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A NEW CLASS OF HIGHER-DIMENSIONAL KALUZA-KLEIN MONOPOLE
AND INSTANTON SOLUTIONS

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A B S T R A C T

We present a new class of instanton solutions to the empty space Einstein equations in even dimensions. These instantons can be turned into higher dimensional Kaluza-Klein monopoles by adding the time co-ordinate in a trivial way. The solutions are constructed as radial extensions of circle bundles over a homogeneous Kähler manifold M . In the case of non-trivial bundles the regularity condition at the origin can only be met if the Kähler manifold is the complex projective space, the simplest example being the four-dimensional self-dual Taub-NUT solution. The trivial bundles give rise to regular Euclidean black hole solutions with the topology of $\mathbb{R}^2 \times M$.

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1. - INTRODUCTION

Kaluza-Klein theories constitute a seductive attempt to unify the ordinary gauge theories with gravity in a purely geometric way¹⁾. The starting point is some higher dimensional ($d=m+n$) Einstein theory possibly augmented with some antisymmetric tensor fields.

In the Kaluza-Klein framework one focuses on classical solutions which correspond to d -dimensional fibre bundles over an m -dimensional spacetime with an n -dimensional compact fibre. Such a solution is said to exhibit spontaneous compactification in that it possesses an intrinsic energy (unification) scale associated with the inverse size of the compact fibre, below which the theory effectively describes a gauge theory in m dimensions of the Yang-Mills type, covariantly coupled to gravity and some scalars. The gauge group corresponds to the isometry group of the compact fibre.

Though this physically interesting situation arises already at the level where the bundle is trivial, it is important to see whether also non-trivial bundles exist, or more in general, whether there are solutions with local compactification, by which we mean: any solution where the geometry of the fibre does depend non-trivially on the co-ordinates of the base manifold. One may think of cosmological solutions²⁾ (non-trivial time dependence), black holes^{3),4)} (non-trivial spatial dependence), but also of monopole - or instanton-type solutions⁵⁾⁻⁷⁾.

In this paper we discuss a new class of solutions with local compactification which allow various interpretations. They are most easily described as radial extensions of circle bundles over a compact homogeneous Kähler manifold M , i.e., they are solutions in $2n+2$ Euclidean dimensions where $\dim_{\mathbb{R}} M = 2n$. If one imposes regularity, one finds that in the case of the bundle being non-trivial, the base manifold M is restricted to $\mathbb{CP}^n$ (n arbitrary). In this way one obtains a class of even dimensional gravitational instantons which are the higher dimensional generalizations of the $d=4$ self-dual Taub-NUT solution⁸⁾. As these solutions are Ricci flat, one may add the time co-ordinate with opposite signature in the metric⁵⁾, which alters the interpretation from a $(2n+2)$ -dimensional instanton to a $(2n+3)$ -dimensional Kaluza-Klein monopole. The trivial bundles have a regular extension for any homogeneous Kähler manifold, and generate Euclidean Schwarzschild type solutions with topology $\mathbb{R}^2 \times M$. These may be turned into neutral Kaluza-Klein solitons.

This paper is organized as follows. In Section 2 we recall some basic properties of Kähler manifolds. In Section 3 we discuss various aspects of circle bundles (over Kähler manifolds), whereas in Section 4 we present the new results obtained by adding the radial co-ordinate. We conclude with a brief discussion and outlook.

2. - KÄHLER MANIFOLDS

We recall some of the basic properties of Kähler manifolds⁹⁾. Consider a Hermitian manifold M . Using the co-ordinates $z^a = x^a + iy^a$, $\bar{z}^a = x^a - iy^a$ ($a=1, \dots, n=\dim_{\mathbb{C}} M$), we may write the metric as

$$ds^2 = 2 h_{a\bar{b}} dz^a \otimes d\bar{z}^b, \quad (2.1)$$

i.e., $h_{ab} = h_{\bar{a}\bar{b}} = 0$. Due to the Hermiticity of h certain connection coefficients vanish

$$\Gamma_{b\bar{c}}^a = \Gamma_{bc}^{\bar{a}} = 0, \quad (2.2)$$

implying that the type of a tensor is preserved under parallel transport. On the Hermitian manifold, consider the real two-form K of type $(1,1)$ (the fundamental form) defined by

$$K \equiv i h_{a\bar{b}} dz^a \wedge d\bar{z}^b. \quad (2.3)$$

A Kähler manifold is a Hermitian manifold for which K is closed,

$$dK = 0, \quad (2.4)$$

K is then called the Kähler form. The property that K is closed has important consequences:

(i) one may introduce a Kähler potential $G = G(z, \bar{z})$ such that

$$h_{a\bar{b}} = \partial_a \partial_{\bar{b}} G(z, \bar{z}). \quad (2.5)$$

It is customary to write

$$G(z, \bar{z}) = \log V(z, \bar{z}) , \quad (2.6)$$

where V is the Bergman kernel function (defined for bounded domains in complex space⁹⁾);

(ii) it is easy to see that (2.4) is equivalent to

$$\Gamma_{bc}^a = \Gamma_{bc}^{\bar{a}} = 0 . \quad (2.7)$$

The only surviving connection coefficients are therefore Γ_{bc}^a and $\Gamma_{bc}^{\bar{a}}$; (iii) from the physics point of view it is worth noting that if K is not exact it implies the existence of a non-trivial solution to the source free Maxwell equations on the Kähler manifold, with the field strength F given by

$$F = \text{const.} \times K , \quad (2.8)$$

this follows from the observation that K is a harmonic form, i.e., $dK = 0$ (by definition) and $\delta K = 0$ [this is true because $*K = K \wedge \Lambda \dots \wedge K$, with $(n-1)$ factors, and K is closed].

The Ricci curvature^{*)}

$$R_{ab} = R^c_{acb} + R^{\bar{c}}_{a\bar{c}b} , \text{ etc.} \quad (2.9)$$

takes a particularly simple form

$$R_{a\bar{b}} = -\partial_a \partial_{\bar{b}} \log \sqrt{\det h} . \quad (2.10)$$

For compact, homogeneous Kähler manifolds, one has in addition that

$$R_{a\bar{b}} = c h_{a\bar{b}} , \quad (2.11)$$

where c is some positive constant, depending on the size of the manifold. From Eqs. (2.5) and (2.6), it follows that V is inversely proportional to $(\det h)^{1/c}$. Borel proved¹⁰⁾ that the compact homogeneous Kähler manifolds are of a type G/H where G is a compact semisimple Lie group and H the centralizer of a torus in G .

*) We employ the same convention as the one used in Ref. 8).

Given G , these may be obtained from a sequence of maximal embeddings

$$H_i \times U_i^{(i)} \subset H_{i-1} \quad (i=1, \dots, \text{rank } G; H_0 \equiv G) . \quad (2.12)$$

The Kähler manifolds are then the quotient spaces

$$M^{(i)} = G / U_1^{(1)} \times \dots \times U_i^{(i)} \times H_i \quad (i=1, \dots, \text{rank } G), \quad (2.13)$$

where the isotropy group is indeed the centralizer of the torus $U_1^{(1)} \times \dots \times U_i^{(i)}$ by construction.

A class of simple examples is obtained for $i = 1$, the so-called irreducible Hermitian symmetric spaces of the compact type: $SU_n/U_{n-1} \approx CP^n$, $SO_n/SO_{n-2} \times SO_2$, SO_{2n}/U_n , Sp_{2n}/U_n , $E_6/SO_{10} \times U_1$ and $E_7/E_6 \times U_1$, they are a subclass of the irreducible Riemannian globally symmetric spaces⁹⁾.

3. - CIRCLE BUNDLES OVER KÄHLER MANIFOLDS

We briefly consider the classification problem for a principal U_1 -bundle E over a manifold M ¹¹⁾.

For every Lie group G there exists an (infinite dimensional) universal bundle which is a principal G -bundle over BG (the classifying space)

$$\begin{array}{ccc} G & \hookrightarrow & EG \\ & \downarrow & \\ & BG & \end{array} \quad (3.1)$$

There is an explicit way to construct such a bundle (Milnor construction^{11a)}). According to the classification theorem the G -bundles over M are classified by the homotopy classes of mappings

$$f: M \rightarrow BG . \quad (3.2)$$

For $G = U_1$ one has that $BG = CP^\infty$. To determine the homotopy class $[M, BU_1]$ of mappings (3.2) one uses the fact that CP^∞ is the Eilenberg-MacLane space $K(Z, 2)$ ^{11b)}. The space $K(\pi, m)$ has the defining properties that

$$\pi_m(K) = \pi, \quad \pi_i(K) = 0 \quad (i \neq m), \quad (3.3)$$

where π is an Abelian group and $m > 1$. Now,

$$\pi_2(CP^\infty) = \mathbb{Z}, \quad \pi_i(CP^\infty) = 0 \quad (i \neq 2), \quad (3.4)$$

and indeed $CP^\infty = K(\mathbb{Z}, 2)$.

Furthermore, from cohomology theory^{11b)} it follows that

$$[M, K(\pi, m)] = H^m(M, \pi), \quad (3.5)$$

the m^{th} cohomology group of M with coefficients in π .

In other words, we obtained that

$$[M, BU_1] = [M, CP^\infty] = [M, K(\mathbb{Z}, 2)] = H^2(M, \mathbb{Z}). \quad (3.6)$$

Note that $[M, BU_1]$ is in fact a group. To determine whether $H^2(M, \mathbb{Z})$ is non-trivial it is convenient to apply the Hurewicz isomorphism theorem¹²⁾ stating that if M is simply connected, the second homology group is given by

$$H_2(M, \mathbb{Z}) = \pi_2(M), \quad (\pi_1(M) = 0). \quad (3.7)$$

A relation between $H_2(M, \mathbb{Z})$ and $H^2(M, \mathbb{Z})$ can be established by embedding $H_2(M, \mathbb{Z})$ in $H_2(M, \mathbb{R})$ which is the dual of $H^2(M, \mathbb{R})$. With the embedding torsion elements (i.e., finite cyclic groups) will get lost. So, modulo finite Abelian elements one has that

$$\pi_2(M) = H_2(M, \mathbb{Z}) \cup H^2(M, \mathbb{Z}) \cup \mathbb{Z}^{b_2(M)}, \quad (3.8)$$

where $b_2(M)$ is the second Betti number,

$$b_2(M) \equiv \dim(H_2(M, \mathbb{R})) = \dim H^2(M, \mathbb{R}). \quad (3.9)$$

Let us return to the homogeneous Kähler manifolds of the form $M^{(i)} = G/(H_1 \times U_1^{(1)} \times \dots \times U_1^{(i)})$. The second homotopy group of $M^{(i)}$ can easily be determined from the appropriate exact sequence. Using the fact that for semi-

simple G , $\pi_2(G) = 0$ and $\pi_1(G)$ contains only finite cyclic factors, one obtains that

$$\pi_2(M^{(i)}) = \text{Ker}[\pi_1(H) \rightarrow \pi_1(G)] = Z^{(1)} \oplus \dots \oplus Z^{(i)} \oplus C_i \quad (3.10)$$

where the last term represents the torsion elements that may occur [coming from $\pi_1(H_1)$]. The second Betti number for the manifold $M^{(i)}$ equals $b_2(M^{(i)}) = 1$, implying that the second cohomology group $H^2(M, Z)$ has at least i generators. As will become clear later on, (3.10) is basically the topological criterion for the existence of Abelian Kaluza-Klein monopoles, where M corresponds to the spatial boundary of the spacetime (base) manifold. It is amusing to note the striking similarity between (3.10) and the criterion for monopoles in spontaneously broken gauge theories which also reads $\pi_2(M=G/H) \neq 0$, however, in that case G refers to the unifying group and H to the unbroken subgroup. At this point this remarkable fact appears to be a mere coincidence.

In what follows we are only interested in bundles which are associated with the Kähler form K , because those satisfy additional geometrical constraints which arise in our construction. We recall the fundamental result that the second cohomology group is generated by the curvature two-form Ω associated with a connection on the bundle as specified by the Chern-Weil theory. For U_1 bundles one has furthermore that only the first Chern class

$$C_1(\Omega) = -\frac{1}{2\pi i} \Omega \quad (3.11)$$

can be non-vanishing. That is, if Ω is not exact it will be in a non-trivial class of the second cohomology group. From (3.10) it is clear that the topological charge has at least i components. Because the n -fold wedge product of the Kähler form K with itself is non-vanishing (in fact proportional to the volume form on M), K generates the diagonal subgroup $Z \subset Z^{(1)} \oplus \dots \oplus Z^{(i)}$ of $H^2(M, Z)$. The corresponding bundles are characterized by the n^{th} Chern number

$$C_n(E) = C_1^n(E) = \int_M C_1(\Omega) \wedge C_1(\Omega) \wedge \dots \wedge C_1(\Omega) \quad (3.12)$$

What do the bundles with curvature equal to the Kähler form look like? In the previous section the homogeneous Kähler manifolds were specified to be of the form (2.12). The lowest non-trivial bundle E over such M is the manifold

$$E \simeq G/H_i \quad (3.13)$$

with typical fibre $U_1^{(1)} \times \dots \times U_1^{(i)}$. For example, in the case $M = \mathbb{CP}^n = SU_{n+1}/SU_n$ one obtains that

$$E \simeq SU_{n+1}/SU_n \simeq S^{2n+1} \quad (3.14)$$

i.e., they are the odd dimensional spheres.

We conclude this section with a brief remark about a Riemannian structure on the bundle E ¹³⁾. Given a connection ω on the bundle,

$$\omega = d\alpha + A \quad (3.15)$$

where α ($0 < \alpha < 2\pi$) is the co-ordinate along the fibre and A a one-form on M related to the Kähler form $K = F = dA$ (locally), a natural metric is given by

$$ds_E^2 = ds_M^2 + R(\omega \otimes \omega) \quad (3.16)$$

where R^2 is some positive function on M . This particular form of the metric is based on the decomposition of $T_x E$, the tangent space at $x \in E$, in horizontal and vertical subspaces and using the fact that the horizontal subspace which, by definition is annihilated by ω , is isomorphic to $T_{\pi(x)}^M$.

4. - RADIAL EXTENSIONS

In this section, we will exhibit a new class of solutions to the Euclidean Einstein equations in even dimensions based on the circle bundles we discussed in the previous section.

The idea is to add a radial co-ordinate, i.e., to consider a metric of the form

$$ds^2 = A^2 dr^2 + C^2 ds_M^2 + B^2(\omega \otimes \omega) \quad (4.1)$$

(A,B,C are functions of r only), and investigate whether there are solutions to the Einstein equations with zero cosmological constant. That is, we demand the space to be Ricci flat. The Ricci tensor reads

$$R_{rr} = -2n \left(\frac{C'}{A} \right)' \frac{1}{AC} - \left(\frac{B'}{A} \right)' \frac{1}{AB} , \quad (4.2a)$$

$$R_{\alpha\alpha} = - \left[\left(\frac{B'}{A} \right)' \frac{1}{AB} + 2n \frac{B'}{AB} \frac{C'}{AC} \right] + \frac{1}{4} \left(\frac{B}{C^2} \right)^2 \left[F_{a\bar{c}} F^{a\bar{c}} + F_{\bar{a}c} F^{\bar{a}c} \right] , \quad (4.2b)$$

$$R_{a\bar{b}} = \frac{1}{C^2} \bar{R}_{a\bar{b}} - \frac{1}{2} \left(\frac{B}{C^2} \right)^2 F_{a\bar{c}} F_{\bar{b}d} h^{d\bar{c}} - h_{a\bar{b}} \left[\left(\frac{C'}{A} \right)' \frac{1}{AC} + \frac{C'}{AC} (2n-1) + \frac{B'}{AB} \frac{C'}{AC} \right] , \quad (4.2c)$$

where $\bar{R}_{a\bar{b}}$ denotes the Ricci tensor of the $2n$ -dimensional Kähler manifold given in (2.10), and $F_{a\bar{b}} = 2ih_{a\bar{b}}$ as is evident from (2.3) and (2.8). Vanishing of the Ricci tensor yields the following equations:

$$2n \left(\frac{C'}{A} \right)' \frac{1}{AC} + \left(\frac{B'}{A} \right)' \frac{1}{AB} = 0 , \quad (4.3a)$$

$$2n \left(\frac{B^2}{C^4} - \frac{B'}{AB} \frac{C'}{AC} \right) - \left(\frac{B'}{A} \right)' \frac{1}{AB} = 0 , \quad (4.3b)$$

$$\left(\frac{B^2}{C^4} - \frac{B'}{AB} \frac{C'}{AC} \right) - \left(\frac{C'}{A} \right)' \frac{1}{AC} + \frac{4}{C^2} - \left(\frac{C'}{AC} \right)^2 (2n-1) - 3 \frac{B^2}{C^4} = 0 . \quad (4.3c)$$

where we have chosen the constant c in (2.11) equal to 4. To arrive at these equations the Kähler property is essential. From the sum of (4.3a) and (4.3b) one learns that without loss of generality one may choose

$$C^2 = r^2 - q^2 , \quad (4.4)$$

where q is some constant and $r > |q|$, which yields the well-known simplification

$$AB = q > 0 . \quad (4.5)$$

Combining the first two equations with (4.3c) one obtains the following equation for A

$$A' - \frac{1}{2r} \left(\frac{(2n-1)r^2 + q^2}{c^2} \right) A + \frac{2A^3}{r} = 0, \quad (4.6)$$

which can be solved explicitly:

$$A^2 = 4 \frac{(r^2 - q^2)^n}{r} \left[\int_q^r \frac{(s^2 - q^2)^n}{s^2} ds + \beta \right]^{-1}, \quad (4.7)$$

where β is some integration constant which will be fixed by the boundary conditions, to which we turn next.

(i) $r \rightarrow \infty$. For large r one finds that

$$A^2 \rightarrow \frac{2n-1}{4}, \quad B^2 \rightarrow \frac{4q^2}{2n-1} \quad \text{and} \quad C \rightarrow r. \quad (4.8)$$

These asymptotic values are independent of β .

It is clear that the asymptotic behaviour of the space is just a U_1 -compactification, over some non-compact space with boundary M, where the internal circle has a radius R_∞ :

$$R_\infty^2 = B^2(\infty) = \frac{4q^2}{2n-1}, \quad (4.9)$$

so

$$q = \frac{1}{2} \sqrt{2n-1} R_\infty \quad (4.10)$$

is of the order of the Planck length.

(ii) Let us now turn to the asymptotic behaviour at the origin where $r \rightarrow q$. A little calculation shows that if β is not equal to zero, $\lim_{r \rightarrow q} A^2 = 0$. This means that the radius of the internal circle blows up. We want to find a regular solution, so we put $\beta = 0$ and ask whether this is sufficient to guarantee a regular solution. Since we want to study this question for solutions associated with any homogeneous Kähler manifold, it is most convenient to look at the Riemann curvature tensor directly. In real co-ordinates, the curvature two-form for small $\rho^2 \equiv r^2 - q^2$ takes the form

$$R^a_o \rightarrow 0, \quad (4.11a)$$

$$R^\alpha_o \rightarrow 0, \quad (4.11b)$$

$$R^a_b \rightarrow \frac{1}{\rho^2} \left[\left\{ \frac{n+1}{4} \bar{R}^a_{bcd} e^c \wedge e^d - \frac{1}{4} (F^a_b F_{cd} + F^a_c F_{bd}) e^c \wedge e^d \right\} - e^a \wedge e^b \right], \quad (4.11c)$$

$$R^a_\alpha \rightarrow \frac{1}{\rho^2} \left[\left\{ \frac{1}{4} F^a_b F_{bd} e^\alpha \wedge e^d \right\} - e^a \wedge e^\alpha \right], \quad (4.11d)$$

where $\bar{R}^a_b$ is the Riemann curvature tensor of M.

Regularity of the solution requires that the coefficient of the pole terms vanish. Here one should observe that (4.11c) and (4.11d) can be combined, and the regularity condition reads

$$\tilde{R}^p_q = e^p \wedge e^q \quad (p, q = m, \alpha) \quad (4.12)$$

where $\tilde{R}^p_q$ is now the curvature two-form of the bundle space E. Condition (4.12) just says that this bundle space should be a sphere. In other words, regular solutions are obtained for the cases where $M = CP^n$, where the apparent singularity at $r = q$ is just a co-ordinate singularity. For the other Kähler manifolds the solution has a physical singularity at the origin. It is through imposing regularity that we enforce the "quantization" of q (or $q\alpha$). As a simple example, we may consider the case of $M = CP^1 \approx S^2$. We find

$$A^2 = \frac{1}{4} \frac{r+q}{r-q}, \quad C^2 = r^2 - q^2, \quad AB = q. \quad (4.13)$$

This is exactly the self-dual Taub-NUT solution in $d = 4$ ⁸⁾, whose interpretation in $d = 5$ Kaluza-Klein context was first given by Sorkin, and Gross and Perry ⁵⁾. Our solutions are clearly a higher dimensional generalization of this solution. Indeed, since our spaces are solutions to the empty space Einstein equations in $d = 2n+2$ we may add the time co-ordinate in a trivial way, so that

$$ds^2 = -dt^2 + A^2 dr^2 + C^2 dS_M^2 + B^2 (\omega \otimes \omega) \quad (4.14)$$

does describe a regular magnetic type soliton in $(2n+3)$ -dimensional space compactified to some $(2n+2)$ -dimensional spacetime with as peculiar spatial boundary CP^n rather than an ordinary S^{2n} .

As far as the other homogeneous Kähler manifolds are concerned, we should be surprised that they lead to exact solutions (though point singular) anyway.

Because the coefficient functions A, B and C only depend on the dimension of the Kähler manifold it is clear that the singularity if it is present is a naked singularity as no horizon appears.

We conclude this section with a brief discussion of the trivial bundles, i.e., we consider now metrics of the form:

$$ds^2 = A^2 dr^2 + C^2 ds_M^2 + B^2 d\alpha^2$$

where we have set the U_1 -connection equal to zero. The only property we will need of M is that it is an Einstein space:

$$\bar{R}_{mn} = \lambda \bar{g}_{mn} \quad (\lambda > 0)$$

Note that this is true for compact homogeneous Kähler manifolds. Instead of (4.3) we find in this case

$$2n \left(\frac{C'}{A} \right)' \frac{1}{AC} + \left(\frac{B'}{A} \right)' \frac{1}{AB} = 0, \quad (4.15a)$$

$$-2n \frac{B'}{AB} - \left(\frac{B'}{A} \right)' \frac{1}{AB} = 0 \quad (4.15b)$$

$$\left[\frac{\lambda}{C^2} - \left(\frac{C'}{AC} \right)^2 (2n-1) \right] - \left(\frac{C'}{A} \right)' \frac{1}{AC} - \frac{B'}{AB} \frac{C'}{AC} = 0 \quad (4.15c)$$

where we have dropped the F^2 -terms.

By virtually the same manipulations we can solve the equations and obtain

$$A^2 = \frac{2n-1}{\lambda} \left[1 - (R_0/r)^{2n-1} \right]^{-1}, \quad C=r, \quad AB=q, \quad (4.16)$$

where R_0 is the integration constant. q is related to R_∞ through

$$q = \sqrt{\frac{2n-1}{\lambda}} R_\infty \quad (4.17)$$

The solution is an $(m+2)$ -dimensional Euclidean black hole solution ($m=\dim M$). To investigate the behaviour at $r = R_0$ we introduce a new co-ordinate

$$\lambda^2 = r - R_0 \quad (4.18)$$

yielding for small λ

$$ds^2 \simeq R_0 (d\lambda^2 + \lambda^2 [d(\frac{2q}{R_0} \alpha)]^2) + R_0^2 ds_M^2. \quad (4.19)$$

In order to have a regular solution $(2q/R_0)\alpha$ should be 2π periodic, so we must choose R_0 such that:

$$R_0 = 2q = 2 \sqrt{\frac{2n-1}{\lambda}} R_n. \quad (4.20)$$

We conclude that for the trivial bundle the apparent singularity is the usual bolt singularity⁸⁾. The topology of the space is $R^2 \times M$. It is worth noting that the simplest case where $M = CP^1 = S^2$ corresponds to the usual $d = 4$ Euclidean Schwarzschild solution.

It is interesting to speculate on the role of these black hole solutions in the stability question of the Kaluza-Klein vacuum. We argue that the black holes with $M = S^n$ describe the semi-classical instability of the Kaluza-Klein vacuum. This is essentially Witten's argument⁷⁾ applied to the higher dimensional Kaluza-Klein theories.

5. - DISCUSSION AND OUTLOOK

We have exhibited a new class of explicit solutions to the empty space Einstein equations. These solutions are radial extensions of both trivial and non-trivial U_1 -bundles over homogeneous Kähler manifolds.

For the non-trivial bundles to be regular one has to choose the base manifold to be CP^n . These correspond to $(2n+2)$ -dimensional instantons or $(2n+3)$ -dimensional Kaluza-Klein monopoles. The other choices for M lead to a naked singularity at the origin.

The trivial bundles lead to regular Euclidean black hole solutions with topology $R^2 \times M$. However, we can find a larger class of such bundles because the only thing we need to know in order to write down the explicit solution is that the base manifold is an Einstein space.

The radial extensions of both the trivial and non-trivial bundles have the property that the bundle structure breaks down at the location of the monopole in that the radius of the internal circle goes to zero ("seals off the would-be boundary"⁷⁾). This mechanism seems to be essential in obtaining a regular and geodesically complete manifold.

The fact that the simplest example for the regular non-trivial bundle is the $d = 4$ self-dual Taub-NUT solution, raises an interesting question. It is well known that this solution is a particular simple case of the more general Gibbons-Hawking multi-centered metrics⁸⁾. It is clear that the construction of this general class depends strongly on the self-duality constraint. This constraint has no obvious generalization to the higher dimensional cases we have considered, though our solution could serve as an interesting starting point in the search for such a constraint.

At the moment we have little to say about conceivable uses of our new solutions in physically relevant Kaluza-Klein scenarios.

An interesting problem is to find non-Abelian analogues of what we have described above. This is presently under investigation.

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