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LOW ENERGY THEOREMS FROM THE EFFECTIVE LAGRANGIAN

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A B S T R A C T

It is shown that the anomalous effective Lagrangian which describes pions coupled to electromagnetism incorporates in a systematic way all the information on the low energy behaviour of $\gamma\pi$ interactions derived from the current algebra, PCAC and the low energy theorems.

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Photon-meson interactions have been the subject of detailed investigations in the heyday of current algebra¹⁾. The joint information concerning the low energy behaviour and gauge invariance of the amplitudes put very strong constraints on their structure.

A more modern (and QCD based) point of view on the same topics is the effective Lagrangian approach advocated by Wess-Zumino²⁾ and Witten³⁾.

In this note we show that the gauged effective Lagrangian first explicitly constructed by Witten^{3),4)} accounts not only for the well-known $\pi^0 \rightarrow \gamma\gamma$ amplitude and the $\gamma \rightarrow \pi^+\pi^-\pi^0$ process but also (and perhaps more impressively) reproduces all the content of the low-energy theorem first deduced by Adler et al. for the $\gamma\gamma \rightarrow \pi^+\pi^-\pi$ processes, and in particular the effective vertices conjectured by these authors in order to explain the fact that their results are naturally incorporated in the effective Lagrangian formulation.

Let us sketch Adler et al.'s results, following their original notation for the sake of easy comparison. They introduce the following amplitudes

$$\mathcal{M}(\gamma(k_1) + \gamma(k_2) \rightarrow \pi^0) = i k_1^\alpha k_2^\beta \varepsilon_1^\gamma \varepsilon_2^\delta \varepsilon_{\alpha\beta\gamma\delta} F^\pi \quad (1)$$

$$\mathcal{M}(\gamma(k_1) \rightarrow \pi^0(q_0) + \pi^+(q_+) + \pi^-(q_-)) = -i \varepsilon_1^\alpha q_0^\beta q_+^\gamma q_-^\delta \varepsilon_{\alpha\beta\gamma\delta} F^{3\pi} \quad (2)$$

$$\mathcal{M}^{000} \equiv \mathcal{M}(\gamma(k_1) + \gamma(k_2) \rightarrow \pi^0(q_0) + \pi^0(q'_0) + \pi^0(q''_0)) \quad (3)$$

$$\mathcal{M}^{0+-} \equiv \mathcal{M}(\gamma(k_1) + \gamma(k_2) \rightarrow \pi^0(q_0) + \pi^+(q_+) + \pi^-(q_-)) \quad (4)$$

and assume the following current algebra pion-pion scattering amplitude

$$\begin{aligned} \mathcal{M}(\pi^a \rightarrow \pi^b(q_b) + \pi^c(q_c) + \pi^d(q_d)) = & \frac{i}{f_\pi^2} \{ \delta_{bc} \delta_{ad} [(q_b + q_c)^2 - m_\pi^2] + \text{perm.} \\ & - x [(q_b + q_c)^2 + (q_b + q_d)^2 + (q_c + q_d)^2 - 3m_\pi^2] (\delta_{bc} \delta_{ad} + \delta_{bd} \delta_{ac} + \delta_{cd} \delta_{ab}) \} \quad (5) \end{aligned}$$

where x is proportional to the isotensor component of the " σ term" and therefore sensitive to the isospin dependence of chiral symmetry breaking⁵⁾. They consider the contribution of the pion-pole diagrams, depicted in Fig. 1, and show that in order to reproduce the amplitudes given in Eqs. (3) and (4) dictated by gauge invariance and low energy theorems (that an amplitude involving soft-pions

vanishes when the four-momentum of the neutral pion vanishes), they need to introduce a contact term (Fig. 2) described by an "ad hoc" effective Lagrangian

$$\mathcal{L}_{\text{eff}} = (1-x) \frac{1}{16 f_\pi^2} F^\pi F_{\alpha\beta} F_{\gamma\delta} \varepsilon_{\alpha\beta\gamma\delta} \pi^0 \vec{\pi} \cdot \vec{\pi} - \frac{1}{2} e F^{3\pi} (\partial_\alpha A_\beta) A_\delta \varepsilon_{\alpha\beta\gamma\delta} \partial_\tau \pi^0 \vec{\pi} \cdot \vec{\pi}$$

However, consistency of their results (gauge invariance and vanishing in the soft- π^0 limit) requires the identity

$$e F^{3\pi} = \frac{1}{f_\pi^2} F^\pi \quad (7)$$

representing their main model-independent prediction.

Our point of view is that the Wess-Zumino effective Lagrangian is the correct one for the low energy interactions of the QCD pseudo-scalars. We shall focus on the two-flavour case, in which the resulting expression is especially simple:

$$\Gamma = \int d^4x \frac{f_\pi^2}{4} \text{Tr} [\partial_\mu U \partial_\mu U^\dagger + i e A_\mu \tau_3 (L_\mu - R_\mu) - \frac{e^2}{2} A_\mu A_\nu (\tau_3 U \tau_3 U^\dagger - 1)]$$

$$- \frac{N_c}{48\pi^2} \varepsilon_{\mu\nu\rho\sigma} e A_\mu \frac{1}{3} \text{Tr} (L_\nu L_\rho L_\sigma) + \frac{N_c}{48\pi^2} e^2 F_{\mu\nu} A_\rho \varepsilon_{\mu\nu\rho\sigma} \frac{1}{4} \text{Tr} \tau_3 (L_\sigma + R_\sigma) \quad (8)$$

where N_c is the number of colours, and

$$U = \exp \frac{i}{f_\pi} \vec{\tau} \cdot \vec{\pi} \quad f_\pi \simeq 94 \text{ MeV}$$

$$L_\mu = \partial_\mu U U^\dagger \quad R_\mu = U^\dagger \partial_\mu U \quad (9)$$

We can immediately expand Eq. (8) and obtain the corresponding effective vertices up to a given number of pions. The result to the order f_π^{-3} is the following:

$$\Gamma = \int d^4x \frac{1}{2} (\partial_\mu \pi^a - e A_\mu \varepsilon^{3ab} \pi^b) (\partial_\mu \pi^a - e A_\mu \varepsilon^{3ab'} \pi^{b'}) - \frac{1}{6} f_\pi^2 (\vec{\pi} \wedge \partial_\mu \vec{\pi})^2$$

$$+ \frac{N_c}{96\pi^2} \varepsilon_{\mu\nu\rho\sigma} \frac{e^2}{f_\pi} F_{\mu\nu} F_{\rho\sigma} \pi_0 - \frac{1}{2} m_\pi^2 \vec{\pi} \cdot \vec{\pi}$$

$$- \frac{N_c}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \frac{e}{f_\pi} F_{\mu\nu} \varepsilon^{abcd} (\partial_\mu \pi^b - e A_\mu \varepsilon^{3bd} \pi^d) (\partial_\nu \pi^c - e A_\nu \varepsilon^{3cd'} \pi^{d'}) \quad (10)$$

where we have introduced for convenience a mass term for the pions.

One immediately recognizes that Eq. (10) implies

$$F^\pi = \frac{N_c}{12\pi^2} \frac{e^2}{f_\pi} \quad F^{3\pi} = \frac{N_c}{12\pi^2} \frac{e}{f_\pi^3} \quad (11)$$

and therefore, Eq. (7) immediately follows. Let us now compute the four-pion vertex:

$$\mathcal{M}(\pi^a \rightarrow \pi^b + \pi^c + \pi^d) = -\frac{i}{6f_\pi^2} \left\{ \delta^{ab} \delta^{cd} \left[2(p^a + p^b)(p^c + p^d) - 4(p^a p^b + p^c p^d) \right] + \text{perm.} \right\} \quad (12)$$

Using four-momentum conservation, this result can be rephrased in the form

$$\mathcal{M} = \frac{i}{f_\pi^2} \left\{ \delta_{ab} \delta_{cd} \left[\frac{2}{3} (p_c + p_d)^2 - \frac{1}{3} (p_b + p_d)^2 - \frac{1}{3} (p_b + p_c)^2 \right] + \text{perm.} \right\} \quad (13)$$

This corresponds to Eq. (5) for the special choice of the x -parameter:

$$x = \frac{1}{3} \quad (14)$$

Thus, the $I = 0$ $\pi\pi$ scattering length for our effective Lagrangian is

$$a_0 = \frac{\eta_\pi}{6\pi f_\pi^2} \quad (15)$$

The piece in (10) which gives a direct $2\gamma-3\pi$ vertex is

$$\mathcal{L}_{2\gamma 3\pi} = \frac{N_c e^2}{48\pi^2} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} A_\rho \left(\frac{2}{3f_\pi^3} \right) \left[\vec{\pi} \wedge (\vec{\pi} \wedge \partial_\sigma \vec{\pi}) \right]_0 \quad (16)$$

Before turning this into a form which can easily be compared to that of Ref. 1), let us rewrite it in terms of the charged fields

$$\frac{N_c}{48\pi^2} \frac{e^2}{3f_\pi^3} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} A_\rho \left[\pi^0 (\pi^+ \partial_\sigma \pi^- + \pi^- \partial_\sigma \pi^+) - 2\pi^+ \pi^- \partial_\sigma \pi^0 \right] \quad (17)$$

Thus there is no $3\pi^0$ final state in the contact term. The same is true for the pole terms as well. The first pole term involves the anomalous $\pi^0-2\gamma$ vertex, and the non-anomalous 4π vertex. Written in terms of the charged components, the 4π interaction term is

$$\begin{aligned} \mathcal{L}_{4\pi} = & \frac{1}{3f_\pi^2} \left[(\pi_0)^2 \partial_\mu \pi^+ \partial_\mu \pi^- + \pi_+ \pi_- (\partial_\mu \pi_0)^2 - \pi_0 (\partial_\mu \pi_0) (\pi_+ \partial_\mu \pi_- + \pi_- \partial_\mu \pi_+) \right] \\ & + \frac{1}{6f_\pi^2} \left[2\pi_+ \pi_- \partial_\mu \pi_+ \partial_\mu \pi_- - (\pi_+)^2 \partial_\mu \pi_0 \partial_\mu \pi_- - (\pi_-)^2 \partial_\mu \pi_+ \partial_\mu \pi_- \right] \end{aligned} \quad (18)$$

Thus the $3\pi^0$ state would not be produced in the first pole diagram. The second pole diagram involves the anomalous $\gamma \rightarrow 3\pi$ vertex and the non-anomalous $\gamma 2\pi$ vertex (the charge vertex). It can immediately be seen from Eq. (10) that the 3π final state in the anomalous $\gamma 3\pi$ vertex is $\pi^+\pi^-\pi^0$. Thus

$$\mathcal{M}^{000} = 0 \quad (19)$$

This is one of the new results in this scheme. Now, by partial integration, Eq. (16) can be turned into

$$\frac{N_c e^2}{24\pi^2 f_\pi^3} \left[\frac{1}{12} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \pi_0 \vec{\pi} \cdot \vec{\pi} - \varepsilon_{\mu\nu\rho\sigma} \partial_\mu A_\nu A_\rho \vec{\pi} \cdot \vec{\pi} \partial_\sigma \pi_0 \right] \quad (20)$$

and, using Eq. (11),

$$\frac{1}{24} \frac{F_\pi}{f_\pi^2} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \pi_0 \vec{\pi} \cdot \vec{\pi} - \frac{1}{2} e F_\pi^3 \varepsilon_{\mu\nu\rho\sigma} \partial_\mu A_\nu A_\rho \vec{\pi} \cdot \vec{\pi} \partial_\sigma \pi_0 \quad (21)$$

Equation (21) reproduces Eq. (6) with the special value $x = 1/3$, Eq. (13), predicted by the non-anomalous part of our effective Lagrangian. Next, computing Fig. 1 and Fig. 2, we get the following result for $\gamma\gamma \rightarrow \pi^+\pi^-\pi^0$ amplitude:

$$\mathcal{M}^{0+-} = \frac{i N_c e^2}{12\pi^2 f_\pi^3} \varepsilon_{\alpha\beta\gamma\delta} \left\{ k_1^\alpha k_2^\beta \varepsilon_1^\gamma \varepsilon_2^\delta \left(\frac{2}{3} - \frac{2(q_+ + q_-)^2 - (q_+ + q_0)^2 - (q_- + q_0)^2}{3[(q_+ + q_- + q_0)^2 - m_\pi^2]} \right) \right. \\ \left. - \varepsilon_1^\rho \varepsilon_2^\tau \left[\frac{(2q_+ - k_2)^\tau}{(q_+ - k_2)^2} k_1^\alpha (q_- - k_2)^\rho q_-^\delta - \frac{(2q_- - k_2)^\tau}{(q_- - k_2)^2} k_1^\alpha q_+^\rho (q_- - k_2)^\delta + (k_1 + k_2, \beta\gamma\tau) + (k_1 - k_2)^\alpha q_0^\delta \delta_{\beta\gamma\tau} \right] \right\} \quad (22)$$

It is easy to check the gauge invariance when the pions are on-shell. One can also check that $\mathcal{M}^{0+-} \rightarrow 0$, as $q_\mu^0 \rightarrow 0$.

In conclusion, we have confirmed the expectation that the effective Lagrangian first introduced by Wess and Zumino incorporates in a systematic way all the information derived from the current algebra, PCAC and the low energy theorems, and therefore it is a good starting point and testing ground for the more ambitious program of a low energy effective Lagrangian for QCD including also the vector meson degrees of freedom and, say, vector meson dominance. Predictions of such a model should be tested against our Eqs. (21) and (22) for confirmation. This subject will be discussed in a subsequent paper⁶⁾.

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FIGURE CAPTIONS

Fig. 1 Pion pole diagrams for $2\gamma \rightarrow \pi^+\pi^-\pi^0$. Black blobs are anomalous $\gamma\gamma\pi^0$, $\gamma\pi^+\pi^-\pi^0$ vertices.

Fig. 2 Contact term for $2\gamma \rightarrow \pi^+\pi^-\pi^0$.

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+ ($k_1 \leftrightarrow k_2$)