GRAVITATIONAL AND LORENTZ ANOMALIES IN TWO DIMENSIONS

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ABSTRACT

We present a direct and complete computation of the Lorentz and gravitational anomalies in two dimensions.

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I. Introduction

Recently L. Alvarez-Gaumé and E. Witten ¹⁾ discussed gravitational anomalies in several field theories. It was pointed out afterwards ²⁾⁻⁴⁾ that these anomalies are probably disguised forms of the anomaly of the Adler-Bardeen type, connected to Lorentz (orthogonal in the Euclidean version) transformations. This has also been checked in Ref.5) using the family's index theorem. The form of these anomalies in $4n-2$ dimensions can be determined by an elegant method ^{2),6)} starting from the invariant which gives the singlet anomaly in $4n$ dimensions. This algebraic procedure yields automatically solutions of the related Wess-Zumino consistency conditions ⁷⁾ and therefore candidates for the anomaly. Since the uniqueness of these solutions has not been proved in general, and since the normalization remains undetermined, it is still worthwhile to do a calculation.

We offer here a direct computation of the Lorentz and gravitational anomalies for a Weyl spinor in two dimensions, and compared to the diagrammatical method used in ¹⁾, we try to fill two gaps. We go beyond the weak field approximation used there, and we treat Lorentz and gravitational anomalies simultaneously. That this is necessary follows from the observation ^{3),4)} that the Adler-Bardeen anomaly can be transferred from the Lorentz to the gravitational sector.

In Section II, we present the model and the simplifications of the two-dimensional case. In Section III, we discuss in detail the regularization scheme that we use and the methods to compute anomalies that follow from it. The actual calculations are presented in Section IV, but we refer to the appendices for more details. Conclusions are given in Section V.

II. The model

Our starting point is the Lagrangian for a Weyl fermion in two dimensions, coupled to an external gravitational field by means of the vielbein e^μ_a , and the corresponding (zero torsion) spin-connection

$$\omega_{\mu ab} = -(\partial_\mu e_{a\rho} - \Gamma_{\mu\rho}^\sigma e_{a\sigma}) e^\rho_b$$

$$\text{With } \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma^{mn} = \frac{1}{4} [\gamma^m, \gamma^n]$$

$$\sigma^{01} = \frac{1}{2} \gamma_5$$

we write

$$\mathcal{L} = e i \bar{\psi} (e^\mu_a \gamma^a \partial_\mu + \frac{1}{2} e^\mu_a \gamma^a \omega_{\mu mn} \sigma^{mn}) (\frac{1-\gamma_5}{2}) \psi \quad (1)$$

We shall use extensively the simplifications of the two-dimensional case, in particular the Abelian nature of the Lorentz group. We take $\omega_{\mu 01} = \omega_\mu$ and $\Omega_{01} = \Omega$ for the parameter of the Lorentz transformation. Latin indices are raised and lowered with $\eta^{ab} = (1, -1)$ and $\epsilon^{01} = 1$. The Lagrangian in Eq.(1) is invariant under Lorentz and co-ordinate transformations. To make a Feynman-diagrammatic evaluation of the vertex functional $S(e^\mu_a, \omega_\mu)$, we split $e e^\mu_a = \delta^\mu_a + h^\mu_a$. We moreover modify the original Lagrangian by adding a free right-handed fermion to the Lagrangian and a Dirac mass term which explicitly breaks the chiral symmetry. We then have a new Lagrangian

$$\mathcal{L} = i \bar{\psi} (\not{\partial} + \not{\Pi} + iM) \psi$$

$$\not{\Pi} = (\not{A} + \not{B}) (\frac{1-\gamma_5}{2}) \quad (2)$$

$$\not{A} = h^\mu_a \gamma^a \partial_\mu, \quad \not{B} = \frac{1}{2} e e^\mu_a \gamma^a \omega_\mu \gamma_5$$

In this Lagrangian only the mass term breaks the original (chiral) symmetry, which we summarize as:

Lorentz transformation

$$\delta \psi = - \frac{1}{2} \Omega (\frac{1-\gamma_5}{2}) \psi$$

$$\delta \bar{\psi} = - \frac{1}{2} \bar{\psi} (\frac{1+\gamma_5}{2}) \Omega$$

$$\delta \not{\Pi} = \Omega \not{\Pi} + \frac{1}{2} [\not{\Pi}, \Omega] + [\Omega, \not{\partial}] + \frac{1}{2} (\not{\partial} \Omega) (\frac{1-\gamma_5}{2}) \quad (3)$$

General co-ordinate transformations :

$$\begin{aligned}\delta_\xi \psi &= \left(\frac{1-\gamma_5}{2}\right) \partial_\alpha(\psi) \xi^\alpha \\ \delta_\xi \bar{\psi} &= (\partial_\alpha \bar{\psi}) \xi^\alpha \left(\frac{1+\gamma_5}{2}\right) \\ \delta_\xi \Pi &= \partial_\alpha(\xi^\alpha \Pi) - [\Pi, \xi^\mu] \partial_\mu - [(\bar{\psi} \xi^\mu) \partial_\mu - (\partial_\alpha \xi^\alpha) \bar{\psi}] \left(\frac{1-\gamma_5}{2}\right)\end{aligned}\quad (4)$$

III. Regularization

Following closely the spirit of the original paper of Bardeen ⁸⁾, we regularize the vertex functional S by point splitting. Since we do not envisage the same type of calculation, we introduce a slightly different prescription for the splitting. We write

$$S^{\text{reg.}} = \text{tr} \ln \left[\frac{\not{D} + e^{-\epsilon \not{\partial}} \not{\Pi} + iM}{\not{D} + iM} \right] = \text{tr} \ln \left(\frac{\not{D} e + iM}{\not{D} + iM} \right) \quad (5)$$

which implies a uniform splitting for the vertices, $|Z_K + \epsilon\rangle \langle Z_K|$, whereas Ref. 8) uses a splitting of the form $|Z_K + (\epsilon/n)\rangle \langle Z_K|$ for the K -th vertex in a loop with n vertices. A symmetric average of ϵ is understood. We next compute the variation of S^{reg} under Lorentz and general co-ordinate transformations. This gives

$$\begin{aligned}\delta_\Sigma S^{\text{reg.}} &= \text{tr} \left[-\frac{iM}{2} e^{-\epsilon \not{\partial}} \Sigma \frac{1}{\not{D} e + iM} \right] \\ &\quad - \frac{1}{2} \text{tr} \left[\not{D} e^{-\epsilon \not{\partial}} \Pi (e^{-\epsilon \not{\partial}} - 1) \Sigma - e^{-\epsilon \not{\partial}} \Sigma (e^{-\epsilon \not{\partial}} - 1) \Pi \right] \frac{1}{\not{D} e + iM} \\ &\quad + \int \frac{dx dp}{(2\pi)^2} e^{i\epsilon \cdot p} \Sigma(x)\end{aligned}\quad (6)$$

$$\begin{aligned}\delta_\xi S^{\text{reg.}} &= \text{tr} \left\{ \left[-\frac{iM}{2} e^{-\epsilon \not{\partial}} (\partial_\alpha \xi^\alpha) - \frac{iM}{2} (\xi^\alpha \partial_\alpha + \partial_\alpha \xi^\alpha) \gamma_5 \right] \frac{1}{\not{D} e + iM} \right\} \\ &\quad + \text{tr} \left\{ [e^{-\epsilon \not{\partial}} \Pi (e^{-\epsilon \not{\partial}} - 1) \xi^\mu \partial_\mu - e^{-\epsilon \not{\partial}} \partial_\mu \xi^\mu (e^{-\epsilon \not{\partial}} - 1) \Pi] \frac{1}{\not{D} e + iM} \right\}\end{aligned}\quad (7)$$

as can be computed directly from the symmetry breaking terms in the Lagrangian with some care for the treatment of the lowest order terms or from a calculation analogous to the one in Ref. 8). One sees the symmetry breaking effect of the mass term and of the ϵ -separation. The last term in

$\delta_\Omega S^{\text{reg}}$ originates in the tadpole diagram for S and is due to the fact that the infinite series do not match. In the spirit of Bardeen's paper, one could compute the anomaly from the ϵ -separation effect after subtracting from S the divergent parts by means of appropriate counter terms. It seems more economical to follow a Pauli-Villars type reasoning, which we present in the following way.

We can subtract from S the leading divergences with the counter term

$$S^{(2)} = \int \frac{d^4 p d^4 x}{(2\pi)^4} \text{tr} \ln \left[1 + \frac{i e^{i\epsilon \cdot p} \not{p} \not{\Gamma}^\lambda(x) \not{p} \lambda \left(\frac{1-\gamma_5}{2} \right)}{(p^2 - M^2)} \right] \quad (8)$$

$$\not{\Gamma}^\lambda = \gamma^\lambda_a \gamma^a$$

This can be written as a sum of local, mass independent functionals with infinite coefficients $\int d^4 p e^{i\epsilon \cdot p}$, $M^2 \int d^4 p [e^{i\epsilon \cdot p} / (p^2 - M^2)]$ and a finite part proportional to M^2 . One can verify explicitly that the variation of $S^{(2)}$ cancels all divergences in the right hand side of Eqs. (6) and (7). Alternatively one can rely on power counting to show that this is unavoidable. $S^{(2)}$ is the leading term corresponding to an expansion in external momenta (if Γ^B is considered as proportional to $\partial_\lambda \Gamma^\lambda$ (i.e. as having one extra momentum) and its variation should cancel analogous terms proportional to $1/\epsilon^2$ or M^2 . S can have extra logarithmic divergences with coefficient $\int d^4 p [e^{i\epsilon \cdot p} / (p^2 - M^2)]$, but there are no analogous terms on the right-hand side of Eqs. (6) and (7). This is due to the improved convergence of the mass term, which has leading $M^2 \int d^4 p [e^{i\epsilon \cdot p} / (p^2 - M^2)]$ behaviour and finite corrections, and to the form of the ϵ -separation terms, whose corrections occur as derivatives of linearly divergent expressions, typically $\int d^4 p (d/dp_\lambda)(p_p / (p^2 - M^2))$, which are finite and therefore mass independent. This implies that S can be renormalized completely by subtracting a second counter term which is Lorentz and general co-ordinate invariant, and with a coefficient that we choose as $\int d^4 p e^{i\epsilon \cdot p} / (p^2 - M^2 - \mu^2)$ to allow for a mass zero limit. Indicating the renormalized mass and ϵ -separation terms as $\langle M \rangle^{\text{ren}}$ and $\langle \epsilon \rangle^{\text{ren}}$, we have

$$\delta S^{\text{ren.}} = \langle M \rangle^{\text{ren.}} + \langle \epsilon \rangle^{\text{ren.}}$$

We are interested in the massless Lagrangian given by Eq. (1), which explains the care in our last subtraction. In that limit, only $\langle \varepsilon \rangle^{\text{ren}}$ remains, but since it was mass independent, we can compute it in the massive theory, and even take the limit $M \rightarrow \infty$. Since S^{ren} becomes a local functional in that limit, and since the anomaly is only defined up to variations of local functionals, we see that the anomaly can be computed as

$$\int a\{\pi, \xi\} = - \lim_{M \rightarrow \infty} \langle M \rangle^{\text{ren.}}_{\{\pi, \xi\}} \quad (9)$$

Apart from displaying explicitly the relation to the point splitting method, this shows that a naïve Pauli-Villars method, with only one mass, although insufficient to regulate, works if one develops this mass term in external momenta, and deletes the divergent part in the limit $M \rightarrow \infty$. A last comparison to Eq. (8) shows that the structure of the divergences in this two-dimensional model is similar to the four-dimensional gauge case. The main difference is of course that we have to consider an infinite number of graphs, due to the derivative coupling $\Gamma^\lambda \partial_\lambda$.

IV. Computation of the anomalies

A standard Feynman diagrammatic evaluation can be used for the mass terms in Eqs. (6) and (7). Retaining only the parts which contribute as $M \rightarrow \infty$ one remarks that an infinite number of graphs contribute, since for each type of graph, the insertion of a Γ^A -vertex gives an extra factor $(\Gamma^\lambda p_\lambda)/(\not{p}-M)$ where p is the loop momentum. This indicates that an effective propagator has to be used, which is obtained by summing over an infinite set of graphs. After rationalization, this propagator has the form $1/(p^2 + \not{p}\Gamma^\lambda p_\lambda - M^2)$. To avoid the combinatorics, we shall immediately use an operator language. We first derive an equivalent form for Eqs. (6) and (7) by rationalizing the propagators and by extracting the chiral projectors. The anomalies take the form

$$A(\Omega) = \lim_{M \rightarrow \infty} \text{tr} \frac{M^2}{2} \exp^{-\epsilon \cdot \partial} \Omega \left\{ \frac{1}{\partial^2 + M^2 + e^{-\epsilon \cdot \partial} \not{A} + B} \left(\frac{1+\gamma_5}{2} \right) + \frac{1}{\partial^2 + M^2 + e^{-\epsilon \cdot \partial} \not{A} + B} \left(\frac{1-\gamma_5}{2} \right) \right\} \quad (10)$$

$$A(\xi) = A(\Omega \rightarrow (\partial_a \xi^a)) + \lim_{M \rightarrow \infty} \text{tr} \left[\frac{M^2}{2} e^{-\epsilon \cdot \partial} ((\partial_a \xi^a) + 2 \xi^a \partial_a) \left\{ \frac{1}{\partial^2 + M^2 + e^{-\epsilon \cdot \partial} \not{A} + B} \left(\frac{1+\gamma_5}{2} \right) - \frac{1}{\partial^2 + M^2 + e^{-\epsilon \cdot \partial} \not{A} + B} \left(\frac{1-\gamma_5}{2} \right) \right\} \right] \quad (11)$$

if subtraction of the leading divergence is understood, as explained above. In two dimensions, we can immediately work out the complete Dirac algebra.

$$\begin{aligned} \gamma^a \gamma^b (\delta_a^\sigma e e^\rho_b) \left(\frac{1-\gamma_5}{2} \right) &= \delta_a^\sigma (\eta^{ab} - \epsilon^{ab}) (e e^\rho_b) \left(\frac{1-\gamma_5}{2} \right) = x^\rho \left(\frac{1-\gamma_5}{2} \right) \\ \gamma^a \gamma^b (e e^\rho_a \delta_b^\sigma) \left(\frac{1+\gamma_5}{2} \right) &= x^\rho \left(\frac{1+\gamma_5}{2} \right) \end{aligned} \quad (12)$$

depend only on one of the two indices and are functions of $x^\rho = (e e^\rho_0 - e e^\rho_1)$. The part in Γ^B is worked out in the same way and if we introduce $B = -\frac{1}{2} x^\mu \omega_\mu$, we can rewrite Eqs. (10) and (11) as

$$A(\Omega) = \lim_{M \rightarrow \infty} \text{tr} \left[\frac{M^2}{2} \Omega \left(\frac{1}{\partial_\lambda (x^\mu \partial_\mu + B) + M^2} + \frac{1}{(x^\mu \partial_\mu + B) \partial_\lambda + M^2} \right) \right] \quad (13)$$

$$A(\xi) = A(\Omega \rightarrow (\partial_a \xi^a)) + \lim_{M \rightarrow \infty} \left\{ \frac{M^2}{2} \text{tr} [(\partial_a \xi^a) + 2 \xi^a \partial_a] \left[\frac{1}{\partial_\lambda (x^\mu \partial_\mu + B) + M^2} - \frac{1}{(x^\mu \partial_\mu + B) \partial_\lambda + M^2} \right] \right\} \quad (14)$$

We also dropped the regulator here, since it is only relevant for the part that we do not need, and suppose it to be implicit in the definition of "lim".

Remark that in these formulas, all indices are summed over, even if they are not contracted. They give the anomalies in terms of the variables of the original Lagrangian (1) which can be written as

$$\mathcal{L} = i \bar{\psi} [x^\mu \partial_\mu + B] \gamma^0 (1 - \frac{\gamma_5}{2}) \psi$$

The evaluation of the remaining operator traces can be reduced to an operator ordering problem. If $\hat{q}$, $\hat{p}$ are the usual position and momentum operators, where $\hat{p}$ has differential operator realization $i\partial/\partial q$, then we have for a non-commuting polynomial F

$$\text{tr}(F(\hat{q}, \hat{p})) = \int dx dp \langle p|x \rangle \langle x|F(\hat{q}, \hat{p})|p \rangle$$

and the trace is found by taking the standard ordering ($\hat{p}$'s to the right) for F . Now $\langle x|F(\hat{q}, \hat{p})|p \rangle$ is given by $\langle x|p \rangle F(x, p + i\partial/\partial x)$ where $\partial/\partial x$ acts on all functions of x right of the original position of the corresponding p . This is obvious for any polynomial, and is valid also for propagators where it parallels the expansion in external momenta of the Feynman diagram.

So we use

$$-\partial_\lambda [x^\mu \partial_\mu + B] \rightarrow (p_\lambda + i\partial_\lambda) [x^\mu (p_\mu + i\partial_\mu) + B]$$

and

$$\frac{-1}{\partial_\lambda (x^\mu \partial_\mu + B) + \eta \epsilon} \rightarrow \frac{1}{p_\lambda x^\mu p_\mu - \eta \epsilon} - \frac{1}{p_\lambda x^\mu p_\mu - \eta \epsilon} \{ i\partial_\lambda x^\mu p_\mu + i p_\lambda x^\mu \partial_\mu + (p_\lambda + i\partial_\lambda) B - \partial_\lambda x^\mu \partial_\mu \} \frac{1}{p_\lambda x^\mu p_\mu - \eta \epsilon} + \dots$$

This is also equivalent to the calculation of the symbol of this propagator⁹⁾. In fact, it is easy to show that the method is equivalent to ζ -function regularization¹⁰⁾ if one starts from a regularized determinant

$$-\frac{d}{ds} \Big|_{s=0} \int_0^\infty dt t^{s-1} \text{tr} [\exp t \mathcal{D}(x+n) (1 - \frac{\gamma_5}{2})] \quad (15)$$

The subsequent computation of the anomalies is long and tedious. More details are presented in appendix A. Using $(\partial_\lambda x^\lambda) = -\omega_\lambda x^\lambda$, and introducing $D = \frac{1}{2}(x^0 - x^1)$, we can write the results in the following form

$$a(\Omega) = \frac{-i}{4\pi} \int d\gamma \pi(\gamma) \partial_\lambda \left[-\frac{1}{6} \partial_\rho \left(\frac{1}{D} x^\rho \right) + \frac{1}{12D} \partial_\rho x^\rho \right] \quad (16)$$

$$a(\xi) = a(\Omega \rightarrow (\partial_\alpha \xi^\alpha))$$

$$\begin{aligned} & + \frac{-i}{4\pi} \int d\gamma \left\{ \frac{1}{6} \xi^\mu \partial_\rho \partial_\sigma \partial_\mu \left(\frac{x^\sigma}{D} \right) - \frac{1}{6} \xi^\mu \partial_\rho \partial_\sigma \left(\frac{1}{D} \partial_\mu x^\sigma \right) + \frac{1}{12} \xi^\mu \partial_\rho \partial_\mu \left(\frac{1}{D} \partial_\sigma x^\sigma \right) \right. \\ & \quad + \epsilon_{\mu\alpha} \xi^\mu \left[\partial_\rho \partial_\sigma \partial_\lambda \left(\frac{1}{24} \frac{x^\lambda x^\sigma}{D^2} \right) - \frac{1}{12} \partial_\rho \partial_\sigma \left(\frac{x^\lambda}{D^2} \partial_\lambda x^\sigma \right) \right. \\ & \quad \left. \left. + \partial_\rho \left(-\frac{1}{12D} x^\lambda x^\sigma \partial_\lambda \partial_\sigma D - \frac{1}{48D^2} (\partial_\lambda x^\lambda) (\partial_\sigma x^\sigma) + \frac{1}{12} \frac{x^\lambda}{D^2} \partial_\lambda \partial_\sigma x^\sigma \right) \right] \right\} \end{aligned} \quad (17)$$

These expressions for the anomalies can be simplified by choosing appropriate counter terms. Adding a first counter term

$$(Ct.1) = \frac{-i}{4\pi} \int d\gamma \left[\frac{1}{6} \omega_\lambda \partial_\rho \left(\frac{1}{D} x^\rho \right) + \frac{1}{24} \omega_\lambda \left(\frac{x^\rho \omega_\rho}{D} \right) \right] \quad (18)$$

we transform the Lorentz anomaly to the form

$$a(\Omega) = \frac{-i}{4\pi} \int d\gamma \pi(\gamma) \epsilon^{\lambda\rho} \partial_\lambda \omega_\rho \quad (19)$$

Then $[a(\xi) + \delta_\xi(Ct.1)]$ is the new form of the gravitational anomaly, but this can be cancelled completely by using a second Lorentz invariant counter term

$$(Ct.2) = \frac{-i}{4\pi} \int d\gamma \left[-4 \frac{\partial_\rho D}{D} \epsilon^{\lambda\rho} \omega_\lambda + \frac{2}{D} \partial_\lambda (x^\sigma \omega_\sigma) + \frac{2}{D} (x^\sigma \omega_\sigma) \omega_\lambda + \frac{x^\sigma}{D^2} \partial_\lambda \partial_\sigma D \right] \quad (20)$$

More details are given in appendix B.

V. Conclusion

We have computed simultaneously Lorentz and gravitational anomalies in two dimensions. The final result can be exhibited as

$$a(\pi) = -\frac{i}{96\pi} \int dy \pi^{ab} \partial_\lambda \omega_{\rho ba} \epsilon^{\lambda\rho}$$

$$a(\xi) = 0$$

which is a pure Lorentz anomaly of the Adler-Bardeen type and coincides with the result of the algebraic method. It has been pointed out ^{3),4)} that this form can also be seen as a pure gravitational anomaly, using the Wess-Zumino Lagrangian with the vielbein as Goldstone field. In two dimensions, one can even see more directly that the Lorentz anomaly can be transformed away, since

$$\delta \pi \int \frac{\partial_\lambda \pi}{D} \epsilon^{\lambda\sigma} \omega_\sigma = \int (\partial_\lambda \pi) \epsilon^{\lambda\sigma} \omega_\sigma$$

The method can in principle be used in arbitrary dimensions, but two dimensions seem to set the practical limit for a direct calculation. Although this case was special, with an Abelian Lorentz group, we do not expect to find any deviations from the algebraic method in higher dimensions.

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Appendix A

A straightforward application of the expansion of the propagators (15), followed by repeated partial integrations in y and p , allows us to express the anomalies as

$$\begin{aligned} \alpha(\eta) = & \int \frac{dy dp}{(2\pi)^2} \left[-\frac{\eta}{x^3} [p_\sigma p_\tau B(\partial_\lambda x^\lambda) - p_\sigma p_\tau B^2] \right. \\ & + \frac{1}{x^2} \left[-\frac{1}{12} (\partial_\mu \partial_\lambda \eta) x^\mu - \frac{1}{12} \eta \partial_\tau \partial_\lambda x^\lambda - \frac{1}{12} (\partial_\mu \eta) (\partial_\lambda x^\mu) \right] \\ & + \frac{\eta}{x^3} \left[\frac{1}{6} x^\mu \partial_\lambda \partial_\mu x^\sigma p_\sigma p_\tau + \frac{1}{6} (\partial_\tau x^\mu) p_\mu p_\sigma (\partial_\sigma x^\tau) \right. \\ & \left. \left. - \frac{1}{12} p_\sigma p_\tau (\partial_\lambda x^\tau) (\partial_\rho x^\lambda) + \frac{1}{4} (\partial_\lambda x^\lambda) (\partial_\rho x^\rho) p_\sigma p_\tau \right] \right] \end{aligned} \quad (A.1)$$

$$\begin{aligned} \alpha(\xi) = & \alpha(\eta \rightarrow (\partial_\alpha \xi^\alpha)) \\ & + \int \frac{dy dp}{(2\pi)^2} \frac{1}{x^3} \left[-(\partial_\rho \xi^\mu) p_\mu B(\partial_\lambda x^\lambda) p_\sigma + (\partial_\rho \xi^\mu) p_\mu p_\sigma B^2 \right] \\ & + \frac{1}{x^3} \left[-\frac{1}{12} \partial_\rho \partial_\sigma \xi^\mu p_\mu p_\tau (x^\lambda \partial_\lambda x^\tau + x^\sigma \partial_\lambda x^\tau) \right] \\ & + \frac{1}{x^2} \frac{1}{12} (\partial_\rho \partial_\sigma \xi^\mu) \partial_\mu x^\sigma \\ & + \frac{1}{x^3} (\partial_\rho \xi^\mu) \left[\frac{1}{6} p_\mu p_\tau x^\lambda \partial_\lambda \partial_\sigma x^\tau + \frac{1}{6} p_\lambda p_\tau (\partial_\mu x^\tau) (\partial_\sigma x^\lambda) - \frac{1}{4} p_\mu p_\tau (\partial_\lambda x^\sigma) (\partial_\sigma x^\tau) \right. \\ & \left. + \frac{1}{12} p_\mu p_\tau (\partial_\lambda x^\lambda) (\partial_\sigma x^\tau) - \frac{1}{12} p_\mu p_\tau (\partial_\lambda x^\sigma) (\partial_\sigma x^\lambda) + \frac{1}{4} p_\mu p_\tau (\partial_\lambda x^\lambda) (\partial_\sigma x^\sigma) \right] \end{aligned} \quad (A.2)$$

where we introduced $X = p_\mu x^\mu p_\lambda - 1$. The momentum integrals are brought to standard form.

We need

$$\begin{aligned} C^{\mu\nu} &= \frac{1}{2} (x^\mu + x^\nu) \\ \sqrt{-\det(C^{\mu\nu})} &= D = \frac{1}{2} (x^0 - x^1) = -\frac{1}{2} x^\mu \epsilon_{\mu\alpha} \end{aligned} \quad (A.3)$$

to write

$$\int \frac{dp}{(2\pi)^2} \frac{1}{x^2} \left\{ 1, \frac{p_\mu p_\nu}{X} \right\} = \frac{-i}{4\pi D} \left\{ 1, \left(\frac{C^{-1}}{4} \right)_{\mu\nu} \right\} \quad (A.4)$$

where

$$C^{-1}_{\sigma\tau} = \epsilon_{\sigma\alpha} \frac{x^\lambda}{D} \epsilon_{\lambda\tau} + \frac{\epsilon_{\sigma\tau}}{D} \quad (\text{A.5})$$

Using these formulas, it is not difficult to obtain Eq.(16) for the Lorentz anomaly. In the formulas for the gravitational anomaly, we have an extra index, and therefore, it can be written in many equivalent forms. To obtain Eq.(17), we made repeated use of cyclic identities in two dimensions ($\epsilon^{\lambda\rho} a^\sigma + \text{cyclic} = 0$), and relabelling of indices by

$$x^\mu = x^\rho - 2 \epsilon^{\rho\mu} D \quad (\text{A.6})$$

Appendix B

All the anomalies are expressed as functionals of $(D, x^\lambda, \omega_\rho)$. The transformation laws for these building blocks have the form

$$\left\{ \begin{array}{l} \delta_\Omega x^\alpha = \Omega x^\alpha \\ \delta_\Omega D = \Omega D \\ \delta_\Omega \omega_\lambda = -\partial_\lambda \Omega \end{array} \right. \quad (B.1)$$

$$\left\{ \begin{array}{l} \delta_\xi x^\alpha = \partial_\mu (x^\alpha \xi^\mu) - x^\mu \partial_\mu \xi^\alpha \\ \delta_\xi D = \partial_\mu (D \xi^\mu) + \frac{1}{2} x^\mu \partial_\mu (\xi^\alpha \epsilon_{\alpha\rho}) \\ \delta_\xi \omega_\lambda = \partial_\mu (\omega_\lambda \xi^\mu) + \omega_\mu \partial_\lambda \xi^\mu \end{array} \right. \quad (B.2)$$

With these rules, it is easy to check Eq. (20), if one rewrites the second part of the Lorentz anomaly in Eq. (16) as

$$\frac{1}{12} (\partial_\lambda \Omega) \frac{1}{D} (x^\rho \omega_\rho) = \frac{1}{12} (\partial_\lambda \Omega) \frac{(x^\rho + x^\lambda)}{D} \omega_\rho - \frac{1}{12} (\partial_\lambda \Omega) \epsilon^{\lambda\rho} \omega_\rho$$

Again the gravitational anomaly is much more difficult. To prove the cancellation of the gravitational anomaly, one must guess the counter term. One needs essentially

$$\delta_\xi \int \frac{1}{D} (x^\sigma \omega_\sigma) \omega_\alpha = \int \frac{1}{2} \xi^\mu \epsilon_{\mu\alpha} \partial_\rho \left[\frac{1}{D^2} (x^\sigma \omega_\sigma) (x^\lambda \omega_\lambda) \right] \quad (B.3)$$

$$\delta_\xi \int \frac{1}{D} \partial_\lambda (x^\sigma \omega_\sigma) = \int -\frac{1}{2} \xi^\mu \epsilon_{\mu\alpha} \partial_\rho \left[\frac{1}{D^2} x^\lambda \partial_\lambda \partial_\sigma x^\sigma \right] - \xi^\mu \partial_\lambda \partial_\mu \left[\frac{1}{D} \partial_\sigma x^\sigma \right] \quad (B.4)$$

$$\begin{aligned} \delta_\xi \int \frac{x^\sigma}{D^2} \partial_\lambda \partial_\sigma D &= \int \xi^\mu \epsilon_{\mu\alpha} \left\{ \partial_\rho \left(\frac{x^\lambda x^\sigma}{D^2} \partial_\lambda \partial_\sigma D \right) - \frac{1}{2} \partial_\rho \partial_\lambda \partial_\sigma \left(\frac{x^\rho x^\sigma}{D^2} \right) \right. \\ &\quad \left. + \partial_\rho \partial_\lambda \left(\frac{x^\sigma \partial_\sigma x^\rho}{D^2} \right) - \frac{1}{2} \partial_\rho \left(\frac{x^\sigma}{D^2} \partial_\sigma \partial_\lambda x^\lambda \right) \right\} \\ &\quad - 4 \xi^\mu \partial_\mu \partial_\sigma \partial_\lambda \left(\frac{x^\sigma}{D} \right) + \xi^\mu \partial_\mu \partial_\sigma \left(\frac{1}{D} \partial_\lambda x^\sigma \right) + 2 \xi^\mu \partial_\sigma \partial_\lambda \left(\frac{1}{D} \partial_\mu x^\sigma \right) \end{aligned} \quad (B.5)$$

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