{3}{64\pi^2} \left(\ln \frac{M_K^2}{\mu^2} + 1 \right) + \frac{M_\pi^2}{288\pi^2} \frac{1}{M_\eta^2} \right\} \quad (6.10)$$

Neglecting $\langle \delta r^2 \rangle_S$ in comparison to $\langle r^2 \rangle_S^8$ we obtain

$$\langle r^2 \rangle_S^\pi = 0.55 \pm 0.15 \text{ fm}^2 \quad (6.11)$$

[Alternatively, one may use the estimate for the Zweig rule violating low energy constant L_4^r given in III: $L_4^r = \pm 0.5 \cdot 10^{-3}$ at $\mu = M_\pi$. This leads to $\langle \delta r^2 \rangle_S = -0.1 \pm 0.16 \text{ fm}^2$ and hence gives $\langle r^2 \rangle_S^\pi = 0.50 \pm 0.15 \text{ fm}^2$.]

In II we have shown that the $\pi\pi$ scattering lengths may be used to measure the scalar radius of the pion directly, with the result

$$\langle r^2 \rangle_S^\pi = 0.7 \pm 0.2 \text{ fm}^2 \quad (6.12)$$

which agrees with (6.11) within the errors.

As mentioned above, the quantity $\langle r^2 \rangle_S^\pi$ is related to the ratio F_π/F where F is the value of F_π in the limit $m_u = m_d = 0$, $m_s \neq 0$ (see II.15.11):

$$\frac{F_\pi}{F} = 1 + \frac{1}{6} M_\pi^2 \langle r^2 \rangle_S^\pi + \frac{13}{192\pi^2} \frac{M_\pi^2}{F_\pi^2} \quad (6.13)$$

The value (6.11) predicts

$$F_\pi/F = 1.057 \pm 0.012 \quad (6.14)$$

If one lets all three quark masses tend to zero the pion decay constant tends to F_0 , with (see III.10.7):

$$F_\pi/F_0 = 1.07 \pm 0.12 \quad (6.15)$$

[The uncertainties are larger here, because the Zweig rule violating contributions are weighted with m_s rather than with $\bar{m}$ as in (6.14).]

7. Summary and conclusions

We have calculated the pseudoscalar matrix elements of the vector current octet in an expansion in powers of the momentum transfer and of the quark masses, retaining terms of order t and of order M in the invariant form factors. For $m_u = m_d$ the expansion involves two of the ten coupling constants of the general effective low energy Lagrangian: L_5 and L_9 . The constraints imposed by chiral symmetry on the low energy structure of the form factors are thus exhausted: up to and including terms of order t and M chiral symmetry does not contain any further information.

The low energy representation obtained in this manner of course incorporates the Ademollo-Gatto theorem, is consistent with Sirlin's generalization of this theorem and satisfies the Callan-Treiman relation. The representation predicts specific corrections of relative order M to these low energy theorems. Numerically, the corrections turn out to be tiny.

In the physical region of $K_{\ell 3}$ decay the form factors are found to be linear in t to a good approximation.

We have found a sizeable isospin breaking effect in the $K_{\ell 3}$ form factors, which is neatly confirmed by the data.

The Dashen-Weinstein relation (Dashen and Weinstein 1969) predicts the slope $f_0'(0)$ of the scalar form factor in $K_{\ell 3}$ decays. Its modified version (Dashen, Li, Pagels and Weinstein 1972) predicts this slope at the unphysical point $t_1 = M_K^2 + M_\pi^2$. Both forms of this relation receive a correction at leading order. As can be seen from Fig. 3 the slope $f_0'(t_1)$ differs substantially from the slope in the physical region of the decay (Auviil and Pritchett 1975; Kenny 1977). Note that our method circumvents such difficulties. We are able to calculate the form factors in the physical region - to a given order in the low energy expansion - there is no need to look at special kinematical points and to discuss the corrections to the relations thus found.

With the experimental value of the ratio F_K/F_π (which fixes L_5) and of the slope λ_+ of the form factor $f_+^{K\pi}(t)$ (which determines L_9) we obtain a parameter free prediction for the electromagnetic charge radii $\langle r^2 \rangle_\pi^{2K^+}$, $\langle r^2 \rangle_V^{2K^0}$ and for the slope λ_0 of the scalar $K_{\ell 3}$ form factor. Except for $\langle r^2 \rangle_V^{2K^+}$ and for some recent data on λ_0 the predictions are in good agreement with experiment.

The electromagnetic charge radii turn out to be approximately $SU(3)$ symmetric, because in this case the infrared singularities generated by the Goldstone boson cuts do not have a large residue.

We have also calculated the pion matrix elements of the scalar operators $\bar{u}u + \bar{d}d$ and $\bar{s}s$ in the same low energy expansion. In contrast to the electromagnetic charge radii the scalar radii turn out to break $SU(3)$ very strongly: the infrared singularities happen to have large residues in this case.

Under the assumption that the Zweig rule violating contributions associated with $\langle \pi | \bar{s}s | \pi \rangle$ can be neglected, we predict a value for the scalar radius of the pion which agrees with the rather crude measurement of this quantity obtained from $\pi\pi$ scattering.

Bibliography

- Ademollo, M. and Gatto, R., Phys.Rev.Lett. 13 (1964) 264.
- Auvil, P.R. and Pritchett, P.L., Phys.Rev.Lett. 34 (1975) 555.
- Bačé, M. and Cornwall, D.T., Phys.Rev. D10 (1974) 2694.
- Bég, M.A.B. and Zepeda, A., Phys.Rev. D6 (1972) 2912.
- Behrends, R.E. and Sirlin, A., Phys.Rev.Lett. 4 (1960) 186.
- Biatnik, S., Stahov, J. and Lang, C.B., Lett. Nuovo Cim. 24 (1979) 39.
- Boal, D.H. and Graham, R.H., Phys.Rev. D15 (1977) 1878.
- Callan, C.G. and Treiman, S.B., Phys.Rev.Lett. 16 (1966) 153.
- Cho, Y. et al., Phys.Rev. D22 (1980) 2688.
- Chounet, L.M., Gaillard, J.M. and Gaillard, M.K., Phys.Rep. 4C (1972) 199.
- Dally, E.B. et al., Phys.Rev.Lett. 45 (1980) 232; 48 (1982) 375.
- Dashen, R. and Weinstein, M., Phys.Rev.Lett. 22 (1969) 1337.
- Dashen, R., Li, L.-F., Pagels, H. and Weinstein, M., Phys.Rev. D6 (1972) 834.
- Donaldson, G. et al., Phys.Rev. D9 (1974) 2960.
- Epstein, S.T., Amer.J.Phys. 22 (1954) 613.
- Feynman, R.P., Phys.Rev. 66 (1939) 340.
- Furlan, G., Lannoy, F.G., Rossetti, C. and Segré, G., Nuovo Cim. 38 (1965) 1747.
- Gaillard, M.K., Nuovo Cim. 61 (1969) 499.
- Gaillard, J.M., Gaillard, M.K. and Vannucci, F., in: Weak interactions (Inst. Nat. de Phys. nucl., Paris 1977, M.K. Gaillard and M. Nikolic (eds)) p. 61.
- Gasser, J. and Leutwyler, H., Phys.Rep. 87C (1982) 77 (this reference is quoted as I);

- Gasser, J. and Leutwyler, H., "Chiral perturbation theory to one loop", Preprint CERN TH-3689 (1983), to be published in Ann. of Phys. (this reference is quoted as I);
- "Chiral perturbation theory: Expansion in the mass of the strange quark", Preprint CERN TH-3798 (1983) (this reference is quoted as III).
- Geddes, H.B. and Graham, R.H., Phys.Rev. D21 (1980) 749.
- Gourdin, G., Phys.Rep. 11C (1974) 29.
- Hearn, A.C., Computer algebra program REDUCE, version 1983.
- Hellman, H.: "Einführung in die Quantenchemie", Deuticke Verlag, Leipzig, 1937.
- Heyn, M.F. and Lang, C.B., Z.Phys. 7C (1981) 169.
- Isgur, N., Rubinstein, H.R., Schwimmer, A. and Lipkin, H., Phys.Lett. 89B (1979) 79.
- Kenny, B.G., Phys.Rev. D15 (1977) 3481.
- Langacker, P. and Page's, H., Phys.Rev.Lett. 30 (1973) 630.
- Leutwyler, H., "Models of local current algebra", in: Proceedings of the International University Week on Nuclear Physics of the Karl-Franz University, Schladming, Austria, February 26 - March 9, 1968 (Acta Phys. Austr. Suppl. V (1968) 320); "Current algebra and lightlike charges", in: First International Summer School for Theoretical Physics, University of Karlsruhe, July 22 - August 2, 1968 (Springer Tracts in Modern Physics 50 (1969) 42).
- Leutwyler, H. and Roos, M., "Determination of the elements V_{us} and V_{ud} of the Kobayashi-Maskawa matrix", CERN TH.3830, to be published in Z. Phys. C.
- Mathur, V.S., Okubo, S. and Pandit, L., Phys.Rev.Lett. 16 (1966) 371, 601 (E).
- Mathur, V.S. and Okubo, S., Phys.Rev. D1 (1970) 3468; D2 (1970) 619.
- Mathur, V.S. and Yang, T.C., Phys.Rev. D5 (1972) 246.
- Molzon, W.R. et al., Phys.Rev.Lett. 41 (1978) 1213.
- Onne, R., Phys.Rev.Lett. 16 (1966) 215.
- Omnes, R., Nuovo, Cim. 8 (1958) 316.

- Particle Data Group, Review of Particle Properties, Phys.Lett. 111B (1982).
- Perez-y-Jorba, J.B. and Renard, F.M., Phys.Rep. 31C (1977) 1.
- Sirlin, A., Ann.Phys. 61 (1970) 294; Phys.Rev.Lett. 43 (1979) 904.
- Suzuki, M., Phys.Rev.Lett. 16 (1966) 212.
- Wada, S., Phys.Lett. 49B (1974) 175.
- Weinberg, S., Phys.Rev.Lett. 17 (1966) 616.
- Yamada, K., Phys.Lett. 97B (1980) 156.
- Zovko, N., Fortschr.Phys. 23 (1975) 185.

Figure caption

Fig. 1: The function $h_0(x)$, which describes the dependence of $f_+^{K\pi}(0)$ on the quark mass ratio, $x = M_\pi^2/M_K^2 = 2\bar{m}/(m_s + \bar{m})$, see eqs. (1.4, 2.1). In accordance with the Ademollo-Gatto theorem, $h_0(x)$ is proportional to $(x-1)^2$ near $x = 1$. The quadratic approximation $h_0(x) \approx \frac{1}{6}(x-1)^2$ however fails to represent $h_0(x)$ at the physical point $x \sim 0.07$.

Fig. 2: We plot the vector and scalar form factors $f_+^{K\pi}(t)$ and $f_0^{K\pi}(t)$ which describe $K_{\ell 3}$ decay.

Fig. 3: We plot the normalized slopes of the vector and scalar form factors in $K_{\ell 3}$ decay. Curve ①: The normalized slope $M_\pi^2 f_+^{K\pi'}(t)$. Curve ②: The normalized slope $M_\pi^2 f_0^{K\pi'}(t)$. Near the πK -threshold $t_0 = (M_K + M_\pi)^2$, the vector form factor behaves as $f_+^{K\pi}(t) = f_+^{K\pi}(t_0) + O[(t - t_0)]$, whereas $f_0^{K\pi}(t) = f_0^{K\pi}(t_0) + O[(t - t_0)^2]$. The slope of the scalar form factor is therefore singular at $t = (M_K + M_\pi)^2$.

Fig.1

Fig.2

Fig.3

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