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POSITIVITY OF ENERGY FOR A CLASS OF SCALAR-TENSOR
THEORIES OF GRAVITY

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A B S T R A C T

It is shown that for a class of scalar-tensor theories of gravity, positivity of the total energy follows from the reasonable requirement that the gravitational coupling which they define is finite, positive and asymptotically constant. The result extends without difficulty to a related unified theory for gravity and the electroweak fields.

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The main purpose of this letter is to show that, as a consequence of the positive energy theorems for general relativity, theories in the class described by Eq. (1) below have positive energy in four dimensions. The zero value corresponds to the Minkowski space-time which, in view of the same theorems, is stable¹⁾.

In the context of the renewed interest in scalar-tensor theories of gravity²⁾⁻⁵⁾ and also in relation to a recent unified theory for the gravitational and electroweak fields⁶⁾, the following type of action will be considered

$$S(g, \phi) = L^{2q+2-n} \int d^n x \sqrt{-g} [-g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}]^q R. \quad (1)$$

The space-time dimension n as well as the exponent q are left arbitrary and the pair will be viewed as a point $P(n, q)$ in the appropriate plane. R is the Riemannian curvature scalar ($R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \dots$) and g is the determinant of the metric $g_{\mu\nu}$ with Lorentz signature $(-, +, +, \dots)$. The scalar field ϕ may have internal indices and $\phi_{,\mu}$ denotes the fully covariant derivative of ϕ . To keep track of dimensionalities, we take ϕ dimensionless and let the dimensionful parameter L (= length) carry the scale. The action (1) is then scale invariant on the line $P(n, (n-2)/2)$, on which the exponent of L vanishes.

Our interest in the action (1) stems from the fact that it seems relevant to a number of fundamental issues. It has been shown that the action at $P(4, \pm 1)$ leads to a theory which fully incorporates Mach's principle into general relativity³⁾. Further, the scale invariance of the action at $P(4, 1)$ as well as possible internal symmetries are spontaneously broken without the addition of any potential⁴⁾. Finally, if we consider R to be the curvature scalar in a Riemann-Cartan geometry and adopt a particular interpretation for the torsion, we are led to a unified theory for the gravitational and electroweak fields⁶⁾.

We may now turn to our main objective, namely the proof of positivity of the total energy for the class of theories we consider. A first (and obvious) difficulty comes from the non-minimal coupling of the scalar ϕ to gravity which renders the energy theorems not directly applicable. Of great relevance here is the possibility of employing a Weyl transformation as done by Deser⁷⁾ in the context of a Brans-Dicke type of action. Here, however, diagonalization of the

action (1) is not as straightforward in view of an apparent singularity at $P(4,1)$, as we will see.

Under a Weyl transformation of the metric

$$g^{\mu\nu} = f^m \bar{g}^{\mu\nu} \quad (2)$$

which leaves the co-ordinates unchanged, one can verify that

$$\begin{aligned} R_{\mu\nu} = & \bar{R}_{\mu\nu} + \frac{m^2}{4}(m-2) \left[(\ln f)_{,\mu} (\ln f)_{,\nu} \right. \\ & \left. - (\ln f)_{,\lambda} (\ln f)_{,\sigma} \bar{g}^{\lambda\sigma} \bar{g}_{\mu\nu} \right] \\ & + \frac{m}{2} \left[(m-2) (\ln f)_{\mu;\nu} + (\ln f)_{\lambda;\sigma} \bar{g}^{\lambda\sigma} \bar{g}_{\mu\nu} \right] \end{aligned} \quad (3)$$

from which $R = g^{\mu\nu} R_{\mu\nu}$ follows as

$$\begin{aligned} R = & \left[\bar{R} - \frac{m^2}{4}(m-1)(m-2) (\ln f)_{,\mu} (\ln f)_{,\nu} \bar{g}^{\mu\nu} \right. \\ & \left. + m(m-1) (\ln f)_{\mu;\nu} \bar{g}^{\mu\nu} \right] f^m. \end{aligned} \quad (4)$$

The scalar function f must be regular (without poles or zeros) and positive everywhere. We have set $\partial_{\mu} (\ln f) = (\ln f)_{,\mu}$ and all barred quantities, including the semi-colon are defined with respect to the metric $\bar{g}_{\mu\nu}$. The action (1) can now be written as

$$S = L^{(m+1)(2q+2-m)} \int d^n x \sqrt{-\bar{g}} f^{\frac{m}{2}(2q+2-m)} (-\bar{g}^{\mu\nu} \phi_\mu \phi_\nu)^q. \quad (5)$$

$$\left[\bar{R} - \frac{m^2}{4} (m-1)(m-2) (\ln f)_\lambda (\ln f)_\lambda \bar{g}^{\lambda\lambda} \right]$$

such that the new scalar field $\bar{\phi}$, defined as

$$\bar{\phi} = m \sqrt{\frac{(m-1)(m-2)}{2}} L^{1-\frac{m}{2}} \ln f \quad (6)$$

with

$$f = -L^2 \bar{g}^{\mu\nu} \phi_\mu \phi_\nu \quad (7)$$

will be coupled minimally to gravity if we also require that

$$f^{\frac{m}{2}(2q+2-m)} (-\bar{g}^{\mu\nu} \phi_\mu \phi_\nu)^q = 1. \quad (8)$$

Our task may now seem impossible because the value of m following from Eq. (8), namely

$$m = \frac{-2q}{2q+2-m} \quad (9)$$

has a pole at $P(4,1)$, which is precisely the point of main interest. One can show, however, that the transformation (2) is, in fact, still regular at $P(4,1)$. Indeed, from Eqs. (2) and (7), we find

$$f^{m+1} = -L^2 g^{\lambda\rho} \phi_\lambda \phi_\rho \quad (10)$$

in terms of which we rewrite (2) as

$$\bar{g}_{\mu\nu} = \left(-L^2 g^{\lambda\rho} \phi_\lambda \phi_\rho \right)^{\frac{m}{m+1}} g_{\mu\nu} \quad (11)$$

or, in view of (9)

$$\bar{g}_{\mu\nu} = \left(-L^2 g^{\lambda\rho} \phi_\lambda \phi_\rho \right)^{\frac{2g}{m-2}} g_{\mu\nu} \quad (12)$$

which is regular at $P(4,1)$ as claimed. To see that the new field $\bar{\phi}$ defined by Eq. (6) is also regular at $P(4,1)$ we write it as

$$\bar{\phi} = \frac{m}{m+1} \sqrt{\frac{(m-1)(m-2)}{2}} L^{1-\frac{m}{2}} \ln f^{m+1} \quad (13)$$

or, in view of Eqs. (9) and (10),

$$\bar{\phi} = \sqrt{\frac{2(m-1)}{m-2}} g L^{1-\frac{m}{2}} \ln (-L^2 g^{\mu\nu} \phi_\mu \phi_\nu) . \quad (14)$$

The final form of the action is simply

$$S(\bar{g}, \bar{\phi}) = \int d^m x \sqrt{-\bar{g}} \left(L^{2-m} \bar{R} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{\phi}_\mu \bar{\phi}_\nu \right) \quad (15)$$

and the resulting field equations are also regular at $P(4,1)$ even when re-expressed in terms of $(\phi, g_{\mu\nu})$. Provided that $g_{\mu\nu}$ is a four-dimensional asymptotically flat metric, it follows from Eq. (12) that the only additional requirement for the applicability of the positive energy theorem, is that the quantity

$$K^{-1} = L^{2(q-1)} (-g^{\mu\nu} \phi_{,\mu} \phi_{,\nu})^q \quad (16)$$

is finite, positive and asymptotically constant.

We will now briefly discuss our result. From the physical interpretation of κ in the original action (1) at $P(4,q)$ and comparing with Eq. (12) we arrive at the interesting result that the function which induces the appropriate Weyl transformation is essentially the gravitational coupling of the original theory. This observation makes unnecessary any additional requirements concerning that function, in the sense that, for any theory following from the action (1) with reasonable gravitational coupling, there will be a well-defined Weyl transformation leading to the diagonal form (15) and thus the original physical system will have positive total energy. The ground state of the action (15) is Minkowski space-time¹⁾ which we describe by

$$\left(\bar{g}_{\mu\nu} = \eta_{\mu\nu} , \quad \bar{\phi} = \bar{\phi}_0 \right) \quad (17)$$

with $\bar{\phi}_0$ constant. In view of Eqs. (12), (14), (16) and (17), it follows that the ground state of the original action at $P(4,q)$ is characterized by

$$\left(g_{\mu\nu} = \eta_{\mu\nu} , \quad \kappa = L^2 \exp \left[-\frac{\bar{\phi}_0 L}{\sqrt{3}} \right] \right) . \quad (18)$$

This result is in agreement with the one derived for $q = 1$ by a less rigorous argument in Ref. 4).

As a final remark, we note that the above results can be extended without any difficulty to the action

$$S(g, \phi) = \int d^4x \sqrt{-g} \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \overset{\circ}{R} + \frac{1}{4} F_{\lambda\sigma} F_{\sigma\epsilon} g^{\lambda\sigma} g^{\epsilon\epsilon} \right) \quad (19)$$

which results from Eq. (1) at $P(4,1)$ when the geometry is enlarged to Riemann-Cartan in the context of the unified theory mentioned earlier. $\overset{\circ}{R}$ is the Riemannian part of the curvature and the second term in (19) is the familiar

kinetic term for the gauge fields. Positivity of energy now follows from the observation that $\sqrt{-g} F^2$ is invariant under the Weyl transformation (2) in four dimensions and thus diagonalization of the action (1) can proceed exactly as in the previous case.

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