PHASE TRANSITIONS AND ZERO MASS EXCITATIONS IN THE
1+1 DIMENSIONAL $Z(p)$ SYMMETRIC SPIN MODELS

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A B S T R A C T

Low temperature kink excitations of selfdual $Z(p)$ symmetric quantum chains are related to the mass gaps in the high temperature phase. It is shown that they provide the massless physical excitations of the intermediate phase. The numerically calculated kink spectrum is presented for chains of different lengths for $p=5,7$ and 9 . The order-to-Coulomb transition is found to be of second order for $p=5$ ($\nu=2.19$) and of infinite order for $p=7$ and 9 .

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In this Letter we report on a study of the spectrum of kink excitations in $Z(p)$ symmetric 1-dimensional quantum chains. Special attention was paid to the question of what are the physical excitations responsible for the existence of an extended critical line [1] in these models.

Recent investigations of Abelian models with variant actions both of local [2] and global [3] symmetries call for a refined physical framework allowing a quantitative understanding of the transition between the massive and massless phases. Arguments were even raised [4] against the equivalence of the original XY-model and a dilute lattice gas of vortices around the Kosterlitz-Thouless point. Hamiltonian studies can give complementary insights into the physical mechanism underlying the transitions in such models.

The low temperature ordered phase of $Z(p)$ models disappears through the stabilisation of domain walls [5]. It is suggestive therefore that the kinks of strength k ($k=1, \dots, [p/2]$), which are superpositions of configurations with net angular difference of $k=2\pi/p$, become massless at the transition point, making domain formation advantageous.

Indeed one can consider an anisotropic $Z(p)$ model in two dimensions defined by the partition function

$$Z_p = \sum_{\{l_x\}} \prod_{\langle x, x+e_1 \rangle} f_{l_x - l_{x+e_1}}^{(1)} \prod_{\langle x, x+e_2 \rangle} f_{l_x - l_{x+e_2}}^{(2)}, \quad (1)$$

where $f_{l-l'}$ denotes the Boltzmann factor for the values l and l' of two neighbouring $Z(p)$ variables along the α -axis. Next one changes

$f_{l_x - l_{x+e_1}}^{(1)}$ to $f_{l_x - l_{x+e_1} + k}^{(1)}$ for links starting from $\underline{x} = (x_1, 0)$, ($0 \leq x \leq T$). In this way one forces on the system a domain wall of strength k and length T . We denote by $\tilde{Z}_p^{(k)}$ the partition function of the altered system. The line tension σ_k is defined as

$$\sigma_k = - \lim_{T \rightarrow \infty} \ln \frac{\tilde{Z}_p^{(k)}}{Z_p} \cdot \frac{1}{T}. \quad (2)$$

Using selfduality of the model and following closely the derivation done for the Ising case [6], one finds the relation:

$$\sigma_k = - \lim_{T \rightarrow \infty} \ln \left\langle e^{i \frac{2\pi}{p} k (l_{\underline{x}^*} - l_{\underline{x}^* + T \underline{e}_1})} \right\rangle_* \cdot \frac{1}{T} \equiv \xi_k^{(1)*-1} \quad (3)$$

Here $\langle \quad \rangle_*$ refers to averages calculated in a $Z(p)$ spin model with periodic boundary conditions, the dual Boltzmann factors and on the dual lattice $\underline{x}^*$.

When one performs the infinite anisotropy limit [7] for both the original and the altered model the following Hamiltonian is arrived at [1a]:

$$\hat{H} = \frac{g}{2a} \left[\sum_{i=1}^N \frac{1 - \cos \frac{2\pi}{p} \hat{L}_i}{1 - \cos \frac{2\pi}{p}} - x \sum_{i=1}^N \cos \frac{2\pi}{p} (\hat{\ell}_i - \hat{\ell}_{i+1}) \right] \quad (4)$$

($x=2/g^2$), $[\hat{L}_i, \hat{\ell}_j] = (p/2\pi) \delta_{ij}$, to be solved however with different boundary conditions

$$\ell_{N+1} \equiv \ell_1 + \frac{2\pi}{p} k \quad (5)$$

in the latter case.

Eq.(3) holds in this limit too. σ_k goes over into the mass of the quantum kink defined by the difference of the respective ground state energies of the $k=0$ and $k \neq 0$ sectors^{*)}:

$$\sigma_k \rightarrow m_{\text{kink}}^{(k)} = \frac{g}{2a} (E_0(k) - E_0(p.b.c.)) \equiv \frac{g}{2a} M_{\text{kink}}^{(k)}(x). \quad (6)$$

The inverse correlation length governing the asymptotics of the

correlations between the variables $\left\{ e^{i \frac{2\pi}{p} k \ell} \right\}$ at large "time" T becomes the conventional mass gap: $m_{\text{gap}}^{(k)}$.

Earlier lattice investigations were done by computing $M_{\text{gap}}^{(1)}$ in a truncated Hilbert space spanned by a set of strong coupling ($x=0$) eigenstates [9]. However technically it was not possible to unravel the critical line by the usual Lanczos-tridiagonalisation method because exceedingly large subspaces are needed in the weak coupling regime.

In view of (3) one can choose equally well to search for the zero of $M_{\text{kink}}^{(1)}$, which should occur at x^* , dual to the position of the high

^{*)} The idea of locating phase transitions by the degeneracy of ground states in topologically distinct sectors of the Hamiltonian appears already in [8].

temperature Kosterlitz-Thouless transition found through $M_{\text{gap}}^{(4)}=0$. Duality connects the zeros of the higher kink masses to those of the corresponding mass gaps. It is clear that at least the $k=1$ (the lightest) domain wall will be massless in the whole intermediate region. However the higher kinks a priori can condense in different points along the critical line.

In the present numerical study a weak coupling configurational basis was used to calculate $M_{\text{kink}}(x)$ because the implementation of eq.(5) is then very natural. With the help of our 'basis vector importance sampling' procedure [10] one can trace the behavior of the kink masses along the whole critical line. We have determined the kink masses in the $Z(5)$ model for rings $N=3-7$ and in the $Z(7)$ and $Z(9)$ models for $N=3-6$, to an accuracy of six decimal places. Next we present the analysis of our data with illustrative figures taken from the $Z(7)$ case. $Z(9)$ and the apparent special behaviour of the $Z(5)$ model will be touched on at the end of the paper.

Following Roomany and Wyld [11] we interpret the closed chains containing different numbers N of sites as discretized approximations to a unique underlying continuous system of fixed physical length L . A phenomenological renormalisation group transformation [12] $g'=R(g)$ can be obtained by keeping the mass of the continuum theory fixed under the scale transformation $(N-1) \rightarrow N$ (when $N \gg 1$):

$$m_{\text{kink}}^{(k)} = \frac{1}{2L} g^N M_{\text{kink}}^{(k)}(x, N) = \frac{1}{2L} g' \cdot (N-1) \cdot M_{\text{kink}}^{(k)}(x', N-1) \quad (7)$$

(For $Z(7)$ the quantity $2LM$ is depicted for $N=3-6$ in Fig.1.). The function

$$\Delta g(g) = g - g' \quad (8)$$

(the sign convention is dictated by the expected coupling flow towards $g=0$) contains information about the Callan-Symanzik β -function in an integral form:

$$\int_{a(N)}^{a(N-1)} \frac{da}{a} = \ln \frac{N}{N-1} = \int_g^{g'} \frac{d\bar{g}}{\beta(\bar{g})} \quad (9)$$

Close to the critical point g_c one expects

$$\beta_c(g) \sim -C(g_c - g)^{1+\sigma}, \quad (10)$$

where $\sigma=0$ corresponds to a second order transition ($C=v^{-1}$), while $\sigma \neq 0$ is obtained when an infinite order transition occurs[13]. The $\Delta g(g)$ function is shown in Fig.2 for the lightest (twist=1) kink of the Z(7) model as calculated from the longest chains $N=5,6$. Its magnitude shows the necessity to calculate eigenvalues at least to 6 places accuracy. In an extended region around $\Delta g(g^*)=0$ one has $|g-g^*| \gg |\Delta g(g)|$. Accepting g^* as a first guess for g_c , one can approximate the right-most expression in (9) by a single term:

$$\beta(g) \approx - \frac{1}{\ln N/(N-1)} \Delta g(g). \quad (9a)$$

The expressions (9a) and (10) were analysed using $g_c \approx g^*(N=6) = .7188$. A very good fit was obtained with $\sigma \neq 0$ (see Table). Using σ one can extrapolate the $g^*(N, N-1)$ estimates ($N=4-6$) towards g_c by a finite size scaling formula adapted from Ref.[14] to the Hamiltonian formulation:

$$g_c - g^*(N, N-1) \sim \frac{\text{const.}}{[\ln N \cdot \ln(N-1)]^{1/2\sigma}}. \quad (11)$$

The small value of $\sigma(Z(7))$ results in very fast convergence and practically unchanged g_c . Therefore there is no need for iterative redetermination of σ either. In the Table also the value of g dual to g_c is shown. It agrees reasonably with the transition point between the Coulomb and the disordered phase like the σ values with those obtained in Ref. [9a].

Next we consider the interior of the Coulomb phase. Our findings are best illustrated by the evolution of the ratio

$$R_{\text{kink}}(x, N) = \frac{N \cdot M_{\text{kink}}(x, N)}{(N-1) \cdot M_{\text{kink}}(x, N-1)}. \quad (12)$$

The standard finite size scaling analysis locates the critical point by the criterion $R_{\text{kink}}(x_c, N)=1$. In Fig.3 one should observe that this line is crossed by the R_{kink} -curves at decreasing slope with increasing N . Moreover they develop a plateau of qualitatively well determined extension, which gradually approaches the $R=1$ line from below. The curves turn away from criticality downwards in a coupling region correlated with the high temperature K.-T.-transition. This is the signal that kinks merge with the ground state of the disordered phase. This tendency forms a qualitative evidence for the twist=1 kink being a massless physical excitation in the whole Coulomb regime and condensing into the ground state at g_{KT} .

In Fig.4 the R-ratios of the twist=2 and twist=3 kinks are shown as extracted from the longest chains. In the strict sense of finite size scaling they exhibit gradual condensation (in agreement with a picture put forward by us some time ago [15]). However the ratios approach the $R=1$ line with increasing N and for the displayed best curves they stay within 2% to this value (for the twist=2 kink the deviation is well within .5%). Therefore we expect that in the $N \rightarrow \infty$ limit they become massless in the same point as the twist=1 kink.

In the Table also the results of a fully analogous procedure for the $Z(9)$ group are displayed.

The nature of the $Z(5)$ transition(s) has been debated in the literature [9,16]. Our results do not contradict yet the 3-phase structure, but the limiting transition seems to be of second order from the weak coupling side too (see Table). As Fig.5 shows, the present size rings do not tend to form any plateau. The quantitative study of the β -function cannot accommodate any value of $\sigma > 0$, but gives a perfect linear fit with $\nu=2.19$.

In conclusion, we have constructed the wave functions of the zero mass particles numerically and demonstrated that the phase diagram of the $Z(p)$ -symmetric spin models can be understood fully in terms of kinks. The critical data obtained by the analysis of the $\Delta g(g)$ function support the earlier results based on finite size scaling analysis of the conventional mass gap. This reaffirms that for these values of p one encounters important deviations from the $\sigma=1/2$ prediction of the dilute vortex gas mechanism. For a definite conclusion in the case of the $Z(5)$ model larger chains will have to be studied. Finally, the experience gained here suggests that the study of kink-like excitations in 3-dimensional Hamiltonian lattice QED might shed light on the nature of the photon in the lattice field theoretic framework.

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p	$g_c(\text{kink})$	$\nu(\text{kink})$	$\sigma(\text{kink})$	$g^{\text{dual}}(\text{kink})$	$g_c(\text{gap})$	$\nu(\text{gap})$	$\sigma(\text{gap})$
5	1.16	2.19	0.	1.19	1.20	2.4	0.
7	0.718(8)	-	0.10(6)	1.05	1.05	-	0.06
9	0.464(5)	-	0.11(2)	1.01	1.00	-	0.1

TABLE

Critical couplings and exponents for the $Z(p)$ ($p=5,7,9$) models from kink mass calculations. The indices of the disorder-to-Coulomb transition (first Ref. in [9]) are also tabulated.

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FIGURE CAPTIONS

- Figure 1: $2mL$ (a quantity proportional to the finite volume mass in physical units) vs. the coupling g as approximated on different lattices.
- Figure 2: The RG-transformation $\Delta g(g)$ obtained from Fig.1 ($N=5,6$) by application of eq.(7).
- Figure 3: Formation of the zero mass kink excitation with increasing N (for $Z(7)$ and $\text{twist}=1$).
- Figure 4: Same as Fig.3 for $\text{twist}=2,3$ and $N=5,6$.
- Figure 5: The R-ratio (see eq.(12)) for $\text{twist}=1$ in the $Z(5)$ model.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5