CERN-TH.3955/84

NON-STANDARD 2D CRITICAL STATISTICAL MODELS FROM LIOUVILLE THEORY

J.-L. Gervais
Laboratoire de Physique Théorique
Ecole Normale Supérieure
F-75231 Paris, France

and

A. Neveu
CERN - Geneva

ABSTRACT

The recent construction of the two-dimensional Liouville field theory suggests the existence of a whole new class of critical statistical models. The conformal dimensions are given by a general formula which encompasses the already known models. We give an operator construction of conformal operators, and a related integral representation for the Green's functions. This representation can be interpreted as a Coulomb system where the charge of the vacuum is non-zero, and in which two particular charges, which we call spurions, appear, and correspond to marginal operators. Contrary to the standard models, none of the new ones lies in the Coulomb phase, even though they exhibit scaling.

CERN-TH.3955/84
July 1984

1. - INTRODUCTION

In two-dimensional statistical mechanics models, remarkably simple expressions for the critical dimensions have gradually been discovered by various means. In the present paper, we discuss the following magic formula for the conformal weights

$$\delta_{m,n} = \frac{1}{8t} [1 - 4t^2 (m\eta_+ + n\eta_-)^2], \quad (1.1)$$

the critical dimensions being sums of two terms of this type. t is a parameter which depends on the model, η_+ and η_- are the solutions of the equation

$$2t\eta^2 - \eta + 1 = 0, \quad (1.2)$$

and m and n are integers^{1*}. The magic formula reproduces all the known critical exponents. For example, setting

$$t = \frac{y-2}{y^2}, \quad q^{1/2} = 2 \cos \frac{1}{2} \pi y, \quad (1.3)$$

for the q -state Potts model¹⁾, one finds the well-known exponents^{2),3)}

$$d_{12} = 2\delta_{12} = \frac{1+y}{2-y}, \quad d_{1/y, 1/y} = 2\delta_{1/y, 1/y} = \frac{1-y^2}{4(2-y)}. \quad (1.4)$$

Formula (1.1) also appears in mathematics. Consider the Virasoro algebra

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m}, \\ L_n^\dagger &= L_{-n}. \end{aligned} \quad (1.5)$$

For a given c , irreducible representations of (1.5) are classified by the highest weight ϵ defined by

=====

(1*)

When η_+ and η_- are real, we choose $\eta_+ > \eta_-$.

$$L_n |\varepsilon\rangle = 0, \quad n > 0, \quad (1.6)$$

$$L_0 |\varepsilon\rangle = \varepsilon |\varepsilon\rangle. \quad (1.7)$$

Consider the space of all vectors obtained by repeated applications of L_{-n} 's on $|\varepsilon\rangle$. The matrix of their inner products can be computed entirely from (1.5), (1.6) and (1.7). Kaz⁴⁾ has shown that the determinant of this matrix precisely vanishes for $\hbar < 0$ and

$$c = 1 + \frac{3}{c}, \quad (1.8)$$

$$\varepsilon = \delta_{mn}, \quad m > 0, n > 0. \quad (1.9)$$

It has been shown^{5),6)} that this region $\hbar < 0$ contains a whole sequence of statistical models, including all the known ones.

Motivated by recent developments on the theory of random surfaces, we have developed the quantum Liouville theory⁷⁾⁻¹⁰⁾. This is a conformally invariant theory in the continuum. In its solution, formulae (1.1) and (1.8) appear^{9),10)}, with $\hbar$ positive playing the rôle of Planck's constant. Thus, in view of the standard connection between field theories in the continuum and critical statistical systems, it appears that the class of such systems is wider than previously thought, and this is the point of the present paper.

We have written Eq. (1.1) for arbitrary m and n , but not all values may actually be reached. In Section 2, we discuss the actual values of $\delta_{m,n}$, according to the value of $\hbar$; there are three natural regions, $\hbar < 0$ ($c < 1$), $\hbar > 1/8$ ($1 < c < 25$), $0 < \hbar < 1/8$ ($c > 25$). $\hbar < 0$ has been discussed in Refs. 5) and 6). For $\hbar > 1/8$, q is complex in general, except for the special values $\hbar = 1/6, 1/4, 1/2$, which appear in our solution of the quantum Liouville theory¹¹⁾ in the strong coupling region. $0 < \hbar < 1/8$ corresponds to our solution of the Liouville theory in the weak coupling region^{8),9)}.

In Section 3, we construct conformally covariant operators in terms of two free fields related by a non-linear transformation, using techniques we developed in Ref. 8). Closure of the operator algebra under commutation imposes the values (1.1) for the conformal weights. Integral representations are obtained for the

corresponding vacuum expectation values^{1*}. They have a natural interpretation in terms of a Coulomb gas where the vacuum is charged.

In Section 4, we discuss related statistical models. In particular, the q -state Potts model, with q given by Eq. (1.3), is known to be related, via the random cluster model, to a random surface model [BC-SOS, See Ref. 2) and references therein]. Interpreting the height of the surface as an electrostatic potential gives back the essential features of the above-mentioned Coulomb gas. The known models for $h < 0$ correspond to the Coulomb phase. Changing the sign of h through $h = \infty$ is going through the Kosterlitz-Thouless transition. This makes the statistical interpretation of the new critical models suggested by the Liouville theory particularly intriguing.

2. - THE SET OF CRITICAL DIMENSIONS

The possible values of $\delta_{m,n}$ can be discussed from Eqs. (1.1) and (1.2). Three regions in h are to be distinguished.

The case $h < 0$ has already been extensively discussed^{5),6)}. For completeness, we recall that closure of the operator algebra⁵⁾ forces us to choose

$$h = \frac{2s}{2(r+s)^2}, \quad (2.1)$$

with r and s integers, $r > 0$, $s < 0$. Positivity further restricts⁶⁾ the possible values to $r+s = -1$, and to $m \leq n < r$.

For $h > 1/8$, η_+ and η_- are complex conjugates. Reality and positivity of $\delta_{m,n}$ are achieved only for $m = n = 1$ ($\delta_{1,1} = 0$) and for $m = -n$, which gives:

$$\delta_{m,-m} = \frac{1}{8h} [1 + (8h - 1)m^2]. \quad (2.2)$$

This is the only case where we shall find an infinite set of values for δ . Indeed, we have built¹¹⁾ a local field compatible with this set of dimensions for $1/h = 2, 4, 6$. We notice that these are precisely the only values of $h > 1/8$ for

=====

(1*)

This integral representation has been independently derived by Feigin and Fuks¹²⁾. In the language of the operator Schrödinger equation (3.22), it corresponds to solving for the second solution $\tilde{V}$ in terms of V by the standard quadratures.

which formula (1.3) gives a real q , namely

$$q = -4 \sinh^2 \frac{\pi\sqrt{3}}{2} \quad \text{for } 1/t = 2.6, \quad (2.3)$$

$$q = 4 \cosh^2 \pi \quad \text{for } 1/t = 4. \quad (2.4)$$

For $0 < t < 1/8$, η_+ and η_- are real again. The set of possible values of $\delta_{m,n}$ can be obtained from the spectrum of the Liouville theory⁹⁾; m must be zero, and n varies so that $\delta_{0,n}$ remains positive. For comparison with the Potts model, we give δ in terms of the parameter y of (1.3):

$$\delta_{0,n} = \frac{1}{8(2-y)} (4n^2 - y^2). \quad (2.5)$$

The operator algebra closes under commutation¹⁰⁾ for

$$t = \frac{N}{2(N+1)^2}, \quad N \text{ integer} > 0. \quad (2.6)$$

This corresponds to $q = 4$ for all values of N . We notice the similarity of (2.6) with (2.1). Indeed, for

$$t = \frac{rs}{2(r+s)^2}, \quad r, s, \text{ integer} > 0, \quad (2.7)$$

the operator algebra again closes, but on a larger set of values of δ than those given by (2.5); this set is simply obtained by taking all integer values of m and n , not necessarily positive, which give a positive $\delta_{m,n}$. This set is still finite because η_+ and η_- are commensurate. For general r and s , q can take a dense set of values between 0 and 4.

Let us now comment on the above results for the corresponding critical statistical models. There is a chain of relationships between the Potts model, the random cluster model, the six-vertex model, the BC-SOS model and the Coulomb gas²⁾. This one-parameter family of models is usually parametrized by q . For $0 < q < 4$, they exhibit a second-order phase transition, and correspond to $c < 1$ ($\Delta < 0$). For $q > 4$, the transition is first-order. The values of q given by Eqs. (2.3) and (2.4), together with the construction of local fields in the continuum¹¹⁾, indicate that nevertheless there seem to exist scaling limits in the region $q > 4$. Exactly at $q = 4$, the values of t between 0 and $1/8$ given by (2.6) ($y = 2N+2$), and the construction of the local Liouville field⁹⁾, show the existence of an infinite family of continuous models. The other values of t between

0 and $1/8$ lead back to q between 0 and 4, but the critical exponents are different because δ is a multivalued function of q . Looking at formula (1.1), one sees that as one continues around branch points in the parameter space q at fixed n and m , the sign of δ may change, a value which was unphysical can become physical and vice-versa. One may speculate that this is similar to what happens¹³⁾ for the anharmonic oscillator $\lambda\chi^4$, where all the energy eigenvalues can be obtained from a single one by going around branch points in the complex λ plane.

3. - ALGEBRAS OF CONFORMAL OPERATORS

The two-dimensional conformal algebra is generated by the operators L_n and $\bar{L}_n$ corresponding to the infinitesimal transformations $\delta_n z$ of the variable z

$$\delta_n z = \varepsilon z^{n+1}, \quad n \in \mathbb{Z}, \quad (3.1)$$

and of its complex conjugate $\bar{z}$:

$$\delta_n \bar{z} = \bar{\varepsilon} \bar{z}^{n+1}. \quad (3.2)$$

Their algebra is

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n,-m}, \quad (3.3)$$

$$[\bar{L}_n, \bar{L}_m] = (n-m) \bar{L}_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n,-m}, \quad (3.4)$$

$$[L_n, \bar{L}_m] = 0. \quad (3.5)$$

and one has $L_n^+ = L_{-n}$ and $\bar{L}_n^+ = \bar{L}_{-n}$. The central charge c is a real number. It has recently been shown that this algebra can be used to derive the conformal exponents of soluble statistical models^{5),6)}. In this section, we first briefly recall how (3.3) and (3.4) can be represented in terms of the harmonic oscillators of a free field, then proceed to build simple conformally covariant operators as normal ordered exponentials of this free field, and discuss their algebra.

In a recent paper¹⁴⁾, we have introduced the harmonic oscillator-like operators p_n , $n \in \mathbb{Z}$, with the following commutation relation:

$$[p_n, p_m] = \frac{3n}{c-1} \delta_{n,-m} . \quad (3.6)$$

The zero mode p_0 commutes with all other p_n modes and can be diagonalized. We have shown that

$$L_n = \frac{c-1}{6} \left[\sum_2 N(p_n p_{n-2}) - i n p_n \right] + \frac{c-1}{24} \delta_{n,0} , \quad (3.7)$$

where N means normal ordering, is a representation of (3.3). We also introduce analogous $\bar{p}_n$ modes, commuting with p_n , to represent $\bar{L}_n$. The p_n and $\bar{p}_n$ operators can be considered as the Fourier components of a field $\Phi(z, \bar{z})$:

$$\begin{aligned} \Phi(z, \bar{z}) &= p_0 \ln z + \bar{p}_0 \ln \bar{z} + i \sum_{n \neq 0} \left(\frac{p_n}{n} z^{-n} + \frac{\bar{p}_n}{n} \bar{z}^{-n} \right) , \\ \mathcal{P}(z) &= i z \frac{\partial \Phi}{\partial z} , \quad \bar{\mathcal{P}}(\bar{z}) = i \bar{z} \frac{\partial \Phi}{\partial \bar{z}} , \end{aligned} \quad (3.8)$$

$$L_n = \frac{c-1}{6} \oint \frac{dz}{2\pi i} z^n \left[N(\mathcal{P}^2) + i z \frac{\partial \mathcal{P}}{\partial z} \right] .$$

If $L_0 + \bar{L}_0$ is interpreted as the quantum mechanical Hamiltonian operator for the theory with spatial variable σ in the periodic one-dimensional box $[0, 2\pi]$, one has at time τ ⁸⁾:

$$\Phi(\sigma, \tau) = i p_0(\sigma + \tau) - i \bar{p}_0(\sigma - \tau) + i \sum_{n \neq 0} \left[\frac{p_n}{n} e^{-in(\sigma + \tau)} + \frac{\bar{p}_n}{n} e^{in(\sigma - \tau)} \right] . \quad (3.9)$$

In this sense, Φ is a free field describing left and right moving massless plane waves in 1+1 dimensions. Because of this, it is natural to use the notation

$$\hbar = \frac{3}{c-1} , \quad (3.10)$$

so that the commutation relation (3.6) takes the familiar form. For the application to statistical mechanics, τ is Wick rotated to it, and one recovers the variables

$$z = e^{i\sigma - \tau}, \quad \bar{z} = e^{-i\sigma - \tau}. \quad (3.11)$$

In statistical mechanics, the system lives on the complex z plane. Since the barred and unbarred modes are essentially identical, we shall henceforth discuss only the unbarred modes. One must only remember that the barred modes must be supplied at each stage to ensure that z and $\bar{z}$ remain complex conjugates of each other in all conformal transformations. Thus, $L_0 + \bar{L}_0$ generates dilatations, $L_0 - \bar{L}_0$ generates rotations, $L_{-1} + \bar{L}_{-1}$ translations along the real axis, and $L_{-1} - \bar{L}_{-1}$ translations along the imaginary axis.

The fundamental building block of conformally covariant operators is the normal ordered exponential

$$V(\alpha, z) = z^{(\frac{1}{2} - i p_0)\alpha} \exp - i\alpha \sum_{n>0} \frac{p_n}{n} z^n \exp i\alpha \sum_{n>0} \frac{p_n}{n} z^{-n}. \quad (3.12)$$

It is straightforward to check¹⁵⁾ that one has the following commutation relation:

$$\begin{aligned} L_n(p_0 + i\alpha t) V(\alpha, z) - V(\alpha, z) L_n(p_0) \\ = z^n \left[z \frac{\partial}{\partial z} + (n+1) \left(-\frac{1}{2} - \frac{1}{2} t \alpha^2 \right) \right] V(\alpha, z). \end{aligned} \quad (3.13)$$

In Eq. (3.13), we see that one must take different values of the zero mode p_0 in L_n on each side of V , as indicated explicitly by the notations $L_n(p_0 + i\alpha t)$ and $L_n(p_0)$. Hence, it is natural to introduce¹⁵⁾ an operator K which shifts p_0 :

$$p_0 K = K(p_0 - i\alpha t), \quad (3.14)$$

and to redefine $V(\alpha, z)$ to $KV(\alpha, z)$. With this new definition, one has the simple commutation relation

$$[L_n, V(\alpha, z)] = z^n \left[z \frac{\partial}{\partial z} - (n+1)\alpha \right] V(\alpha, z), \quad (3.15)$$

where p_0 is now an operator. Equation (3.15) means that $V(\alpha, z)$ is a conformally covariant operator with conformal weight

$$\delta = -\frac{1}{2}\alpha - \frac{1}{2}t\alpha^2. \quad (3.16)$$

One constructs the analogous operator $\bar{V}(\bar{\alpha}, \bar{z})$ with the modes $\bar{p}_n$. From its transformation properties under dilatations ($L_0 + \bar{L}_0$) and rotations ($L_0 - \bar{L}_0$), the product $V\bar{V}$ has conformal dimension $d = \delta + \bar{\delta}$ and conformal spin¹⁶⁾ $J = \delta - \bar{\delta}$.

At this early stage, it is instructive to compute the expectation value of a product of operators V with different values α_i of α , taken at different points z . We suppose that $|z_1| > |z_2| \dots > |z_{n-1}| > |z_n|$. We take the expectation value between states $|0, p_0\rangle$ which are annihilated by $p_n (n > 0)$, and which have a given value of the zero mode p_0 . One finds, by commuting the exponentials¹⁵⁾:

$$\begin{aligned} \langle 0, p'_0 = p_0 - it \sum_i \alpha_i | V(\alpha_1, z_1) V(\alpha_2, z_2) \dots V(\alpha_n, z_n) | 0, p_0 \rangle \\ = \prod_i z_i^{(\frac{1}{2} - ip_0)\alpha_i} \exp - it \sum_{j>i} \alpha_i \alpha_j L_n(z_i - z_j). \end{aligned} \quad (3.17)$$

The statistical mechanical vacuum must be translation- and dilatation-invariant. Thus, it is $|0, p_0 = -i/2\rangle$ which is the only state annihilated by L_0 and L_{-1} . By hermiticity, Eq. (3.17) then represents a vacuum expectation value only if

$$p'_0 = -p_0 = \frac{1}{2}i, \quad (3.18)$$

so that

$$\sum_i \alpha_i = -1/it. \quad (3.19)$$

If α_i is real, and if (3.17) is multiplied by its complex conjugate, one obtains on the right-hand side the exponential of the electrostatic energy of a system of n charges $\alpha_1 \dots \alpha_n$ in the plane. We interpret conditions (3.18) and (3.19) by saying that the vacuum has the charge of $1/\hbar$. Such a charged vacuum is encountered in the Coulomb gas interpretation of the Potts model²⁾, and we shall have more to say about it later.

The fact that V shifts the value of p_0 , together with the constraint (3.19), severely restricts the number of translationally invariant expectation values which one can compute with it. To overcome this problem, we have introduced⁸⁾ another field $\tilde{F}(\sigma)$, defined by

$$N(P^2) + P_\sigma = \tilde{N}(\tilde{P}^2) + \tilde{P}_\sigma, \quad (3.20)$$

$$p_0 = -\tilde{p}_0, \quad (3.21)$$

where $\tilde{N}$ means normal ordering of the modes of $\tilde{P}$. We have shown that $\tilde{P}$ is also a free field, its modes $\tilde{p}_n$ satisfying the commutation relation (3.6). In the same fashion as P , it can be used to define an operator $\tilde{V}(\alpha, z)$, which has the same value of δ , but which shifts p_0 by the opposite amount.

From the condition (3.18) on the vacuum, and from the action of V on p_0 through Eqs. (3.13) and (3.14), it is natural to consider p_0 as pure imaginary. This is in conflict with the most natural Hermiticity condition $p_n^+ = p_{-n}$, compatible with the condition $L_n^+ = L_{-n}$. We have also shown that for pure imaginary p_0 , the natural Hermiticity condition is $p_n^+ = \tilde{p}_{-n}$.

Equation (3.20) is an operator Ricatti equation, and the two Bloch wave solutions of the corresponding Schrödinger equation are precisely V and $\tilde{V}$ for suitable values of α . Indeed, by direct calculation, one has⁹⁾ for both V and $\tilde{V}$:

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\eta^2 t^2}{4} \right) \delta V(\eta, e^{i\sigma}) = 2\eta^2 t \delta \left[\left(\sum_{n \geq 0} L_n e^{in\sigma} \right) V + \frac{1}{2} L_0 V + \frac{1}{2} V L_0 + V \sum_{n \geq 0} L_n e^{in\sigma} \right], \quad (3.22)$$

if and only if η satisfies the equation:

$$2t\eta^2 - \eta + 1 = 0. \quad (3.23)$$

We see here the two values η_+ and η_- of the charge, which appeared in the Introduction. A family of operators with the conformal weight δ_{mn} of Eq. (1.1) is obtained by taking the product of m_1 operators $V(\eta_+)$, m_2 operators $\tilde{V}(\eta_+)$, n_1 operators $V(\eta_-)$ and n_2 operators $\tilde{V}(\eta_-)$, with

$$m = m_1 + m_2, \quad n = n_1 + n_2. \quad (3.24)$$

We shall denote this operator by

$$V^{m_1, \eta_+ + m_2, \eta_-} \tilde{V}^{m_2, \eta_+ + n_2, \eta_-}(\gamma).$$

The product is defined first by taking the operators at different points and letting them come together, extracting the short distance singular behaviours, according to

$$V(\alpha_1, z_1) V(\alpha_2, z_2) \underset{z_1 \rightarrow z_2}{\sim} (z_1 - z_2)^{-\alpha_1 \alpha_2 \hbar} V(\alpha_1 + \alpha_2, z_2), \quad (3.25)$$

with the same equation for V replaced by $\tilde{V}$. For high Fourier modes, i.e., short distances, P and $\tilde{P}$ are equivalent⁸⁾, and the short-distance behaviour of a product of a V and a $\tilde{V}$ is the same as in Eq. (3.25). This can also be seen from the explicit representation of $\tilde{V}$ which we give below. It follows that the conformal weight of $V^{m_1 \eta + n_1 \eta^-} \tilde{V}^{m_2 \eta + n_2 \eta^-}$ depends only on $m_1 + m_2$ and $n_1 + n_2$.

We now come back to the discussion of the various values of the zero mode. We have seen that there is one translationally invariant vacuum $|0, p_0 = -\frac{i}{2}\rangle$. We shall call ground states those which differ from it only by the value of p_0 . These ground states can be obtained by the following limit:

$$|0, -\frac{i}{2} - i\alpha\hbar\rangle = \lim_{z \rightarrow 0} V(\alpha, z) |0, -\frac{i}{2}\rangle. \quad (3.26)$$

V is the only operator with weight δ which gives a finite limit for the right-hand side of this equation. On the other hand, from (3.7),

$$L_0 |0, -\frac{i}{2} - i\alpha\hbar\rangle = \left[-\frac{1}{8\hbar} (1 + 2\alpha\hbar)^2 + \frac{1}{8\hbar} \right] |0, -\frac{i}{2} - i\alpha\hbar\rangle. \quad (3.27)$$

This equation shows that the spectrum of L_0 coincides with the set of possible anomalous dimensions (3.16). This allows for the possibility of a semiclassical determination of anomalous dimensions for small $\hbar$, by considering L_0 as the Hamiltonian of the quantum Liouville theory in a box⁹⁾.

For a given value of L_0 , there are obviously, in general, two possible solutions of α_+ and α_- for α in Eq. (3.27) or, equivalently, there are two operators $V(\alpha_+, z)$ and $V(\alpha_-, z)$ which have the same conformal dimension. We note that

$$\alpha_+ + \alpha_- = -\frac{1}{c}, \quad (3.28)$$

so that the total shift of p_0 induced by the product $V(\alpha_+, z_1) V(\alpha_-, z_2)$ is precisely 1. This product takes us from the vacuum $|0, -i/2\rangle$ to its adjoint $\langle 0, i/2|$, and one has simply

$$\langle 0, \frac{i}{2} | V(\alpha_+, z_1) V(\alpha_-, z_2) | 0, -\frac{i}{2} \rangle = (z_1 - z_2)^{-2\delta}. \quad (3.29)$$

This is a two-point correlation function of the corresponding critical statistical system.

We shall call isobars operators like $V(\alpha_+)$ and $V(\alpha_-)$ which have the same conformal weight, but differ by the way they shift p_0 . $V(\eta_+)$ and $\tilde{V}(\eta_+)$ are also isobars.

The two solutions η_+ and η_- of Eq. (3.23) are such that $V(-2\eta_\pm, z)$ has weight 1. This fact allows for the integration of this operator, to obtain two conformally covariant operators with dimension zero, S_+ and S_- :

$$S_\pm(z) = \int_{z_0}^z dz' V(-2\eta_\pm, z'). \quad (3.30)$$

One has:

$$[L_n, S_\pm(z)] = \tilde{z} V(-2\eta_\pm, z) - \tilde{z}_0 V(-2\eta_\pm, z_0). \quad (3.31)$$

The terms on the right-hand side of this equation vanish if z and z_0 coincide with the positions of other operators in the expectation value*. Consider, for example:

$$V(\eta_+, z_2) V(\eta_+, z_1) S_+(z_1) |0, p_0\rangle = V(\eta_+, z_2) V(\eta_+, z_1) \int_0^{z_1} dz V(-2\eta_+, z) |0, p_0\rangle \quad (3.32)$$

Because of the short-distance property (3.25), this quantity has the same conformal properties as $V(\eta_+, z_2) V(\eta_+, z_1)$, but it is most interesting to remark that it does not shift p_0 . Since an operator is, in general, entirely defined by its conformal weight and shift of p_0 ⁹⁾, we conclude that

$$\begin{aligned} V(\eta_+, z_2) V(\eta_+, z_1) \int_0^{z_1} dz V(-2\eta_+, z) |0, p_0\rangle \\ \sim V(\eta_+, z_2) \tilde{V}(\eta_+, z_1) |0, p_0\rangle, \end{aligned} \quad (3.33)$$

up to an overall factor which may be a function of p_0 . This fact can be checked directly from the differential equation (3.22). Similarly, one can check that

$$\begin{aligned} V(\eta_+, z_2) \int_{z_1}^{z_2} dz V(-2\eta_+, z) V(\eta_+, z_1) \\ \sim V(\eta_+, z_2) \tilde{V}(\eta_+, z_1) + \tilde{V}(\eta_+, z_2) V(\eta_+, z_1). \end{aligned} \quad (3.34)$$

This means that we have essentially obtained an integral representation of $\tilde{V}(\eta_\pm)$ as the product of $V(\eta_\pm)$ times the operators $S_\pm$ which we call spurions^{2*}.

The introduction of the spurions enables us, in principle, to obtain an integral representation of $V^{m_1\eta_+ + n_1\eta_-} V^{m_2\eta_+ + n_2\eta_-}(z)$ for general positive integer m_2 and n_2 , which contains $m_2 + n_2$ integration variables. The actual determination of the integration paths for a given expectation value of products of such operators becomes quite involved as m_2 and n_2 increase¹⁷⁾.

=====

(1*)

In case the boundary values in Eq. (3.31) give infinity rather than zero, it is to be understood that the integral of Eq. (3.30) should be defined by integration by parts, dropping all end-point terms which do not vanish, just as one defines the ordinary integral representation of the function $\Gamma(z)$ for $z < 0$.

(2*)

This integral representation was independently derived by Feigin and Fuks.¹²⁾ In the language of the operator Schrödinger equation (3.22), it corresponds to solving for the second solution $\tilde{V}$ in terms of V by the standard quadratures.

The spurions appear naturally in the Coulomb gas interpretation of the Potts model (see next section). They enable us to relax the charge conservation condition (3.19) to

$$\sum_i \alpha_i - 2m\eta_+ - 2n\eta_- = -\frac{1}{\epsilon}, \quad m, n \in \mathbb{N}. \quad (3.35)$$

The operators with positive weight which appear in the case $\hbar < 0$ of Ref. 5) correspond to α_i being of the form

$$\alpha_i = m_i \eta_+ + n_i \eta_-, \quad m_i, n_i \in \mathbb{N}.$$

One has $\eta_+ + \eta_- = 1/2\hbar$, so that it is always possible to satisfy Eq. (3.35) for m and n in this case. For $\hbar > 0$, as discussed in Refs. 9) and 10), the operators with positive weight are of the same form, except that m_i and n_i are non-positive: all terms on the left-hand side of Eq. (3.35) have the same sign. Hence, in this case, it is not always possible to find a solution to this equation with m and n positive: only some correlation functions can be computed in this way, the others being more complicated. It is not our purpose to discuss the general case in this paper. We only remark that for small positive $\hbar = (N/2(N+1)^2)$ (N large), which is the weak coupling regime of the Liouville field theory, we have

$$\eta_- = \frac{N+1}{N}, \quad \eta_+ = \frac{1}{N},$$

so that one may compute the vacuum correlation functions of up to $N+1$ operators with $\alpha = -2\eta_-$ (conformal weight 1) with this method. Alternatively, one may have up to N spurions S_- , and still be able to satisfy Eq. (3.35), while no S_+ spurion may be included, $2\eta_+$ being by itself larger than $1/\hbar$.

The differential equation (3.22) satisfied by $V(\eta_{\pm})$ and $\tilde{V}(\eta_{\pm})$ has been used⁹⁾ to derive commutation relations between $V(\eta_{\pm}, z_1)$ and $\tilde{V}(\eta_{\pm}, z_2)$. The method of Ref. 9) can trivially be extended to find that $V(\eta_{\pm})$ and $\tilde{V}(\eta_{\pm})$ essentially commute¹¹⁾. These results, combined with the definition of $V^{m_1\eta_++n_1\eta_-}\tilde{V}^{m_2\eta_++n_2\eta_-}(z)$ by the short-distance expansion (3.25) show that these operators satisfy a closed algebra⁵⁾ under commutation for positive integer m_1, n_1, m_2 and n_2 .

Higher order differential equations for $V^{m_1\eta_++n_1\eta_-}\tilde{V}^{m_2\eta_++n_2\eta_-}(z)$ can be derived by simple extension of the calculation of Ref. 9). For example, it is straightforward to check that one has

$$\begin{aligned} \partial_\sigma^3 A = & 4\gamma^4 t^2 \partial_\sigma A + 4\gamma^2 t \partial_\sigma \left[A \sum_{n \geq 0} L_n e^{-in\sigma} + \sum_{n \geq 0} L_{-n} e^{in\sigma} A \right. \\ & \left. + A L_0 + L_0 A \right] + 4\gamma^2 t (\partial_\sigma A) \sum_{n \geq 0} L_n e^{-in\sigma} \\ & + 4\gamma^2 t \sum_{n \geq 0} L_{-n} e^{in\sigma} \partial_\sigma A + 16\gamma^4 t^2 \left[A \partial_\sigma \sum_{n \geq 0} L_n e^{-in\sigma} + \left(\partial_\sigma \sum_{n \geq 0} L_{-n} e^{in\sigma} \right) A \right] \end{aligned} \quad (3.36)$$

where A can be any of the three operators $z^{20} v^2 \eta(\sigma)$, $z^{20} \tilde{v}^2 \eta(\sigma)$, $z^{20} v \tilde{v} \eta(\sigma)$. When A appears in a correlation function, Eq. (3.36) gives a partial differential equation of order 3 for this correlation function, by pushing L_r to the right, L_{-r} to the left until they annihilate on the vacuum. These differential equations are higher order generalizations of the hypergeometric equation. The spurions automatically give integral representations of the solutions of these equations.

Such differential equations were postulated in Ref. 5) by assuming decoupling of zero norm states. However, as in Ref. 9) and for Eq. (3.36) above, they can be derived by straightforward, if tedious, calculations. They follow from the fact that the vanishing of the Kac determinant means something stronger than the existence of a zero norm state used in Ref. 5); it means that this state is actually zero.

Let us note that the classical limit of Eq. (3.36), keeping terms of order $\hbar$ only on the right-hand side, occurs in the inverse scattering theory of the Korteweg-de Vries equation.

Closure of the algebra of conformal operators has selected the set of all possible conformal weights, the magic formula (1.1) for δ_{mn} . This does not mean that all values of m and n do occur in a given theory. For example, as mentioned in the Introduction, d_{mn} must be positive, and the Hilbert space on which the operators of this section act must have positive norm. In Ref. 6), it has been shown that these requirements are met at the special negative values $\hbar = -(r(r+1)/2)$, $r = 1, 2, \dots$, where the algebra of the $v^{m_1} \eta^{+n_1} \eta - \tilde{v}^{m_2} \eta^{+n_2} \eta$ operators closes on a finite set. For these special values of $\hbar$, the short-distance behaviour of v^{mn} and $\tilde{v}^{mn}$ can actually be less singular than Eq. (3.25) indicates.

In Refs. 9) and 10), by studying the quantum Liouville theory, we have found another mechanism for the closure of the algebra in the case $0 < \hbar < 1/8$. We have shown that a bilinear form in $V(\sigma)$, $V(2\pi-\sigma)$, $\tilde{V}(\sigma)$, $\tilde{V}(2\pi-\sigma)$ can be found, which commutes with itself for $\sigma \neq \sigma'$ and acts on a positive norm Hilbert space.

Closure of the algebra is achieved by the fact that the coefficients of the bilinear form vanish when the corresponding operator would create negative norm states. This bilinear form is the Liouville field $e^{-\phi/2}$. In Refs. 9) and 10), the primary aim is the construction of a consistent string model. It is argued that this model is consistent only for the special values $h = N/2(N+1)^2$, $N = 1, 2, 3, \dots$ where negative integer powers of $e^{-\phi/2}$ are precisely the operators with positive dimension corresponding to the spectrum of the eigenstates of L_0 on which $e^{-\phi/2}$ acts consistently. This consistency condition thus coincides with the condition of the existence of a critical statistical model. In Ref. 11), we extend this result to the three special values $h = 1/6, 1/4, \frac{1}{2}$, discussed in Section 2, where $\eta_{\pm}$ are complex.

4. - RELATED CRITICAL STATISTICAL MODELS

As is well known among specialists of two-dimensional physics, there is a chain of discrete models which are related to the q -state Potts model we mentioned in the Introduction²⁾. The simplest such correspondence is the connection with the random cluster model, which allows one to define the Potts model for arbitrary q . Since this will be explicitly needed, we give a few details. The partition function of the q -state Potts model with nearest neighbour interactions reads

$$Z = \sum_{\{\sigma_i\}} \prod_{\langle i,j \rangle} e^{K \delta_{\sigma_i, \sigma_j}}. \quad (4.1)$$

We take a simple square lattice. The spins σ_i at each site i can point in q directions. $\delta_{\sigma_i, \sigma_j}$ is one if $\sigma_i = \sigma_j$, and zero otherwise. From the obvious identity

$$e^{K \delta_{\sigma_i, \sigma_j}} = (e^K - 1) \delta_{\sigma_i, \sigma_j} + 1, \quad (4.2)$$

one can rewrite

$$Z = \sum_{\text{graphs}} (e^K - 1)^{N_b} q^{N_c}. \quad (4.3)$$

The graphs are obtained by connecting the sites by an arbitrary number N_b of bonds, thereby forming N_c clusters. All sites belong to a cluster. Some

clusters contain only one site. In each cluster, the spins are pointing in the same direction. Denote by N_l the total number of loops in a graph. From Euler's relation, one has

$$N_e = N_s + N_l - N_c, \quad (4.4)$$

where N_s is the total number of lattice sites. This allows us to write (4.3) as

$$Z = q^{\frac{1}{2}N_s} \sum_{\text{graphs}} t^{N_l} q^{\frac{1}{2}N_p}, \quad (4.5)$$

$$t = (e^K - 1) / q^{1/2}, \quad (4.6)$$

$$N_p = N_c + N_l. \quad (4.7)$$

N_p is easily seen to be equal to the number of closed circuits one can draw around each cluster and in each loop. Formula (4.5) can be used to reformulate the model in various ways²⁾. It was also used to argue that the model renormalizes in the scaling limit to a Coulomb gas with vortex-like excitations [see Ref. 2) and references therein]. This will establish the connection with Section 3, and especially with the rather peculiar electrostatic analogue which came out. Before discussing this scaling limit, we show that the random cluster model (4.5) can be looked at as a surface model whose features are already similar to this electrostatic analogue. This particular point draws a lot of inspiration from Ref. 2).

From Eq. (1.3), one can write

$$q^{1/2} = e^{i\pi y/2} - e^{-i\pi y/2}, \quad (4.8)$$

and the factor $q^{N_p/2}$ of (4.5) can be obtained by summing over the two ways to follow each of the N_p closed circuits introduced above, each sense of rotation giving one piece of (4.8). By convention, we choose the factor $e^{i\pi y/2}$ if one turns clockwise around the circuit (more on this later). Next, a discrete variable v is introduced on each lattice site. By convention, one chooses $v = v_0$ at one fixed site taken as the origin. From it, one moves to any other site along an arbitrarily chosen line with the rule that v increases by $\pm \frac{1}{2}$ when one crosses each circuit, with the $+$ ($-$) sign corresponding to the circuits flowing from the right (from the left) when one moves away from the origin along the

path. Since the circuits are closed, v does not depend upon the path, and defines a discrete two-dimensional surface. [These ideas were first used to derive a soluble model of roughening¹⁸⁾.] One may draw an amusing analogy with meteorological maps. Remembering that the wind turns anticlockwise around depressions in our hemisphere, one sees that v would give the atmospheric pressure, the circuits would be identified with the isobar lines and the sense of rotation on the circuits would give the direction of the wind.

In the continuum limit, v will go over to the Coulomb potential, and we can regard the introduction of additional terms $\exp i \pi v_r Q$ in (3.5) as reflecting the presence of an external charge Q . Since only $v_r - v_o$ is well defined, we can only describe the Coulomb energy of two charges Q and Q' at lattice sites r and r' by using (3.5) to introduce the average

$$\langle \exp [i \pi Q (v_r - v_o) + i \pi Q' (v_{r'} - v_o)] \rangle. \quad (4.9)$$

This should not depend upon the choice of the origin. Assuming²⁾ that all sites of the boundary of the lattice belong to the same cluster, and moving the origin from this cluster to some other point, one crosses circuits for which the factors $\exp \pm i \pi y/2$ become:

$$\exp [\pm i \pi (y + Q + Q')/2]. \quad (4.10)$$

This change will indeed be irrelevant if

$$Q + Q' = 0, \quad (4.11)$$

or

$$Q + Q' = -2y. \quad (4.12)$$

The first case is the usual neutrality condition. The second case looks as if there is a charge $2y$ floating around. It corresponds exactly to the peculiar Coulomb gas of Section 3. In this connection, we note that the particular minus sign in Eq. (4.12) is due to our earlier choice of attributing the factor $\exp i \pi y/2$ for clockwise rotations. The opposite choice would have led to $Q + Q' = 2y$ instead. In the meteorology analogue, this second possibility would describe the southern hemisphere.

Going back to the previous choice, we look at the continuum limit, which, according to standard wisdom, is the Coulomb gas partition function

$$Z = \sum_{\{Q\}} \prod_{\vec{r}} P(Q(\vec{r})) \exp \sum_{\vec{r}, \vec{s}} \frac{Q(\vec{r})Q(\vec{s})}{2x} \ln |\vec{r} - \vec{s}|, \quad (4.13)$$

where $P(Q)$ describes the probability distribution of the charges. As is now well known, the physical system described by (4.13) is quite different for $x < 2$ and for $x \geq 2$. For $x < 2$, all internal charges are paired in neutral molecules (Coulomb phase). Hence, if we look at the correlation function of external charges, we can disregard the internal charges, and we only get the exponential of minus the Coulomb energy.

For the two-point function, in order to obey condition (4.12), we put the external charges Q and $-2y-Q$, at points $\vec{r}$ and $\vec{s}$, obtaining

$$|\vec{r} - \vec{s}|^{-Q(2y+Q)/2x}. \quad (4.14)$$

This formula is the direct analogue of (3.29). The associated conformal weight

$$\delta = \frac{1}{8x} [(Q+y)^2 - y^2], \quad (4.15)$$

agrees with the magic formula (1.1) if we let

$$x = 2-y, \quad (4.16)$$

$$Q = (m-1)(y-2) + 2(n-1). \quad (4.17)$$

For clarification, we note that there is a factor $1/4$ between (4.14) and (4.15) since the correlation function decreases with a power $-2d = -4\delta$. Moreover, Q and $-2y-Q$ are on the same footing. Exchanging them amounts to changing the signs of m and n in (4.17). This obviously leaves (1.1) invariant.

According to (4.17), the two spurions S_- and S_+ of Section 3 have charges -4 and $-2y + 4$. They play a special rôle in the scaling limit because they have weight $+1$. They are thus marginal operators¹⁹⁾. Their integral can be added to the action in the scaling limit without destroying its scale invariance. In the

language of the renormalization group, this means that if their coupling constant is non-vanishing, one may obtain a family of fixed points.

In the discrete model, adding any number of ± 4 charges is irrelevant, because $\exp 2i\pi v = 1$, but in the scaling limit, v becomes continuous, and this need not be preserved. In order to derive the scaling limit of correlation functions, one can mimic the construction of Section 3. All relevant charges are such that by adding n charges -4 and m charges $4-2\gamma$ (n, m integers), one can obtain a system with total charge -2γ . One will thus define the associated correlation function as

$$\int \prod_{i=1}^n ds_i \prod_{j=1}^m dt_j < \prod_{\ell=1}^n \exp i\pi Q_{\ell} (v_{s_{\ell}} - v_0)/2 \prod_{\ell=1}^m \exp 2i\pi (v_{t_{\ell}} - v_0) \times \prod_{\ell=1}^m \exp i\pi(2-\gamma)(v_{t_{\ell}} - v_0) > . \quad (4.18)$$

In the same way as for the two-point function, this average does not depend on the origin, but corresponds to the integral representation of Section 3 in the scaling limit. In the Coulomb phase, where pair creation of charges is unimportant, we can consider that (4.18) derives from a partition function where the energy involves the two additional terms:

$$\lambda_1 \int ds \exp -2i\pi (v_s - v_0) + \lambda_2 \int dt \exp i\pi(2-\gamma)(v_t - v_0) , \quad (4.19)$$

(4.16) being the coefficient of λ_1^n, λ_2^m . Since these operators are marginal, they do not spoil the existence of the scaling limit.

Next we turn to $\hbar > 0$ (Liouville field theory region). In the standard Coulomb picture, this corresponds to passing through the Kosterlitz-Thouless transition. For $x \gtrsim 2$, pairs of ± 4 charges dissociate because their energy $(8/x)\text{Ln}R$ becomes smaller than their entropy $4 \text{Ln}R$. One enters a plasma phase. Screening by vacuum pairs becomes essential. In the standard model, Debye screening gives a finite correlation length, indicating that the transition of the discrete model becomes first-order. Nonetheless, whenever it is applicable, Liouville theory indicates that formula (1.1), i.e., naive Coulomb energy, still holds. We are thus led to the conclusion that for the values of $\hbar$ or c mentioned in the Introduction, there do exist systems with second-order phase transitions. At these values, the terms (4.19) must play an essential rôle, not only because they are marginal, but also because plasma screening means that the number of

spurious need not be directly related to the external charges. It is remarkable that the magic formula for the dimension can still hold. On the other hand, our efforts to derive general Green's functions for the physical (positive dimension) operators in the region $\hbar > 0$ have met with considerable difficulties, which indicates that indeed non-trivial statistical effects are taking place. As discussed in the previous section, the simple integral representation of this section is of no use, except in very special cases.

REFERENCES

- 1) F.Y. Wu, Rev. Mod. Phys. 54 (1982) 235.
- 2) M.P.M. den Nijs, Phys. Rev. B27 (1983) 1674.
- 3) M.P.M. den Nijs, J. Phys. A12 (1979) 1857.
- 4) V.G. Kac, Proceedings of the International Congress of Mathematicians, Helsinki (1978); Lecture Notes in Physics, Vol. 94, p. 441 (Springer Verlag, N.Y. 1979).
- 5) A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, CERN preprint TH.3827 (1984).
- 6) D. Friedan, Z. Qiu and S. Shenker, Enrico Fermi Institute preprint 83-66 (1983).
- 7) J.-L. Gervais and A. Neveu, Nucl. Phys. B199 (1982) 59; Nucl. Phys. B209 (1982) 125.
- 8) J.-L. Gervais and A. Neveu, Nucl. Phys. B224 (1983) 329.
- 9) J.-L. Gervais and A. Neveu, Nucl. Phys. B238 (1984) 125.
- 10) J.-L. Gervais and A. Neveu, Nucl. Phys. B238 (1984) 396.
- 11) J.-L. Gervais and A. Neveu, in preparation.
- 12) B.L. Feigin and D.B. Fuks, Funktz. Analiz. 16 (1982) 47.
- 13) C.M. Bender and T.T. Wu, Phys. Rev. D7 (1973) 1620.
- 14) J.-L. Gervais and A. Neveu, CERN preprint TH.3954 (1984).
- 15) Dual theory, ed. M. Jacob (North Holland, Amsterdam, 1974); J. Scherk, Rev. Mod. Phys. 47 (1975) 123.
- 16) J.-L. Gervais and B. Sakita, Nucl. Phys. B34 (1971) 477.
- 17) V.S. Dotsenko and V.A. Fateev, NORDITA preprint 84/8 (1984).
- 18) H. van Beijeren, Phys. Rev. Lett. 38 (1977) 993.
- 19) L.P. Kadanoff, Rev. Mod. Phys. 49 (1977) 267.