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On the Spectrum of Kaluza-Klein Theories
with Non-Compact Internal Spaces

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Abstract

We investigate the conditions under which Kaluza Klein theories with non-compact internal spaces admit a discrete spectrum.

One of the most attractive possibilities for the unified description of the fundamental interactions is based on the idea that space-time has more than four dimensions [1]. In this framework, one starts from a theory of gravity (or supergravity) in $d = 4 + D$ dimensions which is then shown or assumed to exhibit spontaneous compactification to four dimensions such that the remaining D coordinates parametrize a compact "internal" manifold which is sufficiently small so as to be undetectable by present day experiments. However, the continued absence of a realistic model based on these ideas has recently prompted attempts to relax some of the original assumptions [2,3]. In particular, one may envisage the possibility that the D -dimensional internal manifold is non-compact but of finite volume [3]. Such manifolds may admit chiral fermions [4] which cannot be obtained in the conventional formulation [5] and may lead to a resolution of the cosmological constant puzzle [6]. Furthermore, they arise naturally as solutions of higher-dimensional σ -models [7]. It was argued [3] that a small finite volume would prevent the direct observation of internal space and implement that the effective four dimensional theory has a discrete spectrum with a finite mass gap. This is essential for the decoupling of a suitable "low energy" sector. For compact internal spaces this is automatically guaranteed, but for non-compact spaces the discrete spectrum with finite mass gap has to be established explicitly. In this paper we show that suitable boundary conditions on the fields indeed lead to the desired result for a large class of non-compact internal spaces with finite volume.

We will consider non-compact internal spaces with the topology of R^D . The metric is assumed to be block-diagonal^{*}

$$g_{\mu\nu} = \begin{bmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{\alpha\beta} \end{bmatrix} \quad (1)$$

with the four dimensional Minkowski metric $\eta_{\mu\nu}$ and an internal metric of the form

$$g_{\alpha\beta}(\rho, \theta) = \begin{bmatrix} -1 & 0 \\ 0 & -g(\rho)g_{AB}(\theta) \end{bmatrix} \quad (2)$$

Here $g_{AB}(\theta) = g_{AB}(\theta_1, \dots, \theta_{D-1})$ describes a $(D-1)$ -dimensional compact homogeneous space such as the unit sphere S^{D-1} . This guarantees that a compact isometry group is acting on internal space, leading to a compact gauge group in the effective four dimensional theory. For a suitable choice of the function $g(\rho)$, the internal space will be non-compact in the direction corresponding to the coordinate ρ which takes values in the interval $0 \leq \rho < \rho_{\max}$ [†]. We distinguish between geodesically complete spaces [9] for which $\rho_{\max} = \infty$, and geodesically incomplete spaces, for which $\rho_{\max} < \infty$; in the latter case, some scalar combination of curvatures diverges at $\rho_{\max}$ so the point corresponding to $\rho = \rho_{\max}$ does not belong to the manifold. We furthermore require the internal manifold to have

^{*}) We use the conventions and notations of refs. [3,4].

[†]) A further generalization of (1) would replace $\eta_{\mu\nu}$ by $\sigma(\rho)\eta_{\mu\nu}$ [6,8].

finite volume since otherwise the effective four dimensional gravity will not have a finite Newtons constant. It is easily seen from (2) that this requirement amounts to

$$\int_0^{\rho_{\max}} g(\rho) \frac{D-1}{2} d\rho < \infty. \quad (3)$$

The mass spectrum of the theory is determined by the eigenvalues of the kinetic operator on the internal space. Although there are in general spinorial and tensorial excitations with respect to both four-dimensional space time and the internal space, it suffices to consider the scalar Laplacian on the internal manifold since all other relevant operators (Lichnerowicz, etc.) differ from it by terms of lower rank (or, more precisely, by operators which are relatively compact [10] with respect to the scalar Laplacian). Any scalar field of the higher dimensional theory can be expanded according to

$$\phi(x, \rho, \theta_i) = \sum_n \phi^{(n)}(x) \gamma^{(n)}(\rho, \theta_i) \quad (4)$$

where the sum ranges over a complete basis of the vector space of functions on internal space. The harmonics $\gamma^{(n)}(\rho, \theta_i)$ may be factorized into "radial" and "angular" parts, viz.

$$\gamma^{(n)}(\rho, \theta_i) = h_k^{(z)}(\rho) \gamma_k^{(n)}(\theta_i) \quad (5)$$

where k labels the representations of the isometry group and k labels the eigenvalues of the Laplacian for a given value of z . To make the problem well-defined it is necessary to impose certain boundary conditions. The

first specifies the behavior of the fluctuations at the boundary; it is^{*}

$$\int_{M_{4+0}} D_\mu (\phi \partial^\mu \phi) \sqrt{\bar{g}} = 0 \quad (6)$$

with the background covariant derivative D_μ . If the space-time fluctuations die off sufficiently rapidly at infinity, (6) implies a similar condition for the harmonics on the internal space

$$\int_{M_0} \partial_\alpha (\sqrt{\bar{g}} \phi \partial^\alpha \phi) = 0. \quad (7)$$

This condition guarantees the hermiticity of the scalar Laplacian on the internal manifold. Observe that without condition (6), the field equations would not make sense.

The second condition is obtained by noting that (in the semi-classical approximation) only fluctuations with finite total action contribute.

This leads to

$$|S(\phi_0 + \phi) - S(\phi_0)| < \infty \quad (8)$$

where S is the total action of the higher dimensional theory, which includes gravitational and matter terms, and ϕ_0 is the background which is assumed to be a solution of the higher dimensional field equations. (For the scalar considered here we assume $\phi_0 = 0$.) Expanding the

^{*}) Observe that, from (1), $\hat{g} \equiv |\det g_{\mu\nu}| = |\det g_{\alpha\beta}| = g^{D-1} |\det g_{\alpha\beta}|$.

fluctuations ϕ according to (4) (5) we see that (8) implies in particular that

$$\int_0^{\rho_{\max}} d\rho \, g(\rho) \frac{D-1}{2} h^2(\rho) < \infty \quad (9)$$

where the labels on the "radial" eigenfunction $h(\rho) = h_k^{(\lambda)}(\rho)$ will be occasionally omitted. Condition (9) guarantees that these modes are normalizable and therefore have a properly normalized kinetic term in the effective four dimensional action.

With the metric (1), the Laplacian on the internal space assumes the form

$$\bar{D}_{\alpha}^2 = -\frac{\partial^2}{\partial \rho^2} - \frac{D-1}{2} \frac{g'(\rho)}{g(\rho)} \frac{\partial}{\partial \rho} - g^{-1}(\rho) \bar{D}_{\alpha}^2 \quad (10)$$

where a prime denotes a derivative with respect to ρ . Owing to the compactness of the $(D-1)$ dimensional submanifold $\rho = \text{constant}$, the $(D-1)$ dimensional Laplacian $\bar{D}_{\alpha}^2$ has a discrete spectrum, i.e.

$$\bar{D}_{\alpha}^2 Y(\theta) = -CY(\theta) \quad (11)$$

with discrete eigenvalues $C \geq 0$ of finite multiplicity and no accumulation point. (For example, on the sphere S^{D-1} , we have $C = \ell(\ell + D - 2)$, $\ell \in \mathbb{N}$.) It is therefore sufficient to consider the "radial" part of (10), which is

$$h'' + \frac{D-1}{2} \frac{g'}{g} \cdot h' - \frac{C}{g} h + \lambda h = 0 \quad (12)$$

with a certain eigenvalue λ . We next introduce the new variable

$$y(\rho) \equiv g(\rho) \frac{D-1}{4} h(\rho) \quad (13)$$

in terms of which the differential equation (12) can be written in the simple form

$$-y'' + V(\rho)y = \lambda y \quad (14)$$

with the "effective potential"

$$V(\rho) \equiv \frac{D-1}{4} \frac{g''(\rho)}{g(\rho)} + \frac{(D-1)(D-5)}{16} \left(\frac{g'(\rho)}{g(\rho)} \right)^2 + \frac{C}{g(\rho)} \quad (15)$$

The condition (9) is equivalent to

$$\int_0^{\rho_{\max}} y^2(\rho) d\rho < \infty \quad (16)$$

whereas condition (7) is equivalent to the following boundary condition at $\rho_{\max}$: For any given ℓ , the eigenfunctions $h_k^{(\ell)}$ must obey

$$\lim_{\rho \rightarrow \rho_{\max}} \left\{ g^{(D-1)/2} h_k^{(\ell)} \partial_{\rho} h_{k'}^{(\ell)} \right\} = 0 \quad (17)$$

for arbitrary k and k' . This implies that for a given ℓ either all y_k or all combinations $(y_k' - \frac{D-1}{4} \frac{g'}{g} y_k)$ must vanish. Note that the topology R^D implies that there is no contribution to the total divergence (7) from $\rho \rightarrow 0$. The functions $h(\rho)$ obey the above boundary conditions for ρ approaching zero (compare eq. (20) below).

A central observation of this paper is that our choice (1) of the coordinate system and the variable transformation (13) have enabled us to reduce the problem of determining the spectrum in a noncompact Kaluza-Klein theory to a standard eigenvalue problem, namely the motion of a particle

on a half line, whose solution is well known from basic quantum mechanics. (For an elementary discussion see ref. [11], whereas more mathematical aspects can be found in [10], sec X.) In particular, one can show that, for suitable boundary conditions and for a large class of potentials $V(\rho)$, the operator $-\frac{d^2}{d\rho^2} + V(\rho)$ is self-adjoint on a suitable dense domain in the Hilbert space $L^2([0, \rho_{\max}])$ [10, 12]. For such operators, the spectral theorem holds [10, 12]. As a consequence, the continuous part of the spectrum does not have normalizable eigenvectors.

We next will examine the potential (15) in somewhat more detail and we will now discuss several examples. The assumed topology $\mathbb{R}^D$ for internal space means that the fixed point of the isometry group at $\rho = 0$ is included into the manifold. This implies (for a $(D-1)$ dimensional homogeneous space with "radius" scaled to one):

$$\lim_{\rho \rightarrow 0} \frac{g(\rho)}{\rho^2} = 1. \quad (18)$$

A little calculation then shows that, for small ρ ,

$$V(\rho) \rightarrow \left[\frac{(D-1)(D-3)}{4} + C \right] \rho^{-2} \quad (19)$$

and the eigenfunctions h approach

$$h(\rho) \rightarrow h_0 \rho^{\frac{1}{2}(D-2)} \sqrt{(D-2)^2 + 4C - (D-2)} \quad (20)$$

independently of the eigenvalue λ .

The potential (19) becomes exact for flat space, and for this case, the spectrum is just the positive real line $\mathbb{R}^+$ and therefore purely

continuous in accordance with one's expectations. The absence of negative eigenvalues is actually a consequence of the manifest positivity of the Laplacian in its original form (10) and independent of the behavior of $V(\rho)$ at $\rho \rightarrow \rho_{\max}$; it is more difficult to verify in the form (14) (see ref. [13] for a discussion of $1/\rho^2$ potentials and the concomitant mathematical subtleties). The behavior of $V(\rho)$ at $\rho = \rho_{\max}$ is, however, relevant for the possible emergence of a discrete spectrum $\subset \mathbb{R}^+$. We are interested in spaces with finite volume, and to study the associated behavior of $V(\rho)$, we first consider the case $\rho_{\max} = \infty$. Let us choose

$$g(\rho) \rightarrow a\rho^{-[2/(D-1)+\epsilon]} \text{ for } \rho \rightarrow \infty \quad (21)$$

which satisfies (3) for arbitrary $\epsilon > 0$. From (21), we get

$$V(\rho) \rightarrow \left[\frac{3}{4} + \frac{D-1}{2} \epsilon + \frac{(D-1)^2}{16} \epsilon^2 \right] \rho^{-2} + \frac{C}{a} \rho^{[2/(D-1)+\epsilon]} \quad (22)$$

as $\rho \rightarrow \infty$. For $C = 0$ (the "singlet case"), the foregoing discussion shows that there are at most finitely many discrete eigenvalues $0 \leq \lambda \leq V(\rho_0)$ where $V(\rho)$ has a local maximum at $\rho = \rho_0$, while the continuous part of the spectrum extends upwards from $V(\rho_0)$; on the other hand, there is at least one nontrivial eigenvalue corresponding to $h = \text{const}$ and $C = \lambda = 0$ in (12). For $C > 0$ (the "non-singlet case"), we see from (20) that $\lim_{\rho \rightarrow \infty} V(\rho) \rightarrow \infty$ and the spectrum is evidently discrete without accumulation points, all eigenvalues having finite multiplicity. This result does not depend on the intermediate behavior of $V(\rho)$ between 0 and ∞ since, by assumption, the metric is free of singularities there. In conclusion,

the spectrum behaves differently for non-singlets and singlets. In the singlet case, the infinite dimensional vector space of normalizable functions obeying the boundary condition (17) cannot be spanned by the finite number of eigenfunctions with discrete eigenvalues. Nevertheless, there is always a finite mass gap separating the lowest mode from the rest of the spectrum.

For the case of finite $\rho_{\max}$ we study a power law behavior

$$\lim_{\rho \rightarrow \rho_{\max}} g(\rho) \rightarrow a(\rho_{\max} - \rho)^\alpha. \quad (23)$$

The volume of internal space is finite for $\alpha > -\frac{2}{D-1}$. For ρ near $\rho_{\max}$ we write $g(\rho)$ in the form

$$g(\rho) \rightarrow a\omega^{-\alpha} \sin^\alpha \omega\rho, \quad \rho_{\max} = \frac{\pi}{\omega}. \quad (24)$$

The compact sphere S^D is then obtained as a limiting case for $\alpha = 2$, $a = 1$ if the homogeneous $D-1$ dimensional subspace is S^{D-1} . In this case (24) is exact and the radius of S^D is ω^{-1} . For general a and α the manifold cannot be extended to $\rho = \rho_{\max}$ and is therefore noncompact. For ρ near $\rho_{\max}$ the effective potential becomes

$$V(\rho) \rightarrow \frac{D-1}{4} \omega^2 \left\{ \left[\frac{D-1}{4} \alpha - 1 \right] \sin^{-2} \omega\rho - \frac{D-1}{4} \alpha \right\} + \frac{C}{a} \omega^\alpha \sin^{-\alpha} \omega\rho. \quad (25)$$

The boundary condition (17) guarantees that the spectrum is discrete. In many cases it also implies automatically that the eigenfunctions are normalizable. Let us demonstrate this for the interesting case $\alpha = 2$, where a finite number of chiral fermions is obtained [3] for $D = 2$, $\alpha > 1$.

Independently of the eigenvalue λ the eigenfunctions must approach near $\rho_{\max}$:

$$h(\rho) \rightarrow h_1 (\sin \omega\rho)^{\frac{1}{2}(\sqrt{(D-2)^2 + 4C/a} - (D-2))} \quad (26)$$

and are therefore normalizable. The mass spectrum contains for every ℓ quantum number infinitely many discrete eigenvalues without accumulation point. The lowest eigenvalue $\lambda = 0$ is separated by a finite mass gap from the higher excitations. This spectrum resembles very much what one obtains from a compact internal space. In conclusion, we have shown that Kaluza-Klein theories with non-compact internal spaces do not necessarily lead to disastrous physical consequences if sufficient care is taken with boundary conditions. The standard approach of dimensional reduction to an effective four dimensional theory can be implemented. Interesting non-compact internal spaces can lead to a discrete mass spectrum with finite mass gap.

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