CERN-TH.3947/84

DIMENSIONAL REDUCTION BY MONTE CARLO

David J.E. Callaway and Roberto Petronzio
CERN -- Geneva

A B S T R A C T

A Monte Carlo method for mapping a field theoretical or statistical system to a new theory embedded in a space-time of lesser dimensionality is presented. Typically, the critical properties of the dimensionally reduced system depend upon the details of the mapping. As an example, the two-dimensional Ising model is mapped to a one-dimensional Ising model with long-range forces and a phase transition. Systems with long-range interactions and known exponents can thus be constructed with this procedure.

I. PROLEGOMENA

Since the work of Kaluza [1] and Klein [2] the idea of mapping a given space-time to one with fewer dimensions has occupied the minds of theoretical physicists. Nor, it seems, has the idea lost its immediacy in the intervening half century (see, e.g., Ref. [3] for a review of recent work). The general philosophy in the Kaluza-Klein approach is to assume that the additional dimension(s) become irrelevant (for example by collapsing and becoming compact). An alternative philosophy is to integrate out the other dimensions (and the fields which depend upon them) in the path integral which defines the field theory. Such a new approach is practical only if a reliable scheme for performing these integrations is available. One candidate for this scheme is presented here.

A brief detour into the lore of the real-space renormalization group [4] is a necessary preliminary for a complete understanding of the dimension-reducing mapping. Recall that in this style of renormalization the system under consideration is first divided up into "blocks" or sets of the original (or "site") variables. Block variables are then defined by averaging in some fashion over these site variables. The change of variables and integration over the site degrees of freedom alters the form of the action governing the resulting effective system; by following the way in which this procedure produces a new action during repeated blocking transformations the critical properties of the system can be determined [4].

It is no surprise that techniques useful in the real-space renormalization group are applicable to dimensional reduction. In this latter case the desired dimension is integrated out of the d -dimensional system, and the "block" variables are just the remaining degrees of freedom in the $(d-1)$ -dimensional system. The resemblance of dimensional reduction to the real-space renormalization group should not be pushed too far however, since dimensional reduction is a one-step mapping operation while the virtues of the renormalization group become evident only after repeated application.

An important ingredient in the dimensional reduction scheme is a way to extract the action for the reduced ("block") system. As is shown below, a simple modification of a known technique [5] for the Monte Carlo renormalization group is adequate

for this purpose. The model chosen here for dimensional reduction is the canonical two-dimensional Ising model with only nearest-neighbour couplings. The resulting reduced system is a one-dimensional Ising model with a phase transition produced by long-range interactions. The critical exponents of the reduced system (determined by finite-size scaling [6]) depend upon the way in which the integration is performed.

One might very well question the paradigm of dimensional reduction selected here. The two-dimensional Ising model with only nearest-neighbour couplings has the virtue of being soluble in closed form [7]. Additionally it possesses a phase transition at finite temperature. On the other hand, the one-dimensional Ising model has no phase transition in the absence of long-ranged interactions [8]. The two models could hardly be more different.

The successful consideration of such an extreme example however shows that this form of dimensional reduction is likely to be applicable to a wide variety of field-theoretic and statistical systems. The fact that the resulting critical exponents depend in a controllable fashion upon the definition of the integration procedure also suggests that systems with long-range interactions and known exponents can be constructed (almost) at will.

II. FORMALISM

At this point it is appropriate to begin a detailed discussion of the method. The original ("site") system in d Euclidean dimensions is governed by an action (read "Hamiltonian" for a statistical system) denoted by S . The action for the dimensionally reduced system (embedded in $d-1$ Euclidean dimensions) is signified by S' . As stated above, the specific system to which the method outlined here is applied is the two-dimensional Ising model with nearest neighbour couplings. This model is reduced to a one-dimensional model which turns out to have highly nonlocal interactions.

The statistical variables A for the two-dimensional system are written as $\{\sigma_{i,j}\}$, where the indices i and j refer to the two discrete Cartesian coordinates on a square lattice. The corresponding statistical variables for the one-dimensional

Ising system are in turn given by $\{\sigma_i'\}$, where i signifies the lone discrete Cartesian coordinate. The action for the two-dimensional system is then

$$S = -K \sum_{i,j} \sigma_{i,j} (\sigma_{i+1,j} + \sigma_{i,j+1}) - C \quad (1)$$

while the corresponding action for the reduced system is

$$\begin{aligned} S' &= -K_1' \sum_i \sigma_i \sigma_{i+1} \\ &\quad - K_2' \sum_i \sigma_i \sigma_{i+2} \\ &\quad \dots \\ &\quad - C' \end{aligned} \quad (2)$$

where the ellipsis (...) refers to all possible interactions of all possible numbers of spins (e.g. four-spin interactions). As usual, the Ising variables $\{\sigma_{i,j}\}$ and $\{\sigma_i\}$ only take on values of plus or minus unity.

The action S' for the reduced system is determined by integrating out the extra degrees of freedom, viz.,

$$e^{-S'\{\sigma_i\}} = \text{Tr}_{\{\sigma_{i,j}\}} \mathbb{P}[\{\sigma_i\}, \{\sigma_{i,j}\}]^* \quad (3)$$

$$* \exp[-S \{\sigma_{i,j}\}]$$

where the projection operator $\mathbb{P}[\dots]$ in this calculation is taken to be a product of delta functions,

$$\mathcal{P}[\{\sigma_i\}, \{\sigma_{i,j}\}] = \prod_k [\sigma_k - f_k\{\sigma_{i,j}\}] \quad (4)$$

In the fashion Eq. (4) the sole constraint on the projection operator

$$\text{Tr}_{\{\sigma_i\}} \mathcal{P}[\dots] = 1 \quad (5)$$

is satisfied. This constraint is mandated by the requirement

$$\langle 0|0 \rangle \equiv Z = \text{Tr}_{\{\sigma_{i,j}\}} e^{-S\{\sigma_{i,j}\}} \quad (6a)$$

$$= \text{Tr}_{\{\sigma_i\}} e^{-S'\{\sigma_i\}} \quad (6b)$$

which ensures that the partition function Z (or vacuum-to-vacuum transition amplitude $\langle 0|0 \rangle$) is preserved by the mapping.

The constraint Eq. (4) is easily implemented [5] in a Monte Carlo simulation. Indeed the form of Eq. (4) is such that it allows a choice to be made separately for each statistical variable in the reduced system. The first option is that a given variable (in the reduced system) is held fixed. Thus for this option the configuration space of the site system is restricted. In practice the constraint is preserved by arranging the initial site configuration $\{\sigma_{i,j}\}$ according to its dictates ($f_k\{\sigma_{i,j}\} = \sigma_k$), then rejecting any (tentative) update of the system which threatens to alter any of the $\{f_k\}$.

The second way to implement the constraints Eq. (4) is to allow the site variables in the given block k to fluctuate freely. The block variable σ_k is then set equal to $f_k\{\sigma_{i,j}\}$. Thus the block variable σ_k and its subsumed site variables both fluctuate in accord with their respective actions S' and S .

Since each of the block variables $\{\sigma_i\}$ individually can be chosen to fluctuate or not, a convenient method [5] for the extraction of the reduced couplings $\{K', C', \dots\}$ can be constructed. This construction relies upon the fact that the dimensionally reduced action S' can be expanded in terms of coupling constants $\{K'_i\}$ and known functions $\{S_i\}$,

$$S' = \sum_i K'_i S_i \{\sigma_i\} \quad (7)$$

All but one (call it σ_0) of the $\{\sigma_i\}$ are fixed in a given background pattern. The sole fluctuating spin σ_0 then takes on values of plus or minus one according to a probability distribution determined by the nonzero $S_i\{\sigma\}$ and their couplings K'_i . For example, suppose that the background pattern is chosen in such a fashion that all but one (call it S_0) of the $\{S_i\}$ containing σ_0 are zero. Then the ratio of the number of times $\rho_+[\rho_-]$ that the fluctuating spin σ_0 takes the value of plus [minus] one is given by

$$\rho_+/\rho_- = \exp [K'_0 (S_0 \{\sigma_0=+1\} - S_0 \{\sigma_0=-1\})] \quad (8)$$

Since the function S_0 is known, it is a simple matter to determine K'_0 given ρ_+ and ρ_- . In general, not all of the functions $\{S_i\}$ containing σ_0 can be cancelled by a single background configuration. However, the results from several background configurations can be combined to separate all couplings.

III. RESULTS

By construction, the above dimensional reduction preserves the partition function Z of Eq. (6). Thus the free energy $F \sim \ln Z$ of the original system and of its dimensionally reduced counterpart are the same. As phase transitions are

traditionally defined in terms of singularities in the free energy, they also should survive the process of dimensional reduction.

What is not immediately self-evident however is to what extent the singularities in the free energy are manifest in the reduced couplings $\{K'_i\}$ and thus to what extent they influence the dynamics. For example, if a pathological projection operator in which all of the $f_k\{\sigma_{i,j}\}$ [cf. Eq. (4)] are independent of the $\{\sigma_{i,j}\}$ is used, the result is that all of the $\{K'_i\}$ are identically zero. The singularities in the free energy are contained entirely in the constant C' , and the $\{\sigma_i\}$ fluctuate randomly. The dynamics of the original system do not survive this type of dimensional reduction.

By contrast, with the two projection operators used for this calculation, enough information is passed on to the reduced system so that a phase transition is clearly evident. However, the critical behaviour (and in particular the critical exponents) seem to depend strongly upon the choice of projection operator. Thus some of the information on the phase transition must pass into the constant C' . The two projection operators are:

- 1) "Majority rule":

$$f_i = \text{sign} \left(\sum_j \sigma_{i,j} \right) \quad (9a)$$

This "majority rule" projection operator can be thought of as the result of a vote amongst all the spins in a column. In this calculation, the extent of the system in both directions is always taken to be an odd number of lattice sites, so no "tie breaking" is necessary.

- 2) "Decimation"

$$f_i = \sigma_{i,1} \quad (9b)$$

For the "decimation" projection operator, the $\{f_i\}$ are simply set equal to the site spins in a chosen row, arbitrarily labelled "1".

Evidence for the existence of a phase transition in each case is provided by a finite-size scaling analysis. (Finite-size scaling is discussed in [6]).

The magnetic susceptibility for a two-dimensional square lattice of side L is given by

$$\chi_{2D} = \frac{\langle \sigma_{2D}^2 \rangle - \langle \sigma_{2D} \rangle^2}{L^2} \quad (10a)$$

where

$$\sigma_{2D} \equiv \sum_{i=1}^L \sum_{j=1}^L \sigma_{i,j} \quad (10b)$$

The corresponding result for a one-dimensional line of length L is,

$$\chi_{1D} = \frac{\langle \sigma_{1D}^2 \rangle - \langle \sigma_{1D} \rangle^2}{L} \quad (11a)$$

with

$$\sigma_{1D} = \sum_{i=1}^L \sigma_i \quad (11b)$$

Near the critical temperature T_c the susceptibility scales like

$$\chi \propto [T - T_c]^{-\gamma} \quad (12a)$$

The critical exponent γ is positive if a divergence exists. The temperature (and thus the critical temperature) is typically a complicated function of the couplings of the system. In the two dimensional system, $T \propto K^{-1}$, and so

$$\chi_{2D} \propto (K - K_c)^{-\gamma} \quad (12b)$$

since an analytic redefinition of the temperature does not change the degree of divergence of the susceptibility.

In an infinite system, the magnetic susceptibility diverges in the fashion of Eqs. (12). However in a finite system the susceptibility has a maximum when the correlation length is the size of the system. Within this same critical region the correlation length ξ scales like

$$\xi \propto [T - T_c]^{-\nu} \quad (13)$$

Therefore for a finite system the maximum value of the susceptibility scales as

$$\chi_{\max}(L) \propto L^{\gamma/\nu} \quad (14)$$

so that the ratio γ/ν of critical exponents can be determined by plotting $\ln \chi_{\max}$ versus L for systems of various finite size. Additionally the steady increase of $\chi_{\max}$ with L provides evidence for the divergence of the susceptibility and hence the existence of a phase transition.

In Figure 1 the susceptibility χ at the exact critical point $K^* = \ln \frac{1}{2}(1 + \sqrt{2}) \sim 0.44068 \dots$ is plotted as a function of linear dimension L . For this calculation none of the block spins were fixed, and the dimensional reduction in each case was performed on a square $L \times L$ lattice. The ratio of critical exponents γ/ν obtained is given in the Table.

As might be expected, the known result for the two-dimensional Ising model, $\gamma/\nu = 1.75$, is recovered to high accuracy. The number extracted for the system dimensionally reduced by the majority rule method, $\gamma/\nu = 1.00$, is a bit surprising. This value suggests that the phase transition has become a "mean field" transition, i.e., that fluctuations are unimportant. Evidence for this idea is also furnished from calculations of the block couplings performed by the method outlined above.

In the low temperature (large K) region, all couplings (two-spin, four-spin, six-spin, ...) become of the same size. Thus in this entire region of coupling every spin is strongly coupled to every other spin, no matter how far away, and the mean-field approach should be valid.

For the decimation mapping, the ratio γ/ν equals 0.75. This value can also be derived by a simple argument. Consider Eqs. (10) and (11). At the point of a second order phase transition (which occurs in the original system) the correlation length is large. Since for decimation the reduced system is just a row in the original system, the correlation length there also diverges (in fact, with the same critical exponent). Thus all of the spins are highly correlated and

$$\chi_{2D}(L)|_{\max} \propto L \chi_{1D}(L)|_{\max} \quad (15)$$

Equation (15) in turn implies that the ratio γ/ν in the two-dimensional case is one more than its one-dimensional counterpart, as observed.

Both methods of reduction thus preserve the phase transition, though the critical behaviour of the reduced system depends upon the projection method used. The steady increase in $\chi_{\max}$ with L implies a divergence in the susceptibility, which in turn implies the existence of a phase transition. Note that the susceptibility diverges even in the reduced system, for the reflection $\sigma_i \rightarrow -\sigma_i$ is still a symmetry of the reduced action, and so the magnetization $\langle \sigma_{1D} \rangle$ retains its status as an order parameter. The derivative of the magnetization with respect to its thermodynamically conjugate variable, the magnetic field, gives the (divergent) susceptibility.

Unfortunately the nearest-neighbour coupling is not conjugate to an order parameter, and so the fluctuations of the nearest neighbour term in the reduced action do not diverge. The next step in the program -- calculating more exponents by studying the divergence of the specific heat -- therefore requires knowledge of the entire block action at each system size and becomes prohibitively difficult.

We have also made a preliminary calculation of the ratio γ/ν for a two-dimensional Ising system constructed by a majority rule reduction of a three-dimensional nearest neighbour Ising model. The result is $\gamma/\nu \approx 1.7$. Curiously enough this is consistent with the two-dimensional nearest-neighbour Ising result. Thus it is conceivable that this dimensionally reduced three-dimensional Ising model is in the same universality class as the familiar two-dimensional model. Extraction of the block couplings suggested the possibility that these couplings also were not long-ranged. An attempt to calculate the ratio of critical exponents α/ν from the finite-size scaling of the specific heat was made. Unfortunately, the approach to the infinite system limit for finite-size scaling in the three dimensional Ising model is pathologically slow (see, e.g., [9]) and so no definite conclusions followed.

In conclusion, a method for mapping a field theory or statistical system to a theory embedded in fewer spacetime dimensions has been presented. The technique was successfully applied to the two-dimensional Ising model, and for various projection operators produced one-dimensional systems with long-range forces and novel critical exponents. A method for calculating the action for the reduced system was also presented and shown to give physical insight into the dynamics of the reduced system.

REFERENCES

- 1) Th. Kaluza, Sitzungsver. Preuss, Akad. Wiss. Berlin, Math. Phys. K1 (1921) 966.
- 2) P. Klein, Z. Phys. 37 (1926) 895;
Arkiv. Mat. Astron. Fys. B, 34A (1946); Contribution to 1955 Bern Conf.,
Helv. Phys. Acta Suppl. IV (1956) 58.
- 3) E. Witten, Nucl. Phys. B186 (1981) 412.
- 4) K.G. Wilson, Rev. Mod. Phys. 47 (1973) 773; 55 (1983) 583;
K.G. Wilson and J. Kogut, Phys. Reports 12C (1974) 25;
Th. Niemeijer and J.M.J. van Leeuwen, in "Phase Transitions and Critical
Phenomena", eds. C. Domb and M.S. Green (Academic, New York, 1976) v. 6.
- 5) D.J.E. Callaway and R. Petronzio, Physics Letters 139B (1984), 189;
CERN Preprint TH.3844 (1984), to appear in Nucl. Phys. B (FS).
CERN Preprint TH.3900 (1984), to appear in Physics Letters B.
- 6) M.E. Fisher, in Critical Phenomena, ed. M.S. Green, (Academic, New York, 1971);
M.E. Fisher and M. Barber, Phys. Rev. Letters 28 (1972) 1516;
M. Barber, in Phase Transitions and Critical Phenomena, v. 8, eds. C. Domb
and J. Lebowitz (Academic, New York, 1983) p. 146.
- 7) L. Onsager, Phys. Rev. 65 (1944) 117.
- 8) See, e.g., C.J. Thompson, in Domb and Green, op. cit., v.1.
Also see L. van Hove, Physica 15 (1949) 951;
D. Ruelle, Statistical Mechanics: Rigorous Results, (Benjamin, New York, 1969);
F.J. Dyson, Comm. Math. Phys. 12 (1969) 212;
L.D. Landau and E.M. Lifshitz, Statistical Physics, (Pergamon, London, 1958);
D.J. Thouless, Phys. Rev. 187 (1969) 732.
- 9) R. Pearson, Phys. Reports 103 (1984) 185.

Table

Critical exponent γ/ν from finite-size scaling

Ising	(d = 2)	1.75
Majority rule	(d = 1)	1.00
Decimation	(d = 1)	0.75

Figure caption

Fig. 1 : Plot of $\chi_{\max}$ versus L for various systems. The legend is: filled circles, original two-dimensional Ising model; open circles, one-dimensional system reduced by majority rule; open triangles, one-dimensional system reduced by decimation.