TOPOLOGICAL CHARGE FROM THERMODYNAMICS
IN TWO DIMENSIONS

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A B S T R A C T

It is shown that in the weak coupling limit the partition function of massless $(\text{QED})_2$ in a finite volume is identical to that of a bose field if the boson topological charge is identified with the fermion number. Generalizations to theories with $U(N)$ gauge symmetry, fractional fermion number, and the Schwinger model for any coupling are also discussed.

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1 INTRODUCTION.

In a previous paper^[1] we showed that the thermodynamical properties of $(QED)_2$ and $(QCD)_2$ are those expected from bosonization^[2,3,4]. In particular, by considering fermions and bosons in a finite volume ($=2a$) and with the boundary conditions :

$$(1 \pm i\gamma^1)\psi(\pm a) = 0 \quad ; \quad \phi(\pm a) = 0 \quad (1.1)$$

where ψ is a fermi, and ϕ a bose, field we proved :

$$Z_F^{(q)} = x^{q^2} Z_B \quad (1.2)$$

where $x = \exp(-\beta\pi/4a)$. Z_B is the partition function of a scalar field ϕ satisfying the boundary condition Eq.(1.1), and $Z_F^{(q)}$ is the partition function of fermions satisfying Eq.(1.1), for a system with total fermion number q . (In Ref.1 we referred to q as "charge", but here we shall distinguish between fermion number q , and electric charge q_e .)

The main point of the present paper is to give an interpretation of Eq.(1.2) for $q \neq 0$. In section 2 we shall show that the right hand side is the partition function of a bose field quantized around a topologically non-trivial, classical background field. After having treated the case of noninteracting fermions and integer q , we generalize to $U(N)$ gauge symmetries, fractional q ^[5,6,7], and finally, in section 3, to interactions with an abelian gauge field, i.e. to the confined Schwinger model at $T \neq 0$.

2 QUANTIZATION IN A TOPOLOGICALLY NON-TRIVIAL SECTOR.

In order to understand the significance of the factor x^{q^2} in Eq.(1.2), we rewrite it in the following form :

$$x^{q^2} Z_B = \sum_{n=1}^{\infty} d_n e^{-\beta(E_n + E_0)} \quad (2.1)$$

where E_n are the discrete energy levels of the $q = 0$ system, d_n is the degeneracy factor of the n :th level, and

$$E_0 = (\pi/4a) q^2 \quad (2.2)$$

is the "vacuum-energy" of the system. The energy E_0 can easily be understood in the fermion language, if one remembers that the fermion energy levels in the box are $\omega_n = (2n + 1)\pi/4a$. Thus, if we sum the q lowest levels, we get exactly the energy E_0 in Eq.(2.2). In the boson formulation, we do, of course, not have any fermion levels, and shall instead interpret the energy E_0 in Eq.(2.2) as the energy of a classical background field, ϕ_0 , around which ϕ is quantized. Without any loss of generality we can choose ϕ_0 to be a time-independent field satisfying $\square\phi_0 = 0$, i.e., $\phi_0 = \alpha x + \eta$, where α and η are constants. This field has an energy

$$E_0 = \frac{1}{2} \int_{-a}^a dx (\partial\phi_0/\partial x)^2 = a\alpha^2. \quad (2.3)$$

Identifying the E_0 's in Eqs.(2.2) and (2.3) gives

$$q = 2\alpha a/\sqrt{\pi} = 1/\sqrt{\pi} [\phi_0(a) - \phi_0(-a)] \quad (2.4)$$

so we see that the fermion number q is in fact equal to the topological charge, $Q = \int dx J^0(x)$, corresponding to the boson current $J^\mu = -\epsilon^{\mu\nu} \partial_\nu \phi_0 / \sqrt{\pi}$. Note in particular the correct normalization factor $1/\sqrt{\pi}$ corresponding to the bosonization of a free fermi theory^[2].

The above arguments are easily extended to sectors of nonzero baryon number in theories with a $U(N)$ gauge symmetry. In Ref.1 we showed that in the weak coupling limit,

$$Z_F^{(0,b)} = x^{Nb^2} Z_F^{(0,0)} \quad (2.5)$$

where $Z_F^{(0,b)}$ is the $SU(N)$ -singlet baryon number b partition function corresponding to the decomposition $U(N) = U(1) \otimes SU(N)$. In Ref.1 we also proved that $Z_F^{(0,0)}$ is a product of N bose partition functions, so the factor x^{Nb^2} can be absorbed by assuming that all these boson fields are quantized on a non-trivial background $\phi^n = (\sqrt{\pi}/2a)bx + \eta$. We can then identify b with the baryon number corresponding to $J_\mu^B = -(1/N) \sum \epsilon^{\mu\nu} \partial_\nu \phi^n / \sqrt{\pi}$ with the sum over n from 1 to N and normalized so that N "quarks" carry unit baryon number. (Since we consider only diagonal currents, the complications discussed by Witten^[8] do not arise here.)

Let us now show how the concept of fractional fermion number naturally emerges from the partition function. It is well known that Eq.(1.1) can be generalized to a continuous class of boundary conditions of the form

$$[\exp(i\gamma_5 \vartheta_{\pm}) \pm i\gamma^1] \psi(\pm a) = 0 , \quad (2.6)$$

which are compatible with the Dirac equation and confine the fermions. The corresponding eigenvalues are $\omega_n = (2n+1)\pi/4a - \vartheta/2a$ where $\vartheta = (\vartheta_+ - \vartheta_-)/2$ and n is an integer^[9]. Following the same procedure as in Ref.1 we get

$$Z[q'] = e^{-\beta[E_0 - E_c]} Z_B , \quad (2.7)$$

where

$$E_0 = (\pi/4a)(q' - \vartheta/\pi)^2 \quad \text{and} \quad E_c = \vartheta^2/(4\pi a) . \quad (2.8)$$

The integer q' equals the number of excited quarks minus the number of excited antiquarks. For $\vartheta = 0$, q' is obviously the fermion number, so for $\vartheta \neq 0$ it is natural to define

$$q = q' - \vartheta/\pi \quad (2.9)$$

in order to retain the equivalence between electric and topological charge. We have thus naturally been led to identify $-\vartheta/\pi$ as the fermion number of the vacuum, q_{vac} . In the fermion formulation this can be computed from the spectral asymmetry (see, e.g., Refs. 9 and 10)

$$q_{vac} = -1/2 \lim_{\tau \rightarrow 0} \sum_n E(\omega_n) e^{-\tau|\omega_n|} = -\vartheta/\pi \quad (2.10)$$

where $E(\omega) = 1$ (-1) for $\omega \geq 0$ ($\omega < 0$) and the sum runs over both positive and negative energy values. Thus our definition of q is equivalent to the conventional one, both for integer and fractional fermion number.

It turns out that also E_c in Eq.(2.8) has a simple fermionic interpretation. The Casimir energy corresponding to the boundary condition Eq.(2.6) is

$$E^{cas} = [\vartheta^2 - \pi^2/12]/(4\pi a) \quad (2.11)$$

after the ultraviolet divergence has been subtracted. Thus $E_0 = \pi q'^2/4a - q'\vartheta/2a + \vartheta^2/4\pi a$, is simply the energy of the first q' fermion levels plus the ϑ dependent part of the Casimir energy. Since Eq.(1.1) implies a trivial subtraction of the zero point energy, the ϑ dependent Casimir energy is explicitly subtracted as seen in Eq.(2.7). Note that the ϑ independent term in Eq.(2.11) is also the Casimir energy for a massless scalar field satisfying Eq.(1.1).

It is interesting to notice that Eq.(2.9) remains true in the limit of arbitrarily large a , so even if the background field has zero energy, it can still give rise to a fractional fermion number.

3 THE CONFINED SCHWINGER MODEL AT $T \neq 0$.

We now consider the confined Schwinger model^[11,12] at a finite temperature and in a non-zero charge sector. Using the covariant gauge version^[13,14,15] of the functional integral formalism^[16], the partition function for $(QED)_2$ can be written :

$$Z = \int D\psi D\bar{\psi} DA^\mu Dc Dc^\dagger \exp(-\int d^2x \mathcal{L}) \quad (3.1)$$

where $\int d^2x = \int_0^\beta d\tau \int_{-a}^a dx$, and

$$\mathcal{L} = 1/4(F_{\mu\nu})^2 + (\partial_\mu A_\mu)^2/(2\alpha) + \mathcal{L}_{FP}[c, c^\dagger] + \bar{\psi}(i\partial - eA)\psi. \quad (3.2)$$

Here $F_{\mu\nu}$ and A_μ are the Euclidean space field strength and potential respectively and $\mathcal{L}_{FP}$ the usual Faddeev-Popov ghost Lagrangian. The boundary conditions on ψ are given by Eq.(1.1), A^1 satisfies Dirichlet conditions and c and $c^\dagger$ Neuman conditions at $x = \pm a$ ^[17]. The boundary conditions on A^0 are chosen to satisfy Gauss's law. In the τ direction ψ and $\bar{\psi}$ obey antiperiodic boundary conditions and A^μ , c and $c^\dagger$ periodic ones^[13,14].

Since the gauge-group is Abelian, the ghosts decouple and it is easy to show, using the boundary conditions given above, that the $DcDc^\dagger$ integrations simply give a factor Z_B^{-2} . (This procedure provides a justification for Eq.(20) in Ref.1 where the gauge-degrees of freedom were treated as redundant^[18].)

We now parametrize the gauge-potential by :

$$A_\mu = \epsilon_{\mu\nu} \partial_\nu s + \partial_\mu t \quad (3.3)$$

so that $E = (\partial_\tau^2 + \partial_x^2)s \equiv \partial^2 s$ and $\partial_\mu A_\mu = \partial^2 t$, and change the fermionic variables ψ to ψ' by the transformation :

$$\psi = \exp(i e(s\gamma_5 + t))\psi' \quad (3.4)$$

This decouples the fermions from the gauge field and the Jacobian introduces a mass term for the field ∂s ^[19,20]. The s -dependent part of the action then reads

$$S[s] = 1/2 \int d^2x [(\partial^2 s)^2 + m^2 \partial s \cdot \partial s] \quad (3.5)$$

where $m^2 = e^2/\pi$. Notice that $S[s + \text{constant}] = S[s]$, i.e., the action is invariant under global chiral transformations as it should be.

Next split s into a background field, $s_0(x)$, and a quantum field, $s(\tau, x)$, satisfying Dirichlet boundary conditions, i.e., $s(\tau, \pm a) = 0$. As in section 2, we take s_0 to be a static solution of the equation of motion, i.e., $(\partial^2 - m^2)s_0 = 0$, and find $s_0(x) = \lambda \sinh(mx)$ corresponding to a background electric field $E(x)$. (A solution of the form $\sim \cosh(mx)$ would correspond to a background electric dipole field.)

λ is then related to the renormalized electric charge, e_R , by Gauss law :

$$e_R q = E(a) - E(-a) = 2m^2 \lambda \sinh(ma) \quad (3.6)$$

where q is the, in general, fractional fermion number computed by a gauge-invariant cut-off procedure, i.e., from the spectral asymmetry. $e_R = Z_R e$, with Z_R a renormalization constant to be discussed below. Note, however, that s_0 implies a non-zero $A^0(\pm a) = \lambda m \cosh(ma)$ at the

boundaries. Since the partition function for $q \neq 0$ depends on a constant background potential we fix $A^0(\pm a)$ by a gauge choice. Demanding that $A^0(\pm a)=0$ implies that s_0 should be replaced by

$$s_0^*(x) = \lambda [\sinh(mx) - mx \cosh(ma)] \quad (3.7)$$

After a change of variables in the functional integral

$$s = (\sqrt{-\partial^2})^{-1} \phi, \quad t = (\sqrt{-\partial^2})^{-1} p, \quad (3.8)$$

such that $DA_\mu = D\phi Dp$, the use of Eqs.(3.7) and (3.8) leads to

$$S[s+s_0^*] = S_m[\phi] + \beta E_b + 2m^2 s_0(x) \int_0^\beta d\tau [\partial_1 / \sqrt{-\partial^2}] \phi(\tau, a) \quad (3.9)$$

where $E_b = (m^2/a) s_0^*(a) [s_0^*(a) - \lambda \sinh(ma)]$ and

$$S_m[\phi] = 1/2 \int d^2x \phi (-\partial^2 + m^2) \phi, \quad (3.10)$$

The $D\phi$ integral can now be performed by a suitable shift in ϕ , resulting in a term $E'_b = (m^2/a) s_0^*(a) \lambda \sinh(ma)$ in the action. As in Eq.(2.7), the fermionic integration is done in the (integer) q' sector and gives $\exp(-\beta E_f) Z_B$ where

$$E_f = (\pi q'^2/4a) - (q'e/a) s_0^*(a) \quad (3.11)$$

For $ea = 0$ we have $q'=q$. The second term in Eq.(3.11) comes from the change in the fermion boundary condition due to the transformation (3.4). Finally, by performing the integration over the p -variables, which gives a factor Z_B , we obtain for the partition function

$$Z^{(q')} = \exp(-\beta E_0) Z_B(m), \quad (3.12)$$

where

$$E_0 = (\pi/4a) [q' - 2e s_0^*(a)/\pi]^2 = \pi q^2/4a \quad (3.13)$$

and $Z_B(m)$ is the bosonic partition function of a free scalar field with mass m and Dirichlet boundary conditions. Note, that for given q' , we can, using Eq.(3.6), express q in terms of Z_R , which can be calculated in perturbation theory. For $q=0$, Eq.(3.12) agrees with the results of Refs.1 and 12. Also note, that for $ea \rightarrow 0$, $es_0^*(a) \rightarrow 0$, so in this limit Eq.(3.13) is identical to Eq.(2.2). The same calculation can easily be carried through using the generalized boundary condition Eq.(2.5). We have verified that the level-splittings as well as the ground state energy, E_0 , are given correctly to $O(e^2)$ by Eq.(3.12)^[21].

For $q \neq 0$ we shall again identify E_0 as the energy of a background field ϕ_0 around which a scalar field ϕ is quantized. Standard manipulations starting from the massive scalar Hamiltonian yield :

$$Z = \exp(-\beta E_0) Z_B(m) \quad , \quad (3.14)$$

where $E_0 = (\alpha^2/m) \sinh(ma) \cosh(ma)$ and $\phi_0 = (\alpha/m) \sinh(mx)$ so $(\partial_x^2 - m^2)\phi_0 = 0$. Equating E_0 in Eq.(3.13) with E_0 in Eq.(3.14) gives $\alpha = \sqrt{\pi q / 2a} [\sinh(ma) \cosh(ma) / ma]^{-1/2}$. The topological charge Q as calculated using Eq.(2.4) is

$$Q = q [\tanh(ma)/ma]^{1/2} \quad . \quad (3.15)$$

It is now tempting to consider the m dependent factor as the charge renormalization constant Z_R in order to keep $eQ = e_R q$. In that case we have $Z_R = [\tanh(ma)/ma]^{1/2}$. When $ma \rightarrow 0$, $Z_R \rightarrow 1$ and when $ma \rightarrow \infty$, $Z_R \rightarrow 0$ or $e_R \rightarrow 0$, which is consistent with free field behaviour for small ma and screening of electric charge at large ma , respectively. It is also easy to check, using Eq.(3.6), that the fermion number $q \rightarrow 0$ as $ma \rightarrow 0$. Note that this follows only if we assume $eQ = e_R q$, an assumption that can be checked in perturbation theory^[21].

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