THE LOW LYING SPECTRUM OF THE 2-d GRASSMANNIAN NON-LINEAR MODELS

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We present analytic results for the mass gap values, of the two-dimensional Grassmannian models $SO(p+q)/SO(p) \times SO(q)$, $SU(p+q)/SU(p) \times SU(q) \times U(1)$ and $Sp(p+q)/Sp(p) \times Sp(q)$. The finite volume effective Hamiltonian of these models is constructed to the one-loop order.

A systematic method for extracting estimates of non perturbative parameters of asymptotically free theories, in the finite volume approximation, has been proposed few years ago [1]. In particular, calculations of the mass gap in the case of $O(N)$ and CP^{N-1} models [1,2] seem to be consistent with improved actions lattice Monte Carlo results. Recently the two loop result for the $O(N)$ model has been obtained [3].

In this work we proceed to extend the finite volume method to the case of the 2-dimensional Grassmannian models:

$$S_{pq} \equiv SO(p+q)/SO(p) \times SO(q)$$

$$C_{pq} \equiv SU(p+q)/SU(p) \times SU(q) \times U(1)$$

and $H_{pq} \equiv Sp(p+q)/Sp(p) \times Sp(q)$, where $SO(N)$, $SU(N)$, $Sp(N)$ the orthogonal, special unitary and symplectic-unitary group, respectively.

These symmetric space models play an important role as models for disordered statistical systems [4] and they have been proposed and shown to be asymptotically free in 2-dimensions in Ref. [5]. They are described by the Lagrangian

$$\mathcal{L} = \frac{1}{4f} \text{Tr} \partial_\mu P \partial_\mu P \quad (1)$$

where P is a $(p+q) \times (p+q)$ hermitian projector of rank p . To construct the projector P we introduce a $(p+q) \times p$ matrix

$$Z_a^i, \quad i=1, \dots, p, \quad a=1, \dots, p+q \quad (2)$$

the elements of which are real (for S_{pq}), complex (for C_{pq}) and quaternionic (for H_{pq}) numbers. The matrix Z consists of p orthogonal real, complex, quaternionic vectors respectively,

$$Z^+ Z = I_p \quad (p \times p \text{ unit matrix}) \quad (3)$$

and the projector P is given as

$$P = Z Z^+ . \quad (4)$$

From Eqs (1), (3) and (4) we see that the Lagrangian, written in terms of fields Z , has a local $SO(p)$, $SU(p)$ or $Sp(p)$ gauge symmetry

$$Z(x) \rightarrow Z(x) \cdot h(x) \quad (5)$$

where $h(x)$ is $p \times p$ orthogonal or unitary matrix, with elements real, complex or quaternionic numbers respectively. We also see that the models have correspondingly the global symmetry $SO(p+q)$, $SU(p+q)$ or $Sp(p+q)$.

To treat the three cases together we choose as the origin of these spaces the matrix

$$Z_0 = (e_1, \dots, e_p) \quad (6)$$

with e_a the vector with components

$$(e_a)_b = \delta_{ab} \mathbb{1} \quad , \quad a = 1, \dots, p \quad , \quad b = 1, \dots, p+q \quad (7)$$

where $\mathbb{1}$ is the unity (identity) in the field of complex or quaternionic numbers (according to the model). To construct the spectrum of these theories we go to a finite cylinder $[0, L] \times [0, T]$ with periodic space boundary conditions (b.c.)

$$Z(0, t) = Z(L, t) \quad \forall t \in [0, T] \quad (8)$$

and Dirichlet b.c. in time

$$\begin{aligned} Z(x, 0) &= Z_0 \\ Z(x, T) &= e^{i\theta \mathcal{U}_0} Z_0 \equiv Z_1 \end{aligned} \quad \forall x \in [0, L] \quad (9)$$

where we choose for all three cases

$$(\mathcal{U}_0)_{ab} = \delta_{a,1} \delta_{p+q,b} + \delta_{a,p+q} \delta_{1,b} \quad . \quad (10)$$

The point Z_1 of the symmetric space has been chosen so that it has geodesic distance from Z_0 , [6],

$$d(Z_0, Z_1) = \theta, \sqrt{2}\theta, 2\theta \quad (11)$$

for S_{pq} , C_{pq} and H_{pq} correspondingly.

After these definitions we come to the construction of the effective Hamiltonian, which goes as follows. The amplitude

$$A_T(Z_1, Z_0) \equiv \langle Z_1, T | Z_0, 0 \rangle = \int [dZ] \delta(Z^+ Z - 1) e^{-\int d^p x \mathcal{L}} \quad (12)$$

is obtained in perturbation theory, by expanding around the classical solution which interpolates the b.c. (9). The gauge invariance (5) is broken by the choice of the unitary gauge [5]

$$U = U^\dagger \quad (13)$$

where U is a $p \times p$ matrix part of Z

$$Z = \begin{pmatrix} U \\ V \end{pmatrix} . \quad (14)$$

Then the orthonormality condition (3) implies $U^2 = I - V^\dagger V$ and the independent degrees of freedom are the $q \times p$ components of V

$$V_{a-p}^i = Z_a^i , \quad i=1, \dots, p , \quad a=p+1, \dots, p+q . \quad (15)$$

To calculate the amplitude $A_T(Z_1, Z_0)$ we find first the classical solution

$$Z_{cl}(t) = e^{i\theta \frac{t}{T} u_0} Z_0 \equiv R(t) Z_0 , \quad (16)$$

then we introduce the quantum fluctuations

$$P = R P_2 R^\dagger$$

where

$$P_2 = Z_2 Z_2^\dagger \quad (17)$$

and we expand up to second order in

$$(\phi^i)_b \equiv V_b^i \quad i=1,\dots,p \quad , \quad b=1,\dots,q \quad (18)$$

using the unitary gauge.

After some straightforward algebra we obtain the Lagrangian of the quadratic fluctuations around the classical solution

$$\mathcal{L} = \frac{1}{2f} \text{Tr}_q \left\{ \partial_\mu \phi^{+i} \partial_\mu \phi^i + \frac{\theta^2}{T^2} \left[1 - \phi^{+1} \phi^1 - (\phi_2^{+1})^2 - (\phi_2^1)^2 - \sum_{i=1}^p \phi_2^i \phi_2^{+i} \right] \right\} \quad (19)$$

The tr_q operation is the quaternionic trace if needed. We identify the perturbative fields for the S_{pq} , C_{pq} and H_{pq} cases as follows :

$$S_{pq}: \quad \phi^i = i\sqrt{f} w^i \quad , \quad w^i \text{ real} \quad , \quad i=1,\dots,p$$

$$C_{pq}: \quad \phi^i = i\sqrt{f} (w_R^i - i w_I^i) \quad , \quad w_R^i, w_I^i \text{ real}$$

$$H_{pq}: \quad \phi^i = i\sqrt{f} (w_0^i + w_a^i \sigma^a) \quad , \quad w_\mu^i \quad (\mu=0,1,2,3) \text{ real}$$

where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices

$$\sigma^l \sigma^m + \sigma^m \sigma^l = -2 \delta^{lm}$$

$$l, m = 1, 2, 3$$

$$(\sigma^l)^+ = -\sigma^l \quad .$$

Doing the quadratic functional integrations over ϕ^i , the constraints

$$Z^+ Z = I_p$$

and

$$U = U^+$$

introduce measure terms which in dimensional regularization drop out [5].

The amplitudes $A_T(Z_1, Z_0)$ normalized to 1 for $\theta = 0$, are given for all three cases in terms of the effective action S_{eff} ,

$$A_T(Z_1, Z_0) = e^{-S_{\text{eff}}} \quad (20)$$

$$S_{P2}: S_{\text{eff}} = \frac{\theta^2 L}{2fT} - \frac{N-2}{2} \frac{\theta^2}{T^2} \left(I_0 + \frac{\theta^2}{T^2} I_1 \right) \quad (21a)$$

$$C_{P2}: S_{\text{eff}} = \frac{\theta^2 L}{2fT} - \frac{\theta^2}{T^2} \left(N I_0 + \frac{N+6}{2} \frac{\theta^2}{T^2} I_1 \right) \quad (21b)$$

$$H_{P2}: S_{\text{eff}} = \frac{\theta^2 L}{2fT} - \frac{\theta^2}{T^2} \left[2(N+1) I_0 + (N+10) \frac{\theta^2}{T^2} I_1 \right] \quad (21c)$$

where

$$I_0 = \frac{1}{6} \left(T^2 - \frac{L^2}{4} \right) + \frac{TL}{2\pi} \left[\frac{1}{\epsilon} + \frac{1}{2} \left(\log \frac{\mu^2 L^2}{4\pi} + \gamma_E \right) \right] \quad (22a)$$

$$I_1 = \frac{1}{90} \left(T^4 - \frac{L^4}{16} \right) + \frac{TL^3}{16\pi^3} f(3) \quad (22b)$$

and γ_E is the Euler-Macheroni constant.

Now let us write the θ^2 part of the effective action as

$$S_{\text{eff}}^{\theta^2} = \left(\frac{L}{\alpha f_B T} - \frac{\gamma}{T^2} I_0 \right) \theta^2 \quad (23)$$

where f_B the bare coupling constant. In order to have a finite S_{eff} , the pole terms in f_B and I_0 must cancel. If the relation of f_B and f renormalized is

$$f_B = \mu^\epsilon \cdot f \cdot Z(f)$$

where

$$Z(f) = 1 + \frac{Z_1(f)}{\epsilon} + \frac{Z_2(f)}{\epsilon^2}$$

we find that

$$Z_1(f) = - \frac{\alpha\gamma}{2n} f \quad (24)$$

and so the beta function $b(f) \equiv f^2 \frac{\partial}{\partial f} Z_1(f)$ is up to one loop

$$b(f) = - \frac{\alpha\gamma}{2n} f^2. \quad (25)$$

Now for the three cases we have

$$S_{p2} : \quad \alpha = 2, \quad \gamma = \frac{N-2}{2}$$

$$C_{p2} : \quad \alpha = 2, \quad \gamma = N$$

$$H_{p2} : \quad \alpha = 1, \quad \gamma = 2(N+1)$$

and from (25) we obtain the correct one loop b-functions [7].

Using the renormalized effective action we determine the Hamiltonian by demanding that to leading order in $\frac{1}{T}$ the propaga-

tion or heat kernel equation is satisfied by $e^{-S_{\text{eff}}}$

$$\left[\frac{\partial}{\partial T} + H_{\text{eff}} \right] e^{-S_{\text{eff}}} = 0 \left(\frac{1}{T} \right) \quad (26)$$

Since what we have calculated is really the propagation kernel of a quantum mechanical particle on the spaces S_{pq} , C_{pq} and H_{pq} , the behaviour for small geodesic distances s must be $\sim e^{-s^2/4\tau}$. But we know that $s \sim \theta$ and that

$$\left(\frac{\partial}{\partial \tau} + \Delta \right) e^{-s^2/4\tau} \frac{1}{\tau^{D/2}} \sim 0 \quad (27)$$

where Δ is the Laplace Beltrami operator in the symmetric spaces. So comparing (27) and (26) we find the H_{eff} in all three cases,

$$H_{\text{eff}} = \frac{\alpha}{\lambda} f \cdot \frac{1}{L} \left[1 + \frac{\alpha \gamma}{4n} f \left(\chi_E + \log \frac{\mu^2 L^2}{4n} \right) \right] \cdot \Delta \quad (28)$$

where $\lambda = 4, 2, 1$ for S_{pq} , C_{pq} , H_{pq} . (The different factors arise from the different metric in every symmetric space.)

To find the mass gap we need the lowest non-zero eigenvalue of Δ on the spaces S_{pq} , C_{pq} , H_{pq} . Recently people working on Kaluza-Klein theories have given general formulae for the spectrum of Δ on these spaces [8]. In our case we need representations of the $SO(N)$, $SU(N)$, $Sp(N)$ groups, which contain the trivial representation of their corresponding maximal subgroups $SO(q) \times SO(N-q)$, $SU(q) \times SU(N-q) \times U(1)$, $Sp(q) \times Sp(N-q)$ respectively. Consider first S_{pq} [9], with $q \leq \frac{N}{2}$ for N even or $q \leq \frac{N+1}{2}$ ($q \leq \frac{N-1}{2}$) for N odd and q odd (even). Depending on N and q , there are two

different Young patterns satisfying the above criterion and giving the smallest Δ eigenvalue, and we find [8] ($N = p + q$)

$$\begin{aligned} N \leq 8 \quad , \quad \Delta(S_{pq}) &= p \cdot q \\ N \geq 9 \quad , \quad \Delta(S_{pq}) &= p \cdot q \quad \text{for } q = 1, 2 \\ &= 2N \quad \text{otherwise.} \end{aligned} \quad (29a)$$

It is interesting to see that, for $N \geq 9$, the lowest eigenvalue does not depend on q for $q \geq 3$. Similar examination for H_{pq} gives

$$\begin{aligned} N \leq 4 \quad , \quad \Delta(H_{pq}) &= 4p \cdot q \\ N \geq 5 \quad , \quad \Delta(H_{pq}) &= 4p \cdot q \quad \text{for } q = 1 \\ &= 4N \quad \text{otherwise.} \end{aligned} \quad (29b)$$

In the case C_{pq} the required Young tableaux are the ones with $r_p + r_q$ columns, of which r_p have p rows and r_q have q rows. The additional $U(1)$ symmetry requires zero $U(1)$ total charge, which implies [8] $r_p = r_q = r$, and the lowest ($r = 1$) eigenvalue is

$$\Delta(C_{pq}) = 2q(p+1). \quad (30)$$

The mass gaps finally are

$$S_{pq}: \quad m_0 = \Delta(S_{pq}) \cdot \frac{f}{2L} \left[1 + \frac{p+q-2}{4\pi} f \left(\gamma_E + \log \frac{\mu^2 L^2}{4\pi} \right) \right] \quad (31a)$$

$$C_{pq}: \quad m_0 = 2q(p+1) \cdot \frac{f}{L} \left[1 + \frac{p+q}{2\pi} f \left(\gamma_E + \log \frac{\mu^2 L^2}{4\pi} \right) \right] \quad (31b)$$

$$H_{p2} : m_0 = \Delta(H_{p2}) \cdot \frac{f}{L} \left[1 + \frac{p+2+1}{2n} f (\gamma_E + \log \frac{\mu^2 L^2}{4n}) \right] \quad (31c)$$

The results (31a-c) reproduce for $q = 1$ the known mass gaps, to one loop order, for the $S_{N-1,1} = O(N)$ and $C_{N-1,1} = CP^{N-1}$ models [1,2] (only a rescaling $f \rightarrow \frac{f}{N}$ of the coupling is needed).

To give a numerical estimate we follow Ref. [1] in looking for the minima of the universal curve

$$\frac{m_0(L)}{\Lambda_{MS}} = F(m_0 \cdot L) \quad (32)$$

that is when the mass gap becomes less dependent on the scale of the box L . Here Λ_{MS} is the R.G. invariant scale of the model

$$\Lambda_{MS} = \mu e^{-1/b_0 f} (b_0 f)^{-b_1/b_0^2} \quad (33)$$

where b_0 and b_1 are the first two coefficients of the b-function

$$b(f) = -f^2 (b_0 + b_1 f + \dots) \quad (34)$$

Now, given the mass gap (ref. 31a-c),

$$m_0 = \frac{1}{L} [a_{00} f + f^2 (a_{11} \log \mu^2 L^2 + a_{10}) + \dots] \quad (35)$$

the function (32) $F(z \equiv m_0 \cdot L)$ in the MS scheme takes the form

$$\frac{m_0}{\Lambda_{MS}} = e^{\frac{a_{10}}{a_{00} b_0}} \left(\frac{b_0}{a_{00}} \right)^{\frac{b_1}{b_0^2}} e^{\frac{a_{00}}{b_0^2 z}} z^{\frac{b_1}{b_0^2} + 1} \quad (36)$$

The coefficients a_{00} , b_0 , b_1 are positive in asymptotically free theories, so there is a minimum at

$$z_{min} = \frac{a_{00} b_0}{b_1 + b_0^2}$$

and we find

$$\left(\frac{m_0}{\Lambda_{MS}} \right)_{min} = e^{\frac{a_{10}}{a_{00} b_0}} \left(\frac{b_0}{a_{00}} \right)^{\frac{b_1}{b_0^2}} (e \cdot z_{min})^{\frac{b_1}{b_0^2} + 1}. \quad (37)$$

The coefficients b_0 , b_1 have been obtained for all three models [7], and up to this order we need only b_1

$$S_{pq}: \quad b_1 = \frac{2p(N-p) - N}{(2n)^2} \quad (38a)$$

$$C_{pq}: \quad b_1 = \frac{2p(N-p) + 2}{n^2} \quad (38b)$$

$$H_{pq}: \quad b_1 = \frac{4p(N-p) + N}{2n^2} \quad (38c)$$

To get estimates for the infinite volume mass gaps [10,11], we need information for the forward scattering amplitudes of the m_0 -mass particles. Unfortunately, only the $S_{N-1,1}$ case has known 2 to 2 S-matrix and for this model see Refs [10,3]. Recently the $SU(N) \times SU(N)$ chiral model has been solved [12,13]. It will be interesting to see how the present method compares with the exact results. A perturbative calculation of the mass gap of the case of chiral models has been presented in Ref. [14]. We think

that this calculation explicitly violates unitarity in the two point invariant functions, because of the wrong treatment of the zero momentum modes of the system and the consequent absence of exponentiation.

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