NEUTRALIZATION OF ACCELERATOR BEAMS BY IONIZATION OF THE RESIDUAL GAS

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INTRODUCTION

This note was written on the occasion of a lecture. It is a review paper as shown by the long reference list. We have grouped together reports of general interest references 1 to 10 and books references 11 to 13.

On various occasions the neutralization effects have been a limitation to machine performance. In particular the ISR could only reach its nominal performance after an improvement of the vacuum by nearly two orders of magnitude. Some of the difficulties of the first accelerators came from the fact that the vacuum systems were not good enough. In the future the difficulties will rather be linked with the large currents and high densities of the circulating beams.

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REFERENCES
1. NEUTRALIZATION OF A BEAM : A SIMPLE DESCRIPTION

A circulating beam of particles such as protons, ionizes the molecules of the residual gas in the vacuum chamber, electrons and positive ions are created. Positive ions are chased by the electrostatic effect of the beam, electrons are trapped until they neutralize the charge density in the beam. That is until the number of static electron charges is equal to the number of circulating proton charges. The beam is then fully neutralized. The definition of the neutralization factor \( \eta \) follows:

\[
\eta = \frac{n_e}{n_p}
\]

(1)

Where \( n_e \) is the number of trapped electrons around the whole machine and \( n_p \) is the number of circulating particles.

If the protons are debunched and if the electron accumulation takes place all around the machine, the local densities are constant and the above relation still holds for each meter of the machine at any time. If \( s \) is the azimuthal distance along the machine one can define a linear density of electrons,

\[
\frac{dn_e}{ds}
\]

and a corresponding local neutralization factor

\[
\eta(s) = \frac{2\pi R}{n_p} \frac{dn_e}{ds}
\]

(2)

Where \( 2\pi R \) is the machine circumference.

then

\[
\eta = \frac{1}{2\pi R} \int_0^{2\pi R} \eta(s) \, ds
\]

(3)

We have conserved the above definition for bunched beams, as we will see \( \eta \) cannot exceed one in bunched or unbunched beams.

In the case of circulating beams of particles of negative charge the difference is that the positive ions are trapped and the electrons chased.

In order to obtain a few simple numbers let us consider a continuous proton beam of 1 A, with a constant density over its round cross-section of radius \( a = 1 \) cm (we call it our nominal beam)

\[
I = 1 \text{ A}
\]

\[
a = 10^{-2} \text{ m}
\]

the corresponding linear density of charge is

\[
\lambda = \frac{dn_e}{ds} = \frac{I}{\pi a^2 c} = 3.10^{-8} \text{ C.m}^{-2}
\]

(4)
the linear density of particles is

\[ \frac{\lambda}{e} = \frac{dn}{ds} = 2.10^9 \text{protons m}^{-1} \]  

The electric field at the edge is obtained via the Gauss theorem:

\[ E = \frac{\lambda}{2\pi \varepsilon_0 a} = 5.10^8 \text{V.m}^{-1} \]  

The magnetic field at the edge is obtained via the Ampere's law

\[ B = \frac{I u_0}{2 \pi a} = 2.10^{-3} \text{Tesla} \]  

The direct space charge force on a circulating proton is

\[ F = e (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]  

using \( \varepsilon_0 \mu_0 c^2 = 1 \) (mks unit relation) we obtain

\[ F = F_e + F_B = F_e (1 - \beta^2) = F_e \frac{1}{\gamma^2} \]  

where \( \beta, \gamma \) are the relativistic factors of the proton and \( F_e \) is the electrostatic force.

The fact that the two forces counteract is called the relativistic cancellation.

With neutralization the electrostatic force is changed from \( F_e \) to \( F_e (1 - \eta) \) and the magnetic force is unchanged. Then

\[ F = F_e \left( \frac{1}{\gamma^2} - \eta \right) \]  

The force which we have calculated at the edge of the beam is in effect proportional to the distance to the center of the beam. The corresponding local quadrupole will have a strength \( k \) (with the courant and Snyder definition)

\[ k(a) = - \frac{1}{E} \frac{dF}{da} \]  

Where \( E = \gamma E_0 \) is the circulating beam energy. It is convenient to simplify the formulae by using the classical proton radius.

\[ r_p = \frac{e^2}{2 \pi \varepsilon_0 m_p c^2} = 1.53 \times 10^{-14} \text{m}. \]  

and the local proton volume density:

\[ d_p = \frac{dn}{ds} \cdot \frac{1}{\pi r_p^2} = 6.4 \times 10^4 \text{m}^{-3} \]
so that with (6) (10) (11) (12) and (13)

\[ k(s) = -\frac{2\pi}{\gamma} \frac{d}{dp} \frac{1}{\gamma^2} \eta \]  

(14)

The corresponding tune shift will be

\[ \Delta Q = \frac{1}{4\pi} s(s).k(s).2\pi R \]  

(15)

where \( s(s) \) is the usual Twiss parameter. With our nominal beam, at medium energy \( (\gamma = 10) \) and for a local \( s(s) \) of 20 m this will give for a machine or 1000 m circumference

\[ \Delta Q = 0.1 \eta \]

The machine performance will be limited for values of \( \eta \) larger than a few percent.

2. THE IONIZATION PROCESS

The circulating beam interacts with the electrons of the molecules of the residual gas or with the electrons trapped in the beam. In turn the trapped electrons interact with the molecules in many different ways. We will limit our study to the effects which directly interest the accelerators that is:
- transfer of energy to free electrons
- ionization cross-section

2.1 Transfer of energy to free electrons

An approximate estimation can be obtained by calculating the electrostatic interaction between a free electron and the primary particle (Ref. 11, Jackson p.430). The energy transfer \( E'(b) \) is a function of the impact parameter \( b \) Figure 1.

![Figure 1. The impact parameter b](image-url)
Equation 13.2 of Jackson where the field \( a \) the electron is obtained by a Lorentz transformation, can be rewritten in mks units and using the classical electron radius,

\[
\rho_e = \frac{1}{m_e c^2} \frac{e^2}{4 \pi e_0}
\]

(16)

\[
E'(b) = \frac{2m_e c^2}{E}\frac{r^2}{b^2}
\]

(17)

or

\[
b^2 = \frac{r^2}{\frac{1}{e} E'} \frac{2m_e c^2}{E'}
\]

(18)

Where \( \beta c \) is the velocity of the primary particle and \( E'(b) \) is the transferred energy, expressed in ev.

The cross-section \( d \sigma \) for energy transfer between \( E' \) and \( E' + dE' \) writes

\[
d\sigma = 2\pi b \, db
\]

(19)

or

\[
d\sigma = 2\pi \frac{m_e c^2}{E^2} \frac{r^2}{E'} \frac{dE'}{r^2}\]

(20)

one sees immediately an unphysical situation for \( E' = 0 \) (b large, distant collisions) and \( E' = \infty \) (b small, close collisions). The difficulty is solved by defining a minimum energy \( E'_{\text{min}} \) and a max. energy \( E'_{\text{max}} \). The maximum energy \( E'_{\text{max}} \) can be obtained from pure kinematic considerations when the minimum energy requires a detailed analysis of the medium in which the interaction takes place.

The above formulae cannot be used for exact ionization cross-section as we will see below but they give a fair description of the phenomenon.

The detailed measurements of \( \frac{d\sigma}{dE'} \) have been made\(^{14,15}\) and are represented below:

![Figure 2](image)

Relative probability of different processes induced by fast (100 keV) electrons in water, as a function of the energy transfer in a collision\(^{16}\). The maximum kinematically allowed energy transfer, \( E_M = 50 \) keV in this case, is also shown.
The ionization event only takes place if the energy transferred is above the ionization potential. Looking at Figure 2, it is clear that most free electrons created in such an event will be left with a rather small energy and therefore trapped by the beam. In reference 15, it has been measured that about 80% of the electrons, have an energy below 45 eV. The average being around a few eV. The ion energy will be smaller in the ratio of the masses and therefore negligible. These energies however should be used in calculating the drift velocities because they are several order of magnitudes higher than the thermal energy (≈ 10⁻² eV for electrons). The proportion of electrons not trapped because they are created with an energy larger than the potential well is negligible (less than 4% for a potential well of a few hundred volts¹⁵).

2.2 Heating of the electron cloud by the beam, cooling by the molecules

The beam passing through the electron cloud will heat the electrons. We see in Figure 2 that the cross-section for energy transfer is large for these events even though the energy transferred is small. On the other hand, the low energy electrons interact with the molecules of the residual gas; the cross-sections for these events are even larger because of the low energy of the electrons (see below). Except for the ISR where the density of molecules is very low due to the ultra high vacuum, the major effect is usually electron cooling by the molecules¹⁵. Some of these interactions create secondary ionization but this effect also is small because most of the electrons do not have an energy larger than the required ionization potential.

2.3 Ionization cross-section

The ionization cross-section depends on the molecule of the residual gas and on the velocity of the ionizing particle but neither on its charge nor on its mass¹⁵. Using the same notation as reference 18 it can be described by the functions (Bethe formula, Jackson p.440) :

\[
\sigma = 4\pi \left(\frac{\hbar}{mc}\right)^2 (M^2 x_1 + CX_2)
\]

\[
x_1 = g^2 \log \left(\frac{g^2}{1 - g^2}\right) - 1
\]

\[
x_2 = g^2
\]

where

\[
4\pi \left(\frac{\hbar}{mc}\right)^2 = 1.874 \times 10^{-28} \text{ m}^2
\]

and

\[
g = v/c
\]

\(M^2\) and \(C\) are characteristics of the molecules. Table I gives a list of values of \(M^2\) and \(C\) for different molecules together with \(Z\) the number of electrons of the molecule. Figure 3 gives a plot of the cross-sections against energy using the above formulae.
TABLE I

Value of the $M^2$ and $C$ constants for calculation of ionization cross-sections

<table>
<thead>
<tr>
<th>Molecule</th>
<th>$M^2$</th>
<th>$C$</th>
<th>$Z$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>H$_2$</td>
<td>0.5</td>
<td>8.1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>N$_2$</td>
<td>3.7</td>
<td>34.8</td>
<td>14</td>
<td>28</td>
</tr>
<tr>
<td>CO</td>
<td>3.7</td>
<td>35.1</td>
<td>14</td>
<td>28</td>
</tr>
<tr>
<td>O$_2$</td>
<td>4.2</td>
<td>38.8</td>
<td>16</td>
<td>32</td>
</tr>
<tr>
<td>H$_2$O</td>
<td>3.2</td>
<td>32.3</td>
<td>10</td>
<td>18</td>
</tr>
<tr>
<td>CO$_2$</td>
<td>5.75</td>
<td>55.9</td>
<td>22</td>
<td>44</td>
</tr>
<tr>
<td>C$_4$H$_8$</td>
<td>17.5</td>
<td>162.4</td>
<td>46</td>
<td>76</td>
</tr>
</tbody>
</table>

Figure 3. Cross-section vs energy

2.4 Ionization rate

The time it takes for one circulating particle to create one ion is given by the obvious formula

$$\tau_m = \frac{1}{d_m \sigma_m \delta c}$$  \hspace{1cm} (22)

Where $d_m$ is the molecular density (m$^{-3}$),
$\sigma_m$ is the ionization cross-section for molecule $m$ (m$^2$),
$\delta c$ is the velocity of the circulating beam (m.s$^{-1}$).
The molecular density $d_m$ is related to the partial pressure $P_m$ in torr by the relation (valid at 20°C).

$$d_m = 3.3 \times 10^{22} \frac{P_m}{m}$$ (23)

If there are several molecules in the residual gas then the total ionization time $\tau_i$ is given by the relation.

$$\frac{1}{\tau_i} = \frac{1}{\tau_m}$$

3. THE ION OR ELECTRON MOTION

The temperature of the molecules of residual gas will be slightly increased by the interactions with the beams which are not of a sufficient energy to produce an ionization. However the reservoir of molecules is so big that the energy in the gas will be the energy related to the temperature that is 300°K. The energy of the electrons acquired through momentum transfer from the circulating beam in the ionization process will be of a few eV and the energy of the ions lower by the mass ratio:

$$\frac{m_e}{A \cdot m_p}$$

Where $A$ is the atomic mass of the ion, $m_e$ and $m_p$ the masses of the electron and the proton.

The electrons and the ions are created at the top (or bottom) of the potential well of the beam which is of a few 100 eV and will be either chased or oscillating in this potential well. Their motion will be influenced by the magnetic fields. The analysis of these different energies and motions is the object of this chapter.

3.1 Energy temperature velocity

The distribution of velocity of molecules of mass $m$ in a gas of density $d_m$ at temperature $T$°K has been calculated by Boltzmann.

$$dn = \frac{1}{2} d_m \sqrt{\frac{m}{2\pi kT}} \cdot e^{-\frac{m}{2kT} (v_x^2 + v_y^2 + v_z^2)} dv_x dv_y dv_z$$

where $\frac{dn}{dv_x dv_y dv_z}$ is the number of particles per unit volume around velocities $v_x, v_y, v_z$ and

$$k = 1.4 \times 10^{-23} \text{ J/°K}$$ or $$\frac{k}{e} = 8.6 \times 10^{-5} \text{ eV/°K}$$

is the Boltzmann constant.
One finds successive mean velocities by integration

\[ \langle v \rangle = v_m = \left( \frac{8kT}{m} \right)^{\frac{1}{2}} \]

\[ \langle v^2 \rangle^{\frac{1}{2}} = v_{\text{rms}} = \left( \frac{3kT}{m} \right)^{\frac{1}{2}} \]

\[ \langle |v_x| \rangle = \langle |v_y| \rangle = \langle |v_z| \rangle = v = \left( \frac{2kT}{m} \right)^{\frac{1}{2}} \]

with naturally

\[ \langle v_x \rangle = \langle v_y \rangle = \langle v_z \rangle = 0. \]

so that

\[ v_m = 2v \quad \text{(Annex I)} \]

The mean kinetic energy is

\[ E = \langle \frac{1}{2} mv^2 \rangle = \frac{1}{2} mv_{\text{rms}}^2 = \frac{3}{2} kT \quad \text{(24)} \]

Table II illustrates for different molecules the relation energy, temperature, velocity.
**TABLE II**

Relation $T$(°K), $E$(eV), $v$($\text{m/s}$) for various molecules

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$v_{r.m.s}$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 300^\circ$</td>
<td>$E = {6.3 \cdot 10^{-12}$ joule $}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_2$</td>
<td>1</td>
<td>$2.7 \cdot 10^4$</td>
<td>$3.7 \cdot 10^8$</td>
</tr>
<tr>
<td>$H_2O$</td>
<td>18</td>
<td>$6.5 \cdot 10^2$</td>
<td>$1.0 \cdot 10^3$</td>
</tr>
<tr>
<td>CO/N$_2$</td>
<td>28</td>
<td>$5.2 \cdot 10^2$</td>
<td>$0.7 \cdot 10^3$</td>
</tr>
<tr>
<td>CO$_2$</td>
<td>44</td>
<td>$4.1 \cdot 10^3$</td>
<td>$0.6 \cdot 10^3$</td>
</tr>
<tr>
<td>e$^-$</td>
<td>1/1836</td>
<td>$1.2 \cdot 10^4$</td>
<td>$1.6 \cdot 10^4$</td>
</tr>
</tbody>
</table>

| $T = 7.8 \cdot 10^4$ °K | $E = 1$ eV |
| $H_2$ | 2   | $9.8 \cdot 10^4$ | $13.5 \cdot 10^8$ |
| N$_2$ | 28  | $2.6 \cdot 10^3$ | $3.6 \cdot 10^3$ |
| e$^-$ | 1/1836 | $6.0 \cdot 10^3$ | $8.3 \cdot 10^3$ |

| $T = 78 \cdot 10^4$ °K | $E = 10$ eV |
| $H_2$ | 2   | $3.1 \cdot 10^3$ | $4.3 \cdot 10^9$ |
| N$_2$ | 28  | $8.3 \cdot 10^3$ | $11.5 \cdot 10^9$ |
| e$^-$ | 1/1836 | $1.9 \cdot 10^4$ | $2.6 \cdot 10^4$ |
3.2 The electric field and the potential well

Before analysing the motion of ion we will first compute the fields which act on the ions and the electrons which have been "just created". In the absence of external fields an electric field is induced by the circulating beam. This field defines a potential and the value of the potential is fixed by the fact that the vacuum chamber is at ground potential. We consider the simplified case of a circular beam in a circular chamber.

\[ a \quad \text{the radius of the beam with uniform density in real space} \]
\[ r \quad \text{the radial variable} \]
\[ r_o \quad \text{the vacuum chamber radius}. \]

We have seen already that the field can be calculated using the Gauss theorem:

\[
E_r = \begin{cases} 
\frac{1}{2\pi\varepsilon_0} \frac{r}{a^2} & r < a \\
\frac{1}{2\pi\varepsilon_0} \frac{1}{r} & r \geq a
\end{cases}
\] (25)

The potential is obtained by integration.

\[ V = -\int E_r \, dr \]

The constant being fixed by the condition

\[ V = 0 \quad \text{for} \quad r = r_o. \]

The result is

\[
V = \begin{cases} 
\frac{r^2}{2a^2} - \frac{1}{2} \log \frac{r}{a} & r < a \\
\log \frac{r}{r_o} & r \geq a
\end{cases}
\] (26)

Figure 4 represents the potential for our nominal beam with 1 amp circulating current, a vacuum chamber of 100 mm diameter and different beam sizes.

Figure 5 represents the value of the central potential \( V_o \) for different ratio \( \frac{a}{r_o} \).
Figure 4. Potential well for different beam sizes.

Figure 5. Potential well depth versus beam size.
Figure 6. Beam potential in ISR. The position of clearing electrodes is indicated by the dots.
The motion of ions or electrons in this field is very simple: in a beam of protons, positive ions are chased and arrive on the wall with an energy $eV_0$, electrons are trapped if their transverse energy is less than $eV_0$. Since the probability of energy transfer larger than the potential well is very small electrons are always trapped.

The detailed calculation of the potential well for non cylindrical geometries has been made. We give in Annex the resulting formulae to be used in practical calculations. Figure 6 is the result of the calculation made for the ISR.

These potential wells have been directly measured.

If the vacuum chamber size or the beam size varies in a long straight section, the electron or ion will drift towards the deepest potential well. The kinetic energy gained in the process can be considerably higher than the thermal energies since the variations of potential can reach several hundred eV while the thermal energies are of the order of $4 \times 10^{-2}$ eV.

3.3. The effect of the magnetic field

The motion of a particle in a uniform magnetic field is simple, it is the well known cyclotron motion. When the field has a gradient perpendicular to the field direction a drift of the particle occurs which is called the gradient drift. If the gradient of the field is in the direction of the field there is a containment effect called the magnetic mirror. This chapter studies these three effects.

3.3.1 Cyclotron motion

In all these problems one separates the velocity in its two components

$$v_{\parallel}$$ parallel to $B$

$$v_{\perp}$$ perpendicular to $B$

![Figure 7. $v_{\parallel}$ and $v_{\perp}$](image)
With
\[ v^2 = v_\parallel^2 + v_\perp^2 \]  \hspace{1cm} (27)

If the field is uniform \( \frac{\partial B}{\partial r} = 0 \), then the velocity along the field \( \vec{v}_\parallel \) is uniform and unchanged. The perpendicular velocity \( v_\perp \) induces a force and therefore an acceleration.

\[ \gamma m \frac{dv_\perp}{dt} = e \vec{v}_\perp \times \vec{B} \]

This is a central force perpendicular to \( \vec{v}_\perp \) which gives a circular motion, the radius \( r \) of the circle is obtained by equating the central acceleration to the centrifugal force.

\[ e \vec{v}_\perp \cdot \vec{B} = m \frac{v_\perp}{r} \]

the angular frequency \( \omega_c \) also called cyclotron angular frequency is

\[ \omega_c = \frac{v_\perp}{r} \]  \hspace{1cm} (28)

which gives

\[ r = \frac{m v_\perp}{eB} \]  \hspace{1cm} (29)

\[ \omega_c = \frac{eB}{m} \]  \hspace{1cm} (30)

With the remarkable result that \( \omega_c \) does not depend on \( v_\perp \) : in a given field the larger the velocity the longer the radius but the frequency does not change for a given particle.

3.3.2 Effect of a transverse gradient (the gradient drift)

We have already seen that

\[ v_\perp = r \frac{eB}{m} = r \omega_c \]

but here

\[ B = B_0 + \frac{\partial B}{\partial x} \cdot x \]

The projection of the velocity on \( z \) writes (with \( x = r \cos \omega_c t \)).

\[ v_z = v_\perp \cos \omega_c t = r \frac{eB_0}{m} \cos \omega_c t + r \frac{eB}{m \frac{\partial B}{\partial x}} \cdot r \cos \omega_c t \cdot \cos \omega_c t \]

then the mean velocity is not zero corresponding to a drift

\[ v_D = \langle v_z \rangle = \frac{1}{2} r \frac{eB}{m} \frac{\partial B}{\partial x} \]
This calculation only applies if the field variation over the cyclotron motion is small, that is, if
\[ r \frac{dB}{dx} \ll B. \]
This effect is called the gradient drift.

It can also be written:
\[ v_D = \frac{1}{2\omega_C} v_x^l \left( \frac{1}{B} \frac{dB}{dx} \right) \]

This gradient can only be created by a curvature of the magnetic field, particles with a velocity parallel to the magnetic field \( v_x \) will have to follow the field lines. This curvature will give an additional drift, so that the final drift writes:
\[ v_D = \frac{1}{\omega_C} (v_x^l + \frac{1}{2} v_x^l) \left( \frac{1}{B} \frac{dB}{dx} \right) \]

(31)

3.3.3 Effect of a longitudinal gradient (the magnetic mirror)

We assume that \( B_x \) changes with \( z \). This gives a set of lines of force as sketched in Figure 8).

![Diagram of magnetic mirror](image)

Figure 8. The magnetic mirror

When the particles move to the right towards higher fields, the field lines are more dense. A variation of flux through the orbit would induce an electromotive force and therefore an exchange of energy between the static magnetic field and the particles. This is not possible so the flux circled by the particle is constant. (The exact demonstration makes use of the action integral (See Jackson p. 422.))

\[ \ast B r^l = \ast B_0 r^l_0 \]

or using the equation of the cyclotronic motion.
\[ v_x^l = v_x^l_0 \frac{B}{B_0} \]
Since the kinetic energy of the particle is conserved \( v' = v_0 \)

So that (Eq. 27)

\[
v_f = v_{o} - v \frac{B}{\gamma \beta_0} \tag{32}
\]

If \( B \) becomes large enough \( v' = 0 \) the motion of the particle is stopped, detailed calculations show that the particle in fact spirals back.

Looking at equation (32) it is clear that the particle will be trapped if \( v_f \) can reach zero that is if

\[
\left| \frac{v_0}{\gamma \beta_0} \right| < \left( \frac{B}{B_0} - 1 \right)^\dagger \tag{33}
\]

With an isotropic distribution of speed at the time of creation of the particle the proportion of particles trapped will be

\[
R = 1 - \frac{B_0}{B} \tag{34}
\]

3.4 **Combined effects of \( \vec{E} \) and \( \vec{B} \) (the cross-field drift)**

We consider the magnetic field of a magnet and the electric field of the beam (Figure 9).

![Cross-field drift](image)

**Figure 9. Cross-field drift**

The electric force is: \( e \vec{E} \)

The magnetic force is: \( e \vec{v} \times \vec{B} \)

at equilibrium

\[
v_f = \frac{E}{B} \tag{35}
\]
This equilibrium can only be reached if \( \vec{v}_L \) can reach \( \vec{E}/B \) that is if

\[
\frac{E}{B} < c
\]

a rather simple analysis shows that indeed if \( \vec{E} < cB \) this equilibrium is always reached. However the time it takes to reach that equilibrium is approximately the time it takes for the field to accelerate the particle to an energy corresponding to the velocity \( \vec{v}_L \). In practice during the acceleration phase where \( \vec{v}_L \cdot B \) is very small compared to \( \vec{E} \), the magnetic field can be neglected. If \( \vec{E}/B > c \) this equilibrium will never be reached, the magnetic field can be neglected. In all the practical cases which will be considered, the motion can be described by a pure acceleration or a pure drift. Where \( \vec{E}/B << c \) the equilibrium is reached in less than a \( \mu s \) and the transverse displacement is less than a \( \mu m \).

4. A FEW EXAMPLES OF ION OR ELECTRON MOTIONS

4.1 Field free section

Let us consider a straight section with enlarged vacuum chamber (Figure 10).

The potential well at the center of the beam is

\[
V_0 = -\frac{1}{2}\varepsilon_0 \left\{ \log_\pi \frac{x^2}{a^2} - \frac{1}{2} \right\}
\]

(36)

giving the curve of potential represented in Figure 10 as a function of the azimuth.

In this example the ions created between B and C will be trapped, the ions created between A and B will drift towards C and continue to the right with an energy between 0 and 28 eV.

4.2 Pure dipole field \( B = 0.1 \) Tesla

Assuming a nominal beam (paragraph 2), the electric field at the edge of the beam is \( 5.10^4 \) Vm\(^{-1} \) so that the cross-field drift velocity varies from zero at the center to \( 5.10^6 \) ms\(^{-1} \) at the edge of the beam. The corresponding energies are 0.12 eV for electrons and 13 eV for \( N_2 \) ions. The cyclotron radius for electrons in the center with energy about 1 eV is 10 microns (1 mm for ions).

Particles with the same charge as the beam are chased out of the potential well in the vertical plane. Particles with opposite charge drift towards the end of the magnet, but are contained in the potential well in the vertical plane.

4.3 Combined function magnet

Let us consider a magnet like the CPS magnet with a field of 0.1 Tesla on the central orbit and a gradient of \( = 0.412 \) Tesla/m on the central orbit.
Beam and vacuum chamber

$2r_0 = 0.15 \text{ m}$

$2a = 0.015$, $2a = 0.009$

Potential at beam center

Figure 10. Potential well in a field free section
We have seen that the typical energy of particles created in the potential well is of a few electron volts, the corresponding velocities are (paragraph 3.1):

- for ions \( \sim 2.6 \times 10^9 \text{ ms}^{-1} \).
- for electrons \( \sim 6.1 \times 10^9 \text{ ms}^{-1} \).

The cyclotron radii in the center are very small. The gradient drift for ions or electrons of 1 eV is of the order of a 100 ms\(^{-1}\) and therefore negligible with respect to the cross-field drift.

4.4 The quadrupole field

If the beam is centered in a quadrupole with gradient 1 T m\(^{-1}\), the cross-field drift at the edge of the beam will be (35)

\[
v_c = 5.10^9 \text{ ms}^{-1}
\]

corresponding to an energy for ions of 350 keV. This energy would have to be provided by the potential well which is of only a few 100 volts, so that the equilibrium required for the cross-field drift will never be reached. The electric field of the potential well dominates the ion motion. The cross-field drift however is possible for electrons.

The gradient drift does not take place because the field is too small.

Particles with the same charge as the circulating beam are chased toward the poles of the quadrupole. In some cases a magnetic mirror effect could insure the containment of these particles.

4.5 Undulator

The field of an undulator is dipolar but with alternatively positive and negative polarity along the azimuth. The main effect in the horizontal plane is the cross-field drift but the fact that the field is alternated gives a possibility of containment. In the vertical plane the field is concentrated in the poles so that a magnetic mirror effect can develop. This mechanism can be effective for electrons or ions depending on the geometry and the beam polarity.

Since the potential well is not required to achieve containment, it is not easy to find the mechanism which limits the accumulation of ions in an undulator.
5. Bunched Beams

All previous studies are in principle only valid for unbunched circulating beams. In fact the bunching only introduces additional effects.

In references 2, 9, 22, 23, one finds a study of the stability of ions in single beam and colliding beam machines, we will only summarise the results of this study in the case of single beam machines.

Here, we have changed the definition of neutralization factor $n$ from the local definition used in Ref. 22 to the average definition given above Eq. 3. This explains the difference in our formulae.

At a given azimuth, an ion sees successively the focusing (or defocusing) forces induced by the bunch passage, followed by a drift time between bunches. If a vertical dipole magnetic field is applied, the horizontal transverse and longitudinal motions will be coupled, the vertical motion however is independent of the magnetic field. The horizontal motion will not be studied here (Ref. 21). It gives similar results in drift spaces; in bending magnet the horizontal motion is usually stable.

**Vertical motion**

The forces induced by the passage of a bunch have been studied by several authors. With the time as independent variable the bunch passage is similar to a focusing (or defocusing) lens; if the bunch is short the effect on the ion can be described by a matrix equivalent to a thin lens. The non-linear effects are not considered here.

Let $y$ and $\frac{dy}{dt} = \dot{y}$ be the position and speed of the ion. Then the passage of a bunch is described by

$$
\begin{pmatrix}
y' \\
y_0'
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
y \\
y_0
\end{pmatrix}
\frac{\Delta}{\mathbf{b} + \mathbf{a}} \frac{\mathbf{c}}{\mathbf{A}}
$$

with

$$
\alpha = \frac{n_p}{n} \frac{\Delta}{\mathbf{b} + \mathbf{a}} \frac{\mathbf{c}}{\mathbf{A}}
$$

(37)

where $n_p/n$ is the charge per bunch, $n$ is the number of bunches, $r_p$ is the classical proton radius, $a, b$ is the beam size, horizontal resp. vertical, $A$ is the atomic weight of the ion, $\beta c$ is the velocity of the circulating beam.
In between bunches the ion drifts freely during the time \( t = \frac{T}{n} \). Where \( T \) is the revolution time, corresponding to the matrix transformation

\[
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\]

One period of the forces applied is described by the matrix product.

\[
M = \begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
a & 1
\end{pmatrix}
\]

The stability is insured if the trace of the matrix satisfies

\[-2 < \text{Tr}(M) < +2\]

that is for

\[-1 < \left(1 + a \frac{T}{n}\right) < +1\]

The \( + \) sign is to be selected for cases where the beam and the ion are of the same sign. Electron beams therefore cannot accumulate electrons and proton beams cannot accumulate positive ions which is obvious.

For opposite sign, the requirement is:

\[1 - a \frac{T}{n} > 1\]

If \( A_m \) is the mass of the ion, one can express the above formulae by the following criterion: all ionic masses larger than a critical mass \( A_c \) will be accumulated. With

\[A_c = \frac{A_m \mu_p}{\mu_n} \frac{\pi R}{8.0 b^2 (1 + \frac{a}{b})}\]  \hspace{1cm} (38)

In most practical cases the critical mass varies between .1 and 100. This explains why electrons \((A = \frac{1}{2000})\) never accumulate in bunched beams of positrons or protons.

In bunched beams of electrons with a large number of bunches (e.g. synchrotron radiation sources) the critical mass can be very small \( \approx 10^{-2} \); then the ion accumulation can be treated as in D.C. beams: the ions are too heavy to "see" that the beam is bunched.

When the critical mass is above 44, the ions usually found in vacuum systems cannot accumulate.
Intermediate cases:

\[ 1 < A_c < 44 \]

require a detailed analysis\(^{11}\).

Note: There is some confusion on the values to be used for \( a \) and \( b \) after a real measurement of particle distribution. They are related to the central density of the distribution of particle. If the measured distributions were rectangular then \( a \) and \( b \) would be the measure of the real half width of the distributions. If the distributions are gaussian, a simple calculation shows that

\[ a = o_x \sqrt{2} \]
\[ b = o_y \sqrt{2} \]

6. CLEARING ELECTRODES

As we have seen in 4, ions (or electrons) created in some point will drift around the circumference of the machine and will accumulate to a certain level of neutralization. The clearing can be done if required by installing in selected places, electrodes with a potential sufficient to suppress the potential well. If one wants to avoid neutralization pockets, the electrodes should be placed at convenient places around the ring. In the example of paragraph 4.1 Figure 9 an electrode placed in A or B will not clear the neutralization pocket BC.

The location of the clearing electrodes in the ISR Figure 6 has been selected to avoid these neutralization pockets. The large list of references on this subject (Ref. 24 to 29) can be used as a guide to the design of clearing electrodes. The design should be made such as to avoid to introduce parasitic impedances especially in electron rings.

Clearing electrodes placed in magnetic fields must be calculated to avoid the cross-field drift.

7. THE LIMIT OF ACCUMULATION

The containment of particles is in general due to the potential well. When charges start to accumulate, they diminish the depth of potential well. The limit is reached when the density of accumulated charges is equal and opposite to the density of circulating charges so that the net resulting charge is null.

\[ 0 = d_i - d_p \]

The neutralization factor \( \eta \) is then at most, equal to 1. The beam is fully neutralized. Clearly the same limitation of neutralization exists in bunched beams.
As we have seen, several clearing effects due to various drifts will limit the local neutralization factor to less than 1. If required, clearing electrodes will collect electrons and reduce the neutralization in appropriate places.

However, in some extreme cases the trapping of the ions or electrons is not due to the potential well, so that one could have local accumulation of charges to neutralization levels higher than 1. We have seen that the undulators are probably very efficient in accumulating ions. Detailed calculations should be made in this particular case. It is not excluded that similar situations are found in the combination of end-fields of the conventional magnets or quadrupoles.

8. THE EFFECTS OF NEUTRALIZATION

There are several effects of neutralization. The particles (ions or electrons) chased from the beam and hitting the wall with the energy of the potential well can create desorption. The ions stored in the beam modify the local pressure and therefore the life-time of stored beams.

The most important effects come from the electric field of the ion (or electron) cloud. This induces a shift and a spread of betatron oscillation frequencies and a coupling of horizontal and vertical motion.

Finally the ion cloud can interact as a whole with the beam and induce instabilities.

8.1 The pressure bump

The mechanism of the pressure bump is rather simple. In a beam of protons the ions are chased out of the potential well and hit the wall with enough energy to induce desorption of \( \eta \) molecules per ion (here \( \eta \) is the desorption factor and not the neutralization factor). If the rate of filling the vacuum vessel with these molecules is higher than the pumping speed, the pressure will increase.

The exact treatment is given in Ref. 30 and 31, it is however possible to obtain a first indication and order of magnitude by equating the pumping speeds to the desorption rate.

The equation of pumping of a vacuum vessel is in the absence of desorption \(^1\) :

\[
\frac{dP}{dt} = - \frac{SP}{V}
\]

where

- \( P \) is the pressure (torr)
- \( S \) is the pumping speed (\( m^3 \) s\(^{-1}\))
- \( V \) is the volume (\( m^3 \))
In terms of molecular density \(d_m\) and linear pumping speed \(s\), this writes

\[
\frac{d}{dt} d_m = -\frac{s}{A} d_m
\]

(39)

Where \(A\) is the cross-section area of the vessel (and not a molecular mass).

The rate of production of ion is \(\frac{l}{e} d_m \sigma\) ions per second and per meter, \((\frac{l}{e}\) is the number of protons per second and \(d_m \sigma\) is the probability of ion production per meter). The desorption rate is therefore

\[
\frac{d}{dt} d_m = \eta \cdot \frac{1}{A} \cdot \frac{l}{e} d_m \sigma
\]

(40)

Combining the two effects gives

\[
\frac{d}{dt} d_m = \frac{d_m}{A} (\eta \frac{l \sigma}{e} - s)
\]

(41)

leading to exponential increase of density or pressure if

\[
\eta \frac{l \sigma}{e} > s
\]

(42)

Ex: the ISR pressure bump at the beginning of operation

\[
\eta = 4
\]

\[
I = 4
\]

\[
\frac{l}{e} = 6.25 \times 10^{24}
\]

gives \(\eta \frac{l \sigma}{e} = 10^{-3} \text{ m}^2\text{s}^{-1}\)

If in some place the effective local pumping speed is reduced below \(10 \text{ m}^{3}\text{s}^{-1}\), the pressure bump can develop.

- The rise time of the pressure is of the order of \(\frac{A}{s}\), that is for standard vacuum chambers a few seconds. This could explain the oscillation of the vacuum pressure observed in slow cycling machine with high currents like the PS.

- The desorption coefficient \(\eta\) depends on the energy of the ion striking the wall and therefore on the potential well depth: \(V_o\). It also depends on the cleanliness of the vacuum chamber.

8.2 Pressure increase due to ion

The residual gas density and composition enter in the calculations of beam life time due to scattering of particles of the beam by the residual gas or of beam emittance growth due to multiple scattering.
The effective density to be considered is the sum of molecular and ionic densities.

We have seen (Eq. 4 and 13) that the circulating beam density is

\[ d_p = \frac{1}{e} \frac{1}{gc \nu s^3} \] \hspace{1cm} (43)

The neutralization factor \( \eta \) gives the ion density

\[ d_i = \eta d_p \]

The molecular density is

\[ d_m = 3.3 \times 10^{12} \] \hspace{1cm} (23)

If we use the data of our nominal beam

\[ d_p = 64 \times 10^{11} \] \hspace{1cm} m\(^{-3}\)

With 10\% neutralization

\[ d_i = 64 \times 10^{11} \] \hspace{1cm} m\(^{-3}\)

Corresponding to a "partial pressure"

\[ P_i = 2 \times 10^{14} \] \hspace{1cm} Torr.

Except for ultra high vacuum systems or very dense beams the increased pressure due to ion accumulation can be neglected \( ^{11, 13, 15} \).

8.3 Tune shifts

The problem of tune shifts induced by neutralization is similar to the problem of tune shifts induced by space charge. One assumes that the transverse distributions of ions are the same as the distributions of circulating particles. The calculations are made with constant real space density (Ref. 2) for a beam of width 2a and height 2b. The calculations are made valid for bunched or unbunched beams by the introduction of a bunching factor B equal to the ratio of average to peak longitudinal linear density. B is less than 1 for a bunched beam and equal to 1 for unbunched beams.

The local density of protons is

\[ d_p = \frac{1}{2\pi R} \frac{1}{\nu a} \cdot \frac{1}{B} \cdot n_p \] \hspace{1cm} (44)

The local density of ions is (\( \eta \) is the neutralization factor)

\[ d_i = \frac{1}{2\pi R} \frac{1}{\nu a} \eta \cdot n_p \] \hspace{1cm} (45)
The electric field of this distribution has been calculated by several authors Ref. 2) in MKS units it writes for the horizontal plane.

\[ \frac{\partial F}{\partial x} = \frac{e}{\epsilon_0} \left( \frac{d_1}{1 + \frac{a}{b}} \right) \]  \hspace{1cm} (46)

The local quadrupole strength as defined by C and S (Ref. 1) writes:

\[ k = \frac{e}{\epsilon_0} \frac{\partial^2 F}{\partial x^2} \]  \hspace{1cm} (47)

Where \( E \) is the energy of the circulating particle.

The corresponding tune shifts are obtained using

\[ \Delta Q = \frac{1}{\pi} \int g(s) k(s) \, ds \]

The tradition is to introduce the classical proton (or electron) radius

\[ r_p = \frac{1}{\epsilon_0} \frac{e^2}{\gamma} \]

(12)

The integration is to be done around the circumference of the machine where \( s \), \( a \), \( b \) vary as a function of \( s \). One usually replaces the quantities by their average around the machine (\( s = R/Q \)). This averaging is partly justified by the fact that the quantity \( \pi a^2 / \gamma \) is an emittance.

\[ \Delta Q_x = r_p \frac{1}{\gamma} \frac{R}{Q_x} \frac{1}{\pi a^2 (a + b)^2} \eta_{np} \]  \hspace{1cm} (48)

\[ \Delta Q_y = r_p \frac{1}{\gamma} \frac{R}{Q_y} \frac{1}{\pi a^2 (a + b)^2} \eta_{np} \]  \hspace{1cm} (49)

The values \( \Delta Q_x \) and \( \Delta Q_y \) are in fact tune spreads as well as tune shifts because the distribution of ion is not uniform in the beam so that the fields are strongly non linear.

The asymmetry of the ion cloud (a \( \neq \) b) and the non linearities also introduces coupling effects. These effects are particularly visible in \( e^+ e^- \) machines Ref. 5, 6, 7.

These effects have been directly measured \(^{7, 14}\).

### 8.4 Instabilities in proton beams

The motion of the electrons (or ions) in the potential well of the circulating beam and of the motion of the circulating beam is the electron cloud provides a feedback mechanism which can drive beam instabilities. Two such instabilities have been detected in proton machines. They received the name of ionic oscillations (Ref. 37) and of electron instabilities (Ref. 38). The detailed theory of these instabilities does not have its place here. The mechanisms however are rather simple, and simplified formulae can be derived to obtain orders of magnitude.
8.4.1 The ionic oscillations

The proton bunches leave ion-electron pairs after their passage. With a bunched beam the light weight electrons are so stirred up in the ionization process that they are lost to the wall almost immediately.

The heavy positive ions even though they are of the same polarity as the beam, will stay longer in the vacuum chamber because they are created with thermal energies and are more difficult to move.\(^1\)

The detailed calculations go through the following steps:

- A closed pattern of oscillation of the beam around the machine is defined:
  \[ x(\theta, t) = a \cos (Q - n) \Omega t + n \delta \]
  which implies that at a given time (e.g. \( t = 0 \)) the center of gravity of the beam is distributed along \( n \) oscillations around the machine and that a given particle \( (\theta = \theta_0 + n \delta) \) oscillates with the betatron frequency \( Q \delta \). For simplicity we define
  \[ \omega = (Q - n) \Omega \]
  then
  \[ x = a \cos (\omega t + n \delta) \]  

- The corresponding pattern of ions will have the same aspect but with a phase shift due to the ion motion in the proton field.
  \[ x_1 = B_0 \cos (\omega t + n \delta) + B_1 \sin (\omega t + n \delta) \]
  We will skip the difficult calculation of \( B_0 \) and \( B_1 \).

- The presence of these ions will induce a force on the proton beam to be inserted in the equation of motion of the proton.
  \[ \ddot{x} + (Q \delta)^2 x = K x_1 \]

- The definition of \( x \) (Eq. 50) no longer satisfies this equation of motion. The technique is to let the two constants \( A \) and \( \omega \) vary slightly to take into account the small extra force introduced.
  \[ a = a + \dot{a} t \]
  \[ \omega = \omega + \dot{\omega} \]
  Then
  \[ \ddot{x} = -a (\omega + \dot{\omega})^2 \cos (\omega t + n \delta) - \dot{a} \omega \sin (\omega t + n \delta) \]  

This can be introduced in (51).
The resolution of cosine terms gives a negligible change in the tune shift. The resolution of the sine terms leads to the rate of rise of the instability

\[ \frac{1}{\tau_r} = \frac{a}{a} \]

where \( \tau_r \) is given as a function of \( B_0 \) and \( B_1 \).

For \( \tau_r > 0 \) the amplitude is unstable. This is only obtained for \( Q - n \) negative which means that only patterns with \( n > Q \) will be unstable. This is a good signature of the instability.

The final result\(^{17} \) is remarkably simple for the dense beams which are common in present accelerators

\[ \tau_r = \frac{8 \cdot 0.8}{(n - Q)} \cdot \tau_i \]

where \( \tau_i \) is the ionization time, \( A \) the atomic mass number.

For example at low energy (\( \gamma \approx 1 \)) in the CPS (\( Q = 6.25, n = 7 \)) with a pressure of \( 10^{-7} \) torr of \( N_i (A = 28) \) one finds

\[ \tau_i = 10 \text{ ms} \quad \tau_r = 20 \text{ ms} \]

8.4.2 Electron proton instabilites

We have seen that the field of the electrons accumulated in a proton beam induced a tune shift \( \delta Q \). This effect can be introduced in the equation of motion of the protons \(^{18, 19} \) (we have selected the vertical motion)

\[ z_p + Q^e n^t z_p = Q^e n^t (z_e - z_p) \quad (53) \]

Where \( z_p \) and \( z_e \) are the transverse positions of the center of gravity of the beam and of the electron cloud and (paragraph 8.3).

\[ Q_{\text{proton}} = 2Q_0 Q = \frac{2n_e}{\gamma} \cdot R \cdot \frac{1}{s + b (a + b)} \cdot n_e \quad (54) \]

In a similar way the equation of motion of the center of gravity of the electron cloud can be written:

\[ z_e = Q_{\text{electron}} (z_p - z_e) \quad (55) \]

with

\[ Q_e = \frac{2 \cdot n_e \cdot R}{\gamma \cdot a (a + b)} \]
As in the previous case we can define a pattern of oscillation of the proton beam.

\[ z_p = A_p e^{i(n \delta - \omega t)} \]  \hspace{1cm} (56)

Note that the introduction of the complex exponential will simplify the calculation of the phase shift and of the rate of rise that was treated with real sine and cosine functions in the previous example.

The electrons have only a local oscillation:

\[ z_e = A_e e^{-i\omega t} \]  \hspace{1cm} (57)

The substitution of (56) and (57) in (53) and (55) gives two homogeneous equations in \( A_e \) and \( A_p \). \( A_e \) and \( A_p \) can be eliminated with the result that the defined quantities must satisfy the resulting equation where the reduced frequency \( x = \frac{\omega}{\Omega} \) has been introduced:

\[ (Q_e^2 - x^2) [Q_p^2 + Q_e^2 - (n - x)^2] = Q_e Q_p \]  \hspace{1cm} (58)

In this equation \( Q_e, Q_p, Q \) and \( n \) are the parameters and \( x \) is the unknown.

If \( x \) is real it means that there exists a real frequency of oscillation of the system of two beams. The system is stable.

If \( x \) is complex the solutions come by pair; one with a positive imaginary part corresponding to a damping of the pattern; one with a negative imaginary part corresponding to an anti-damping of the pattern.

A detailed examination of equation (58) shows that for large values of \( Q_p \) the solutions become complex.

The threshold value of \( Q_p \) is

\[ Q_p^{th} = \frac{(n - Q_e)^2 - Q_e^2}{2Q_e(n - Q_e)} \]  \hspace{1cm} (59)

It defines the threshold of the instability.

Above this threshold the imaginary part of the complex conjugate solutions \( x \) is

\[ \text{Im} \, n = \frac{Q_p}{2 \sqrt{(n - Q_e)}} \]  \hspace{1cm} (60)
The growth is

$$\frac{1}{\tau_r} = \alpha \text{ Im} \frac{n_e}{n_p}$$  \hspace{1cm} (61)

For example, consider the case of the ISR where this instability was first discovered (Ref. 38).

\[
\begin{align*}
N_p & = 6 \times 10^{13} \\
a & = 3 \times 10^{-2} \text{ m} \\
b & = 10^{-2} \text{ m} \\
\gamma & = 16 \\
Q & = 8.75
\end{align*}
\]

Then

$$Q_e = 200 \text{ (Eq. 55).}$$

The most dangerous mode \( n \) will give the smaller threshold \( Q_p^{th} \), this is obtained for \( n = 209 \) (Eq. 59)

$$Q_p^{th} = 0.052$$

Equation (54) gives the corresponding average neutralization required

$$n = \frac{n_e}{n_p} = 2 \times 10^{-3}$$

and equations (60) and (61) give the growth rate

$$\tau_r = 8 \text{ T}$$

Where \( T \) is the revolution period of the ISR

$$\tau_r = 25 \mu s.$$  

The instability is extremely fast and the frequency observed is in general above the bandwidth of pick-up electrodes

$$\frac{1}{2\tau} Q_e \omega = 64 \text{ Mhz}$$

8.4.3 Landau damping

In reality these instabilities are much less predictable than in the above simplified picture. The complexity comes when instead of analysing the behaviour of the beam or of the ion cloud as a whole, one analyses the behavior of each individual particle or ion before averaging the displacements. The fact that neutralization forces are very non linear, that the frequency of
oscillation of electrons depends on the azimuth as well as the number of electrons trapped still complicates the picture. The calculations cannot be made on a single frequency but rather on a distribution of frequencies. It is far easier to explain why a given instability occurs in a given machine than why it does not occur in another machine.

ACKNOWLEDGEMENTS

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ANNEX I

Computation of the mean velocity in thermal motion

By integration from $-\infty$ to $+\infty$ of the Boltzmann equation (paragraph 4.1) over $v_x$ and $v_y$, one obtains the equation

$$\frac{dn}{dv_x} = \frac{d}{m} \sqrt{\frac{m}{2 \pi kT}} e^{-\frac{v_x^2}{2kT}}$$

The mean value of $v_x$ is by definition

$$\langle v_x \rangle = \frac{1}{dm} \int_{-\infty}^{+\infty} v_x \frac{dn}{dv_x} \, dv = 0$$

What we are interested in is the mean velocity in one direction this is:

either

$$\langle v_x \rangle = \left| \frac{1}{dm} \int_{-\infty}^{+\infty} v_x \frac{dn}{dv_x} \, dv \right| = 0$$

or

$$\langle |v_x| \rangle = \frac{1}{dm} \int_{-\infty}^{+\infty} |v_x| \frac{dn}{dv_x} \, dv_x$$

in both cases the result is

$$\langle |v_x| \rangle = \frac{2kT}{\sqrt{\pi m}}$$

and therefore $\langle |v_x| \rangle = \frac{v_m}{2}$ and not $\frac{v_m}{4}$ as quoted in several papers.
ANNEX II

Elliptical vacuum chamber and beam

The detailed calculation of the potential in a rectangular vacuum chamber of width $2w$ and height $2h$ induced by a beam with current $I$, velocity $sc$, height $2b$ and width $2a$ has been made Ref. 19. The following formulae allow the detailed computation of the potential at the center of the beam:

$$U_o = \frac{I}{sc \, e_0}$$

$$n_s = \frac{w}{2w} \, s$$

$$C_s = \frac{U_o \, s}{4bw \, n_s}$$

$$s_s = \frac{\cos n_s (w - a) - \cos n_s (w + a)}{n_s \, a (1 - n_s^2 a^2)}$$

Then the potential in the center is

$$V(0,0) = \int \left( 1 - \frac{\cosh[n_s (h - b)]}{\cosh n_s \cdot h} \right) C_s \sin n_s \, w$$
ANNEX III

Numerical value (MKS units)

c_o = 8.85 \times 10^{-12} \quad \text{(F.m.}^{-1} \text{or C} \cdot \text{m}^{-1})

\nu_o = 4.47 \times 10^{-7} \quad \text{(H.m}^{-1} \text{or A}^{-1} \text{V} \cdot \text{s} \cdot \text{m}^{-1})

c = 3.10^8 \quad \text{(m s}^{-1})

r_p = 1.53 \times 10^{-10} \quad \text{(m)}

re = 2.82 \times 10^{-15} \quad \text{(m)}

m_p = 1.67 \times 10^{-27} \quad \text{(Kg)}

m_e = 9.10 \times 10^{-31} \quad \text{(Kg)}

m_p/e = 1.0 \times 10^{26} \quad \text{(T.s or Kg C}^{-1})

m_e/e = 5.7 \times 10^{12} \quad \text{(T.s or Kg C}^{-1})

\hbar = 6.85 \times 10^{-16} \quad \text{eV} \cdot \text{s}

k = 1.4 \times 10^{-23} \quad \text{J} / {^\circ} \text{K}
REFERENCES


34. O. Gröbner, P. Strubin, "ISR Performance Report - Decay rate due to nuclear scattering pressure as determined from the clearing currents", ISR-VA/OG/sm, 11th July, 1974.


