DO SUPERSYMMETRIES FORM A GROUP ?

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A B S T R A C T

This question is answered affirmatively for the case of supersymmetric quantum mechanics. The group is a nilpotent Lie group entirely determined by the choice of the Grassmann algebra.

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The composition of two symmetry transformations is again a symmetry transformation. In mathematical terms: the set of transformations leaving a given object invariant form a semi-group. This set is a group if and only if all transformations are invertible. For example, the isometries of the Hilbert space

$$\ell^2 = \left\{ (x_1, x_2, x_3, \dots), x_i \in \mathbb{C}, \sum_{i=1}^{\infty} \overline{x_i} x_i < \infty \right\}$$

form only a semi-group, a non-invertible isometry is given by:

$$(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$$

A natural question to ask is whether all symmetries of a given supersymmetric model are invertible. For simplicity, we consider supersymmetric quantum mechanics, which has recently been used to simplify the proof of the Atiyah-Singer index theorem¹⁾.

Let

$$\alpha = \alpha_e \oplus \alpha_\sigma$$

be a real Grassmann algebra (exterior algebra) with even and odd parts α_e, α_σ . We denote its elements by pairs

$$(e, \sigma) \quad e \in \alpha_e, \quad \sigma \in \alpha_\sigma$$

A natural group structure is defined on α :

$$(e, \sigma) \circ (e', \sigma') := (e + e' + \sigma \sigma', \sigma + \sigma') \quad (*)$$

Its neutral element is the origin and the inverse is given by

$$(e, \sigma)^{-1} = (-e, -\sigma)$$

This group is non-Abelian, nilpotent and also a Lie-group because α is a vector space. Its Lie algebra is α once more with commutators:

$$[(e, \sigma), (e', \sigma')] = (\sigma \sigma', 0)$$

Double commutators vanish. This Lie algebra can be written formally as

$$\alpha_e \mathcal{H} \oplus \alpha_\sigma \mathcal{Q}$$

where $\mathcal{H}$ and $\mathcal{Q}$ are a basis of the two-dimensional graded Lie algebra ε_2

$$[H, Q] = 0 \quad \{Q, Q\} = H$$

In other words, the Lie algebra α is the even part of the graded Lie algebra

$$\alpha \otimes \varepsilon_2$$

We now define a representation of the group α . V is the infinite-dimensional vector space of a real $\mathcal{C}^\infty$ function with compact support and values in α .

$$V = \left\{ \mathbb{R} \xrightarrow{\quad} \alpha \atop x \mapsto (x(x), \psi(x)) \right\}$$

The action ρ of α on V is given by

$$\begin{aligned} \rho(\tau, e, \sigma) (x(x), \psi(x)) &= \\ &= \left(\sum_{n=0}^{\infty} \frac{e^n}{n!} \left(\frac{d}{dx} \right)^n [x(x+\tau) - \sigma \psi(x+\tau)] \right), \\ &\quad \sum_{n=0}^{\infty} \frac{e^n}{n!} \left(\frac{d}{dx} \right)^n \left[\psi(x+\tau) + \sigma \frac{d}{dx} x(x+\tau) \right] \end{aligned}$$

$$\tau \in \mathbb{R} \subset \alpha \quad e \in \alpha_e - \mathbb{R} \quad \sigma \in \alpha_\sigma$$

Note that the sums are finite if α is finite-dimensional. For an infinite number of generators, we define α to consist of finite linear combinations of products of generators

Finally, we define the "action" of supersymmetric quantum mechanics

$$S : V \longrightarrow \alpha_e$$

$$(x, \psi) \longmapsto \int dx \left(\frac{dx}{dt} \frac{dx}{dt} + \psi \frac{d\psi}{dt} \right)$$

S is invariant under the group α , i.e.,

$$S[\rho(a)(x, \psi)] = S[x, \psi] \quad \forall a \in \alpha$$

The Lie group $\mathcal{G}$ can, of course, be gauged in the standard way: elements of the gauge group are functions from the real line to $\mathcal{G}$, the group multiplication is given by (*) pointwise.

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