Beam dynamics in linacs

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Abstract
An introduction to beam dynamics in proton linear accelerators in the absence of space charge is given.

1 Introduction

Beam dynamics is the study of the collective behaviour of an ensemble of particles constituting the beam in a particle accelerator. For the purposes of this introduction to the subject, the particle motion in the transverse ($x$ and $y$) and longitudinal ($z$) directions is treated entirely independently which is a perfectly good assumption in the vast majority of cases where space charge forces are absent or negligible.

Section 2 introduces transverse particle dynamics while Section 3 covers the longitudinal dynamics.

2 Transverse dynamics in linacs

Transverse dynamics studies the motion of particles in a plane perpendicular to the average direction of motion of the particles. In a linear accelerator there is a well-defined forward direction and the transverse plane is defined with reference to this. In the majority of cases the motion is studied independently in two orthogonal directions within this plane, usually the horizontal ($x$) direction and the vertical ($y$) direction. Occasionally, where the forces involved are axisymmetric, motion is considered simply in the radial direction.

In the absence of any other tendency for beam particles to move away from the beam axis, the very act of acceleration introduces a transverse perturbation to the motion. Some method of external focusing is necessary, with the most common method being the magnetic quadrupole lens.

2.1 Transverse radiofrequency defocusing

Acceleration within a linac is usually accomplished by means of radiofrequency (RF) electric fields. The details of this process will be expanded in Section 3. The RF fields are generated inside a structure generically referred to as a cavity. There are a great variety of types of cavity however the specific details are not important for analysing the effect on the transverse motion of the beam particles. For the purposes of this analysis the important aspect of an accelerating cavity is that it concentrates high-frequency electromagnetic fields in a relatively small region of space around the beam axis. This region is referred to as a RF gap or accelerating gap.

Figure 1 shows a schematic representation of a RF gap. The solid areas represent parts of the boundary of the cavity between which the electric field lines are shown. It is apparent from the shape of the electric field lines that off-axis particles will experience a radial force. The radial component of the electric field is an unavoidable consequence of concentrating the field into a relatively small longitudinal extent as can be shown by examining Gauss’s law which states that in the absence of free charges the divergence of the electric field must be zero.
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Fig. 1: Schematic representation of a generic accelerating gap

Gauss’s law:

\[ \nabla \cdot \mathbf{E} = 0. \]  \hspace{1cm} (1)

Any change in the magnitude of the longitudinal field will result in an off-axis radial field. This effect is shown in Fig. 2. The longitudinal accelerating component of the field is zero outside the gap, rising to some peak value within the gap. In the regions where the magnitude of the longitudinal field is changing a radial field results.

Fig. 2: Left, typical variation of the longitudinal field in a RF gap. Right, the resulting off-axis radial field component.

It might be supposed that due to the symmetrical nature of the gap the oppositely directed radial electric field in the two halves of the gap will cancel out to give no net radial impulse to the beam. However, for reasons which will be expanded in the section on longitudinal dynamics, the field is usually rising as the particles cross the gap resulting in the outwardly directed radial impulse in the second half of the gap exceeding the inwardly directed impulse in the first half. A net defocusing force is therefore experienced.

2.1.1 Radial RF impulse

A typical linac cavity is cylindrically symmetric and operated in a resonating mode that results in only three non-zero field components: \( E_z \), \( E_r \) and \( B_\theta \) where \( E \) is the electric field, \( B \) is the magnetic flux density and \( z, r \) and \( \theta \) are the longitudinal, radial and azimuthal field components, respectively.
The resulting Lorentz force impulse leads to a change in the radial momentum of a particle of charge $q$ and velocity $\beta c$ in a RF gap of

$$\Delta p_r = q \int_{-L/2}^{L/2} (E_r - \beta c B_\theta) \frac{dz}{\beta c}$$

(2)

where $\pm L/2$ are the extents of the gap. If $E_z$ is independent of $r$ near to the axis, then the divergence and curl relationships of Maxwell’s equations lead to

$$E_r = -\frac{r}{2} \frac{\partial E_z}{\partial z}$$

(3)

$$B_\theta = \frac{r}{2c} \frac{\partial E_z}{\partial t}$$

(4)

which when substituted into Eq. (2) gives

$$\Delta p_r = -q \frac{1}{2} \int_{-L/2}^{L/2} r \left( \frac{\partial E_z}{\partial z} + \frac{\beta}{\gamma c} \frac{\partial E_z}{\partial t} \right) \frac{dz}{\beta c}$$

(5)

By noting that

$$\frac{\partial E_z}{\partial z} = \frac{\partial E_z}{\partial x} + \frac{1}{\beta c} \frac{\partial E_z}{\partial t}$$

(6)

and replacing the longitudinal electric field $E_z$ by

$$E_z = E_a(z) \cos(\omega t + \phi)$$

(7)

where $E_a$ is the accelerating field, $\omega$ is the RF angular frequency and $\phi$ is the phase as the particle passes the centre of the gap, the change in radial momentum in a RF gap is given by

$$\Delta p_r = -\frac{q q_{\text{eff}}}{2\gamma^2 \beta^2 c^2} \sin \phi \int_{-L/2}^{L/2} E_a(z) \cos(kz) \, dz$$

(8)

where $k = 2\pi / \beta \lambda$, $\omega t = kz$.

2.1.2 Radial deflection in a RF gap

The momentum of a particle is given by

$$p_r = mc \beta \gamma r'$$

(9)

where $m$ is the particle mass, $\beta \gamma$ the usual relativistic parameters and

$$r' = \frac{dr}{dz}$$

(10)

Substituting Eq. (9) into Eq. (8) and replacing the integral $\int_{-L/2}^{L/2} E_a(z) \cos(kz) \, dz$ by the effective accelerating voltage $E_{aTL}$ (see Section 3) gives

$$\Delta (\beta \gamma r') = -\frac{q q_{\text{eff}} E_{aTL} \sin(\phi)}{mc^2 \beta^2 \gamma^2 \lambda}.$$  

(11)

The radial deflection in a RF gap is proportional to the accelerating voltage, the radial position and the sine of the RF phase and inversely proportional to $(\beta \gamma)^2$ and the RF wavelength $\lambda$. Stable acceleration requires that $\phi < 0$ resulting in RF defocusing.
2.2 Quadrupole focusing

In order to compensate for the transverse defocusing effect of a RF gap and any inherent divergence in the particle beam, some form of external focusing must be provided. In linacs the magnetic quadrupole lens is by far the most common. Figure 3 shows a cross-section of the poles and magnetic field lines of a quadrupole magnet.

![Cross-section of a magnetic quadrupole showing the four poles and magnetic field lines](image)

**Fig. 3:** Cross-section of a magnetic quadrupole showing the four poles and magnetic field lines

In an ideal quadrupole the vertical component of the magnetic field is linearly proportional to the horizontal position and the horizontal component of the magnetic field is linearly proportional to the vertical position. This results in particle forces which are linear functions of position. The quadrupole gradient $G$ is defined as

$$G = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \frac{B_0}{a} \quad (12)$$

where $B_0$ is the field at the pole tip and $a$ is the distance from the pole tip to the central axis. For a particle moving in $z$ with a velocity $v$ the Lorentz force is

$$F_x = -qvGx, \quad F_y = qvGy. \quad (13)$$

The effect is analogous to that of an optical lens except one which focuses in one direction but defocuses in the other. If $qG$ is positive the lens focuses in $x$ and defocuses in $y$. Despite this apparent deficiency the quadrupole lens can achieve a net focusing effect by combining magnets in sequences with both polarities.

2.2.1 Particle motion in a quadrupole

For a particle with charge $q$ travelling with velocity $\beta c$ the equations of motion in the two transverse coordinates $x$ and $y$ with axial position $s$ are

$$\frac{d^2x}{ds^2} + \kappa^2(s)x = 0$$
$$\frac{d^2y}{ds^2} - \kappa^2(s)y = 0 \quad (14)$$
where
\[
\kappa^2(s) = \frac{|qG(s)|}{m_0 c}.
\] (15)

In an ideal hard-edged quadrupole \(G(s) = G_0\) and simple harmonic motion results. Equation (14) where the restoring force is a linear function of the transverse position is an example of Hill’s equation. A convenient method to study the behaviour of Hill’s equation is through matrix solutions.

### 2.2.2 Transfer matrix representation

Write Eq. (14) in one plane as
\[
x'' + K(s)x = 0
\] (16)

with
\[
| K(s) | = \kappa(s)^2.
\]

The solution to the linear second-order differential equation can be written in matrix form as
\[
\begin{bmatrix}
  x_1' \\
  x_1
\end{bmatrix} =
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  x_0' \\
  x_0
\end{bmatrix}
\]
\[
x' = \frac{dx}{ds}, \quad x'' = \frac{d^2x}{ds^2}.
\] (17)

The 2 \times 2 matrix is called the transfer matrix \(R\):
\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} = R.
\] (18)

For a sequence of elements the total transfer matrix is the product of the individual \(R\) matrices. The order of the multiplication is significant so if the beam is transported through elements 1, 2, 3, ..., \(n\) in order, the total transfer matrix \(R_t = R_n \ldots R_2 R_1\).

In order to study the behaviour of sequences of quadrupole magnets separated by field free drifts it is useful to derive the following transfer matrices.

#### 2.2.2.1 Drift space

For a field free drift of length \(l\)
\[
R = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}.
\] (19)

#### 2.2.2.2 Focusing quadrupole

For a quadrupole of length \(l\) focusing in \(x\)
\[
R = \begin{bmatrix}
  \cos \sqrt{Kl} & \frac{\sin \sqrt{Kl}}{\sqrt{K}} \\
  -\sqrt{K} \frac{\sin \sqrt{Kl}}{\sqrt{K}} & \cos \sqrt{Kl}
\end{bmatrix}
\]
\[
K = \frac{qG}{mc^2\beta\gamma} > 0.
\] (20)
For sufficiently small values of $\sqrt{Kl}$ the quarupole can be approximated as a thin lens where

$$R = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

(21)

$$\frac{1}{f} = |KI| = |\frac{qGl}{mc\beta\gamma}|.$$  

(22)

2.2.2.3 Defocusing quadrupole

For a quadrupole of length $l$ defocusing in $x$

$$R = \begin{bmatrix} \cosh\sqrt{|K|l} & \frac{\sinh\sqrt{|K|l}}{\sqrt{K}} \\ \sqrt{|K|l} \sinh\sqrt{|K|l} & \cosh\sqrt{|K|l} \end{bmatrix}$$

(23)

$$K = \frac{qG}{mc\beta\gamma} < 0.$$  

The thin lens approximation is

$$R = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix},$$

$$\frac{1}{f} = |KI| = |\frac{qGl}{mc\beta\gamma}|.$$  

(24)

2.3 Periodic solutions to Hill’s equation

It is common in linacs that the arrangement of drifts, gaps and quadrupoles is periodic. This means that $K(s)$ in Eq. (16) is also periodic and the solution to Hill’s equation is oscillatory. The general solution is the so-called phase-amplitude form of the solution:

$$x(s) = \sqrt{\epsilon_1}\hat{\beta}(s)\cos[\phi_1 + \phi(s)]$$

(25)

where $\hat{\beta}(s)$ is the amplitude function and $\phi(s)$ is the phase function related to the amplitude function by

$$\phi(s) = \int \frac{ds}{\hat{\beta}(s)}.$$  

(26)

The constants $\phi_1$ and $\epsilon_1$ are given by the initial conditions. Defining two additional functions

$$\ddot{a}(s) = -\frac{1}{2} \frac{d\hat{\beta}(s)}{dv}$$

(27)

$$\ddot{y}(s) = \frac{1+\ddot{a}(s)^2}{\hat{\beta}(s)}$$

(28)

gives the three well-known Courant–Snyder or Twiss parameters $\ddot{a}(s)$, $\ddot{\beta}(s)$ and $\ddot{y}(s)$ which are all periodic with the same period as $K(s)$. Making use of the Twiss parameters the transverse coordinates satisfy the equation

$$\ddot{y}(s)x^2 + 2\ddot{a}(s)xx' + \ddot{\beta}(s)x'^2 = \epsilon_1$$

(29)

which is the equation of an ellipse in the $x$–$x'$ phase space, centred at the origin with an area equal to $A = \pi\epsilon_1$ as shown in Fig. 4.
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\[ x_{\text{max}} = \sqrt{\varepsilon \beta} \]
\[ x'_{\text{max}} = \sqrt{\varepsilon \gamma} \]
\[ \text{slope} = \frac{\bar{a}}{\bar{\beta}} \]

Fig. 4: Ellipse representing the trajectory of a particle in \(x-x')\) phase space

Equation (29) can be written in matrix form as

\[ X^T \sigma^{-1} X = \varepsilon \]

(30)

where

\[ X = \begin{bmatrix} X \\ x' \end{bmatrix} \]

and

\[ \sigma^{-1} = \begin{bmatrix} \bar{\beta} & \bar{\alpha} \\ \bar{\alpha} & \bar{\beta} \end{bmatrix} \]

and

\[ \sigma = \begin{bmatrix} \hat{\beta} & -\bar{\alpha} \\ -\bar{\alpha} & \hat{\gamma} \end{bmatrix}. \]

(31)

Using subscript 1 to denote the beam coordinates at the beginning of a period and subscript 2 to denote the coordinates at the end, then

\[ X_1^T \sigma_1^{-1} X_2 = \varepsilon \]
\[ X_2^T \sigma_2^{-1} X_2 = \varepsilon \]
\[ X_2 = RX_1 \]

which leads to

\[ \sigma_2 = R \sigma_1 R^T \]

(32)

\[ \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ -R_{11}R_{21} & 1 + R_{12}R_{21} & -R_{12}R_{22} \\ R_{21}^2 & -2R_{21}R_{22} & R_{22}^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}. \]

(33)
For a periodic solution with the same Twiss parameters at the beginning and end of a sequence of elements
\[ \sigma_2 = \sigma_1 \]
\[
\begin{bmatrix}
\alpha_2 \\
\beta_2 \\
\gamma_2
\end{bmatrix} =
\begin{bmatrix}
\alpha_1 \\
\beta_1 \\
\gamma_1
\end{bmatrix}.
\]

For such a periodic channel of length \( L \) the periodic transfer matrix \( P \) is given by
\[
P = R(s \rightarrow s+L) = \begin{bmatrix} \cos \sigma + \bar{\alpha} \sin \sigma & \bar{\beta} \sin \sigma \\ -\bar{\gamma} \sin \sigma & \cos \sigma - \bar{\alpha} \sin \sigma \end{bmatrix}
\]
(34)

where
\[
\sigma = \Delta \phi = \int^{L}_{0} \frac{ds}{\beta(s)}
\]
is the phase advance per period. Each particle in the beam lies on its own elliptical trajectory and undergoes the same phase advance \( \sigma \). For stable solutions, \(|\cos(\sigma)| < 1\).

By constructing the total transfer matrix or transporting two orthogonal particles \( \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ x_1' \end{bmatrix} \) through the channel the elements of \( P \), the periodic Twiss parameters and the phase advance can be calculated:
\[
\begin{bmatrix} x_2' \\ x_3' \end{bmatrix} = P \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_3 \\ x_3' \end{bmatrix} = P \begin{bmatrix} 0 \\ x_1' \end{bmatrix}
\]
\[
x_2 = P_{11} x_1 = (\cos \sigma + \bar{\alpha} \sin \sigma)x_1
\]
\[
x_3' = P_{22} x_1' = (\cos \sigma - \bar{\alpha} \sin \sigma)x_1'
\]
\[
2 \cos \sigma = P_{11} + P_{22} = \frac{x_2}{x_1} + \frac{x_3'}{x_1'}
\]
(35)

The Twiss parameters follow directly from the phase advance from Eq. (34).

### 2.4 The smooth approximation

Equation (14) can be written to include the effects of RF defocusing by approximating the defocusing as a continuous force which leads to
\[
\begin{align*}
\frac{d^2 x}{ds^2} + \kappa^2(s)x - \frac{k_{10}^2}{2} x &= 0 \\
\frac{d^2 y}{ds^2} - \kappa^2(s)y - \frac{k_{10}^2}{2} y &= 0
\end{align*}
\]
(36)

where
\[
k_{10}^2 = \frac{2\pi q E_0 \tan(-\phi)}{m c^2 (\gamma \beta)^3 \lambda}.
\]
(37)

The most common focusing arrangement found in linacs is the FODO structure. Each period consists of a focusing quadrupole and a defocusing quadrupole of equal strength with accelerating gaps between them. Figure 5 shows the typical arrangement. The period length is \( 2L \).
In the smooth approximation the particle trajectories become sinusoidal and the phase advance is given by

\[ \sigma^2 \approx \left( \frac{L}{f_q} \right)^2 - \left( \frac{4L}{f_g} \right)^2 \] (38)

where

\[ \frac{1}{f_q} = \frac{qG\ell}{mc\beta\gamma} \]

\[ \frac{1}{f_g} = \frac{\pi qE_0TL\sin(-\phi)}{mc^2(\beta\gamma)^3\lambda}. \]

Substituting for the focal lengths leads to

\[ \sigma^2 \approx \left( \frac{qG\ell L}{mc\beta\gamma} \right)^2 - \frac{\pi qE_0TL\sin(-\phi)(2L)^2}{mc^2(\beta\gamma)^3\lambda}. \] (39)

For stable behaviour the first term on the right-hand side, the quadrupole term, must be greater than the second, RF defocusing term. The \((\beta\gamma)^3\) in the RF defocus term makes it more important at low, typically not highly relativistic, velocities. The phase advance per unit length is given by

\[ \left( \frac{\sigma}{2L} \right)^2 \] (40)

and is independent of the period length. The period length is limited by the requirement that \(\sigma < \pi\).

3 Longitudinal dynamics in linacs

A linac typically consists of a series of RF cavities designed to efficiently accelerate the particles. For the purposes of understanding the longitudinal dynamics the cavities are considered to be simple gaps of the type shown in Fig. 1. Only the on-axis component of the electric field is considered and any radial dependence is ignored.

3.1 Energy gain in a RF gap

For a generic RF gap of frequency \(\omega\) and length \(L\) with an axial electric field \(E_z(r, z, t)\) the field experienced by an on-axis particle is given by

\[ E_z(r = 0, z, t) = E(0, z)\cos[\omega t(z)] + \phi \] (41)
where

\[ t(z) = \int_{0}^{z} \frac{dz}{v(z)}. \]

Taking the origin to be the centre of the gap when \( t = 0 \) and RF phase equal to \( \phi \), then the energy gain is

\[ \Delta W = q \int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \cos\{\omega t(z) + \phi\} \, dz. \quad (42) \]

Using a trigonometric identity the energy gain can be written as

\[ \Delta W = q \int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \left[ \cos(\omega t) \cos(\phi) - \sin(\omega t) \sin(\phi) \right] \, dz \quad (43) \]

which is expressed in the conventional form as

\[ \Delta W = qE_0TL\cos\phi \quad (44) \]

where \( E_0 = \frac{1}{L} \int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \, dz \) is the average on-axis electric field and

\[ T = \frac{\int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \cos(\omega t(z)) \, dz}{\int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \, dz} - \tan(\phi) \frac{\int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \sin(\omega t(z)) \, dz}{\int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \, dz} \quad (45) \]

is the transit time factor.

### 3.2 Transit time factor

If \( E(0,z) \) is symmetric about \( z = 0 \), then

\[ \int_{-\frac{L}{2}}^{\frac{L}{2}} E(0,z) \sin(\omega t(z)) \, dz = 0 \]

and

\[ T = \frac{\int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \cos(\omega t(z)) \, dz}{\int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \, dz} \quad (46) \]

Further, if the change in particle velocity across the gap is small

\[ \omega t \approx \frac{\omega z}{\beta c} = \frac{2\pi z}{\beta \lambda} \]

giving

\[ T \approx \frac{\int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \cos(2\pi z/\beta \lambda) \, dz}{\int_{\frac{L}{2}}^{\frac{L}{2}} E(0,z) \, dz} \quad (47) \]
3.3 Phase stability

In Eq. (43) the value of $\phi$ at which the cavity is designed to operate is called the synchronous phase. A particle arriving at the cavity with the synchronous energy and synchronous phase will also arrive at all subsequent cavities at the synchronous energies and phases. Acceleration only occurs when $\cos(\phi_s)$ is positive:

$$\frac{-\pi}{2} < \phi_s < \frac{\pi}{2}.$$  \hspace{1cm} (48)

Phase stability occurs when the accelerating field is rising in time

$$-\pi < \phi_s < 0.$$ \hspace{1cm} (49)

A particle that arrives earlier than the synchronous phase receives less acceleration than the synchronous particle. A later particle receives more acceleration. The effect is longitudinal focusing which drives the particles towards the synchronous phase. For stability and acceleration

$$\frac{-\pi}{2} < \phi_s < 0.$$ \hspace{1cm} (50)

Figure 6 shows representative particles in the stable part of the RF cycle.

![Figure 6: Phase stable acceleration](image)

3.4 Acceleration by a series of RF gaps

The longitudinal dynamics are studied by treating the linac as a series of thin gaps separated by field free drifts as shown in Fig. 7.

![Figure 7: The arrangement of gaps and drifts for analysing longitudinal dynamics](image)
In Fig. 7, the parameters are

\[ \phi_n = \phi_{n-1} + 2\omega l_{n-1} + \frac{\beta_{n-1} c + \pi}{2} \]

where

\[ N = \begin{cases} 1 & \text{for 0-mode} \\ \frac{1}{2} & \text{for } \pi \text{-mode} \end{cases} \]

\[ l_{n-1} = \frac{N\beta_{s,n}\lambda}{2} \]

If the synchronous velocity is given by \( \beta_{s,n} \), then

\[ L_n = \frac{N}{2} (\beta_{s,n-1} + \beta_{s,n}) \lambda \]  

(51)

and

\[ \Delta(\phi - \phi_s) = 2\pi N \beta_{s,n-1} \left( \frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}} \right) . \]  

(52)

If

\[ \beta - \beta_s = \frac{W-W_s}{mc^2\beta_s\gamma_s} \ll 1 \]  

(53)

then

\[ \frac{1}{\beta} - \frac{1}{\beta_s} \approx -\frac{\beta - \beta_s}{\beta_s^2} \]  

(54)

and

\[ \Delta(\phi - \phi_s)_n = -2\pi N \frac{W_{n-1} - W_{s,n-1}}{mc^2\beta_{s,n-1}^2\gamma_{s,n-1}} . \]  

(55)

The difference in the particle energy is simply the difference in the effective voltage leading to a pair of coupled difference equations in relative energy and phase:

\[ \Delta(W - W_s)_n = qE_0T L_n (\cos \phi_n - \cos \phi_s) \]  

(56)

\[ \Delta(\phi - \phi_s)_n = -2\pi N \frac{W_{n-1} - W_{s,n-1}}{mc^2\beta_{s,n-1}^2\gamma_{s,n-1}} . \]  

(57)

### 3.4.1 Longitudinal equations of motion

The difference equations (55) and (56) can be converted into differential equations by noting that \( s = nN\beta_x\lambda \). Then, by letting

\[ \Delta(\phi - \phi_s) = \frac{d(\phi - \phi_s)}{dn} \]

and

\[ \Delta(W - W_s) = \frac{d(W - W_s)}{dn} \]

leads to

\[ \frac{d(\phi - \phi_s)}{ds} = -2\pi \frac{W - W_s}{mc^2\beta_s^2\gamma_s^2} \]  

(57)
\[ \frac{d(W-W_c)}{ds} = qE_0T(\cos\phi - \cos\phi_s). \] (58)

Combining the two coupled differential equations (57) and (58) leads to a second-order differential equation:

\[ \frac{d^2(\phi-\phi_s)}{ds^2} = -\frac{2\pi qE_0T}{mc^2b_2^2\gamma_2^2\lambda} (\cos\phi - \cos\phi_s). \] (59)

Equation (59) leads in turn to the Hamiltonian of the longitudinal motion

\[ \frac{Aw^2}{2} + B(\sin\phi - \phi\cos\phi_s) = H_\phi \] (60)

where

\[ A = \frac{2\pi}{b_2^2\gamma_2^2\lambda}, \]
\[ B = \frac{qE_0T}{mc^2}, \]
\[ w = \frac{W-W_c}{mc^2}. \]

The Hamiltonian, Eq. (60), is of the form \textit{kinetic energy} + \textit{potential energy} = \textit{constant}. The potential energy term

\[ V_\phi = B(\sin\phi - \phi\cos\phi_s) \] (61)

indicates a potential well for \(-\pi < \phi_s < 0\) as shown in Fig. 8.

![Fig. 8: The potential well for \(-\pi < \phi_s < 0\)](image)

3.5 The separatrix

The separatrix is the name given to the boundary which separates longitudinal phase space into stable and unstable regions. It can be seen from Fig. 8 that the potential well creates a region of stable phase motion which covers

\[ \phi_2 < \phi < -\phi_s. \] (62)

The upper limit at \(\phi = -\phi_s\) is a stationary point where
\[ \frac{d\phi}{ds} = -Aw = 0 \]  

which defines the Hamiltonian constant as

\[ H_\phi = B(\sin(-\phi_s) - (-\phi_s \cos(\phi_s))) \]  

and leads to the equation for the separatrix

\[ \frac{Aw}{2} + B(\sin(\phi) - \phi \cos(\phi_s)) = -B(\sin\phi_s - \phi_s \cos\phi_s). \]  

Figure 9 shows the separatrix in longitudinal phase space and its relationship to the accelerating field.

The lower limit of the separatrix is at \( \phi_2 \) which is given by

\[ \sin\phi_2 - \phi_2 \cos\phi_s = \phi_s \cos\phi_s - \sin\phi_s. \]  

The total phase extent of the separatrix is therefore

\[ \psi = |\phi_s| + |\phi_2| = -\phi_s - \phi_2. \]  

Combining Eqs. (66) and (67) leads to the relationship

\[ \tan\phi_s = \frac{\sin\psi - \psi}{1 - \cos\psi}. \]
The maximum energy extent of the separatrix occurs at \( \phi = \phi_s \). Solving Eq. (65) at this point gives
\[
W_{\text{max}} = \frac{(W-W_0)_{\text{max}}}{mc^2} = \sqrt{\frac{2qE_0 \mu^2 \gamma^4 \lambda}{\pi mc^2}} (\phi_s \cos \phi_s - \sin \phi_s). \quad (69)
\]

3.6 Ellipse representation

If \( \phi - \phi_s \) is small compared with \( \phi_s \) then trigonometric approximations allow the equation of phase motion, Eq. (59) can be written as
\[
\frac{d^2 \phi}{ds^2} + k_0 \left[ (\phi - \phi_s) - \frac{(\phi - \phi_s)^2}{2 \tan(-\phi_s)} \right] = 0 \quad (70)
\]
where
\[
k_0 = \frac{2qE_0 \gamma \sin(-\phi_s)}{mc^2 \mu^2 \gamma^3 \lambda}. \quad (71)
\]

The quadratic term in Eq. (70) reduces the longitudinal focusing at large phase excursions. Similar approximations on the condition that \( |\phi - \phi_s| << 1 \) allows the Hamiltonian to be rewritten
\[
Aw^2 + B \sin(-\phi_s) \sin(\phi - \phi_s)^2 = 2 \left( H_\phi + \phi_s \cos \phi_s - \sin \phi_s \right). \quad (72)
\]

For \( \phi_s < 0 \) this is the equation of an upright ellipse with its centre at \( w = 0 \) and \( \phi = \phi_s \) as shown in Fig. 10.

![Ellipse representing longitudinal motion for small phase oscillations](image)

**Fig. 10:** Ellipse representing longitudinal motion for small phase oscillations

If
\[
\phi_0 = \phi \bigg|_{w=0}
\]
and
\[
\Delta \phi_0 = \phi_0 - \phi_s
\]
then
\[
\frac{w^2}{w_0^2} + \frac{(\phi - \phi_s)^2}{\Delta \phi_0^2} = 1 \quad (73)
\]
where
The area of the ellipse is \( \pi \epsilon \) where \( \epsilon \) is the longitudinal emittance.

\[
\Delta W_0 = W_0 - W_s
\]

\[
\epsilon = \Delta \phi_0 \Delta W_0 = \Delta \phi_0^2 \sqrt{\frac{q E_0 T m c^2 \beta_y \gamma \lambda \Delta \phi_0^2 \sin(-\phi_s)}{2\pi}}.
\]

(75)