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SURVEY OF NUMERICAL METHODS IN FIELD CALCULATIONS

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Abstract - In magnetic field computations, Maxwell's equations are often restricted to the Poisson equation. Several approaches to solving this type of equations are discussed together with the possible field defining potentials. Finite element and finite difference methods are compared, as are the numerical means available for solving the corresponding large systems of matrix equations. Successive overrelaxation, conjugate gradient and preconditioned conjugate gradient method are frequently applied. The integral equation method avoids limitations due to boundary conditions but has other inconveniences due to the fully occupied matrix equation which has to be solved. After mentioning the boundary integral method and techniques to tackle eddy current problems, some examples of proven computer codes for the numerical field calculation are briefly described.

INTRODUCTION

The technique of calculating electromagnetic fields depends entirely on the ability of solving Maxwell's equations. The four Maxwell's equations, established in 1860, are in most cases, however, too general. Thus, the first act in applying them consists of finding the correct constraints for the problem studied. Since analytical solutions can only be found for a very limited number of cases and boundary conditions, it is often necessary to apply numerical methods. As nowadays these methods have become highly developed and as the transcription of such a method into an applicable computer program requires an effort of the order of several man-years, full advantage must be taken of existing software. Thus, in order to make the right choice, one should have some knowledge about the possibilities as well as the limitations of the different methods.

The methods and consequently the programs differ considerably according to the type of solution envisaged. For example, problems concerning permanent magnets are quite different from those of current-excited magnets; field precision of 10^{-3} and 10^{-4} over a wide range of excitation is sometime required, for example in magnets used for particle accelerators; remanence in iron must be treated differently from that in superconducting wires, etc.

Even authors claiming that their programs are widely applicable, cannot usually hide which types of problem they had in mind when developing their software. For two and three dimensions, there are only very few routines available which can solve every field problem. Thus, it is preferable to have at one's disposal a number of methods and programs rather than searching for a "universal" program for solving Maxwell's equations.

In this context, it should be mentioned that not all programs can be run on every computer. This is a general problem, but it is amplified by the size of these programs themselves and their requirements concerning central memory space. Program optimization for one type of computer is not necessarily valid for other types.

Despite these difficulties, a number of numerical programs have been created and have found a certain and

in some cases even a world-wide distribution. Those which this author has had the opportunity to become acquainted with will be characterized briefly at the end of this paper.

CHOICE OF POTENTIAL

For static magnetic field problems, Maxwell's formulae can be reduced to Poisson's equation or, as a further special case, to Laplace's equation. This implies the introduction of either a vector or a scalar potential.

Most numerical methods for solving magnetic field problems concern the solution of the equation

$$\nabla^2 A = -\mu_0 j,$$

where A is the magnetic vector potential defined by $\text{div } A = 0$ and $B = \text{curl } A$.

This elliptical partial differential equation can be solved analytically for only a few boundary conditions; for many, at first sight not even complicated ones, analytical solutions are not available. A further complication may arise from the presence of a ferromagnetic material, characterized by its relative permeability μ_r , which normally is a non-linear function of the induction to which it is exposed. To make things worse, the relative permeability may be directionally anisotropic.

When current carrying conductors are absent, the magnetic scalar potential Ω is sometimes used instead of the vector potential. By applying an additional potential, it is also possible to take into account excitation currents, in which case one solves for a so-called total potential and a reduced potential: the latter is taken for regions which include currents.

It would seem that the choice of potentials is an important decision at the beginning of the development of a method of solution. However, since the systems of equations for the potentials are of similar character [1], especially for the static problems, some programs even allow the free choice between vector and scalar potential. It is only at the stages of applying the boundary conditions and interpreting the results, i.e. when forming the derivatives of the potential, that the corresponding differences have to be respected.

For computing eddy currents, it is necessary to work simultaneously with two potentials, e.g. with the magnetic vector potential and the electric scalar potential. There, the observation of boundary conditions becomes even more difficult since those for the magnetic field differ from those of the eddy currents. The multitude of possible boundary conditions in conjunction with the non-linear behaviour of magnetic materials explains why although several programs exist for two-dimensional eddy-current problems, only certain special cases of three-dimensional problems can presently be treated.

FINITE DIFFERENCES AND FINITE ELEMENTS

In the specialized literature, the method of finite differences has suffered considerable disregard with respect to the method of finite elements. One tends to forget, however, that the former method is still in use and this with continuing good success

[2,3,4,5,6]. The method is based on replacing the differential quotient of the Poisson or Laplace equation by the quotient of differences obtained by applying Taylor's theorem. The partial differential equation is thus approximated by the difference equation, also called (in 2D) the five-point formula [7].

In practice, the problem region is covered with an orthogonal grid of mesh points and for every point, the difference equation is defined. This results in a matrix equation, the solution of which represents the distribution of potential.

$$A x = b$$

$$A = \begin{bmatrix} B & -I & & & 0 \\ -I & B & -I & & \\ & \ddots & \ddots & \ddots & \\ & & I & B & -I \\ 0 & & & -I & B \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 & & & 0 \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ 0 & & & -1 & 4 \end{bmatrix}$$

Matrix A is of order $(M-1)^2$, B is of order $(M-1)$, I is the unit matrix of order $M-1$, with M^2 equal to the number of mesh points. The elements of b are given by the boundary conditions.

Two principle types of boundary conditions must be distinguished:

- Dirichlet boundary conditions, and
- Neumann boundary conditions.

In the first case, the boundaries are determined by fixed values of x, the potential, and in the second, the normal derivatives are given. Dirichlet boundaries can be applied for any shape of boundaries, whereas for Neumann boundaries, one is usually limited to those coinciding with mesh lines.

It is the solving of this matrix equation which is at the heart of the numerical problem. Here, we have two main approaches:

- direct methods,
- relaxation or iterative methods.

Since on the one hand the matrices are usually very large and on the other hand they are sparsely occupied, i.e. not zero only in the neighbourhood of the diagonal, direct methods are not often applied. This speciality of the matrices is taken advantage of by using relaxation methods.

The finite element method has had a great deal of success. Its development has been stimulated by its wide range of applications, where the calculation of electromagnetic field is only one amongst many. The origin of this method lies in the analysis of mechanical structures. It rapidly spread to other fields of application, where one is faced with the description of the behaviour of continua [8,9,10,11].

The starting point of this method is the minimization of the variation integral, or functional, for the solution of a partial differential equation with given boundaries. One establishes a variation integral, normally the local energy content, and tries to distribute it over the problem region such that it becomes a minimum [12]. This is done by splitting the air, conductor

and iron regions into finite elements. In two dimensions, these elements often (but not necessarily) have a triangular shape. The only condition for their distribution is that every single element belongs to only one region. There are no other restrictions to the discretization of the problem. In three dimensions, elements which can be split into tetrahedra are taken. Each of these elements is associated with the variation integral in terms of potential values at each corner point. The minimization is performed for the integral over the total problem region. The system of equations thus obtained is solved as for the finite difference methods, i.e. either by direct methods or by relaxation methods. Instead of the potential, piecewise polynomials of it, also called trial or shape functions, can be used. The variation integral is then minimized with respect to the coefficients of these functions.

The Galerkin method is an extension of the latter one. There, the series is substituted into the differential equation and multiplied by a weight function. By applying Gauss' theorem, a system of non-linear equations is again obtained, the number of which equals that of the unknowns. By properly selecting the weight function, a linear system of equations, which may even be symmetric, can be obtained [13].

As for the finite difference method, the bulk of the computational work for finite elements consists in solving the large system of matrix equations.

Essential to the success of this method has been the wide choice possible for the shape of the elements. Their deformability permits treatment of practically any shape of region, i.e. any configuration of iron, coils and air. Furthermore, by increasing the number of points per element, the precision can be increased (at the cost of increasing the requirements in memory space and computation time).

In Ref. 14, an excellent demonstration of the mathematical equivalence of both methods, finite differences and finite elements, is given.

SOLVING LARGE SYSTEMS OF MATRIX EQUATIONS

Turning now briefly to some typical numerical methods for solving large systems of matrix equations, the algorithm of successive overrelaxation [7] has first to be mentioned. The initial matrix A, built up for example from the five-point formula, is split into

$$A = I - L - U.$$

One corrects the potential values in each point according to the recursion formula:

$$x_{i+1} = (I - \omega L)^{-1} [\omega U + (1 - \omega)I] x_i + \omega(I - \omega L)^{-1} b$$

where:

- x = unknown potential matrix,
- I = unit matrix,
- L = matrix occupied only below its diagonal,
- U = matrix occupied only above its diagonal,
- ω = relaxation parameter.

If $\omega > 1$, the method is called overrelaxation, if $\omega = 1$, it is called Gauss-Seidel method and if $\omega < 1$, it is called underrelaxation.

One can show that the optimum relaxation parameter must be between 1 and 2. For a good SOR program, it is essential to determine the best ω as fast as possible. The relaxation parameter depends mainly on the boundary conditions and the size of the matrix. For simple

cases, where the mesh width h equals $1/M$, ω can be calculated by

$$\omega = \frac{2}{1 + \sin \pi h}.$$

An improvement of the method is the line SOR where the potentials of a row are calculated simultaneously.

The conjugate gradient method consists of a recursion procedure where theoretically an $N \times N$ matrix is solved in exactly N steps. A vector of residuals is found for every iteration cycle by determining a search vector which is conjugate to the preceding iterations:

$$P_{j+1}^T A P_j = 0 \quad (j = 0, 1, 2, \dots, l).$$

Strictly speaking, this last method is a direct one. Moreover, despite its sensitivity to unavoidable rounding errors, the solution is often obtained after considerably less cycles than N . Although only the non-zero matrix elements have to be kept in memory, its space requirements are greater than for SOR. In some cases it is possible to evaluate the iteration parameter directly, however, counterbalancing the previous disadvantage.

By extending the original matrix equation to

$$M^{-1} A x = M^{-1} b,$$

with M being a positive definite matrix, a solution may be obtained faster if the new matrix shows a two-cyclic structure. This approach is called the preconditioned conjugate gradient method.

The incomplete Cholesky-conjugate gradient method (ICCG) [15] belongs to the same family of solution methods. On the symmetric and positive definite matrix A , one applies the Cholesky decomposition

$$A = LL^T,$$

where L is the lower triangular of the matrix. This classical decomposition used in the Gaussian elimination is not very efficient in large sparse matrices, since many of the zero elements become non-zero. One can, however, impose a sparsity pattern on L , preferably the same as that in A . This leaves us with a matrix E of small errors:

$$A = LDL^T + E.$$

With this decomposition, one enters the algorithm of the conjugate gradient method.

Let r_0 , the residual vector, be $r_0 = b - A x_0$, and

p_0 , the search vector, be $p_0 = (LL^T)^{-1} r_0$.

Then,

$$x_{i+1} = x_i + \alpha_i p_i,$$

with $\alpha_i = \frac{r_i^T (LL^T)^{-1} r_i}{p_i^T A p_i},$

$$r_{i+1} = r_i - \alpha_i A p_i,$$

$$p_{i+1} = (LL^T)^{-1} r_{i+1} + \beta_i p_i,$$

with $\beta_i = \frac{r_{i+1}^T (LL^T)^{-1} r_{i+1}}{r_i^T (LL^T)^{-1} r_i},$

$$i = 0, 1, 2, \dots$$

Although the approximation of $(LL^T)^{-1}$ is essential, convergence is extremely fast for most cases. By decomposition into lower and upper triangulars, the method can be applied to any non-singular matrix not being symmetric or positive definite.

INTEGRAL METHOD

For the finite difference and the finite element methods, boundary conditions have to be defined in advance. In many cases, this is easily possible, but in magnetic circuits with widely spread stray fields, i.e. in circuits which can be classified as open ones, predefined boundaries may give a distortion of the results. With the integral method, this requirement is avoided.

It begins with the computation of the field due to the conductors using the Biot-Savart law. The magnetization of the iron and its contribution to the field is computed by means of a relaxation method. A discretization has only to be made for the active parts, for the iron and for non-standard coil shapes. The resulting system matrix, contrary to the methods discussed before, is fully occupied. Although the matrix is considerably smaller, the computation times needed to solve more complicated problems may be longer than for the matrix representing the comparable differential approach.

This method may not always give satisfactory precision in its results and it sometimes exhibits problems of convergence. Nevertheless, it has experienced extensive application, for the simple reason that with GFUN a program package had been created in the Rutherford Laboratory which, for many years, was about the only one available permitting three-dimensional magnetic field computations [16,17]. Although it is limited in the number of iron elements, it has in particular been frequently used for the design of accelerator magnets.

The boundary integral method is related to the one just described [18,19,20]. Limiting oneself to constant permeability and after application of Green's theorem, the magnetization can be interpreted as a surface charge on the interfaces. An integral equation for surface charges is then obtained. In order to take into account variable permeability, a further surface charge must be introduced which is proportional to the gradient of magnetization.

A combination of finite elements with the boundary integral method is used in the so-called hybrid methods. The finite elements are applied to regions with variable permeability while for the surrounding air region, the boundary element method is taken. This is of interest for problems with unbounded air regions. Although the interfacing between inner and outer regions is not always straightforward, its practicability has been demonstrated for a number of cases [21,22,23]. The term hybrid solution, however, is sometimes used for other combined approaches [24].

METHODS FOR EDDY CURRENT PROBLEMS

Scanning the list of contributions to COMPUMAG Conferences for papers about eddy current computations, one realizes that almost all report extensions of methods applied to static fields. At Argonne, for example, the program EDDYNET makes use of the integral equation method in conjunction with the representation of conducting materials by means of a superposed wire grid [25,26]. PE2D from the Rutherford Laboratory is an example of a two-dimensional finite element program capable of handling eddy currents [27]. PROF1 from Darmstadt permits to solve eddy current applications by means of finite differences [6,28,29].

Important for the understanding of each algorithm is not only the numerical method which is used for the time varying field computations but also the equations and the imposed limitations, such as time function,

materials and geometries, which have been chosen when initially setting up these equations.

If the method starts off with the vector potential, the defining equations are

$$\mathbf{E} = \text{curl } \mathbf{A} \quad \text{and} \quad \mathbf{E} = -\text{grad } \phi - \frac{\partial \mathbf{A}}{\partial t}.$$

From this follows:

$$\text{curl} \left(\frac{1}{\mu} \text{curl } \mathbf{A} \right) + \sigma \frac{\partial \mathbf{A}}{\partial t} = -\sigma \text{grad } \phi.$$

Usually, $\text{div } \mathbf{A} = 0$ is imposed to make the solution unique [30].

For steady-state eddy current problems, i.e. those for which a harmonic hence sinusoidal time function of the currents can be taken and a linear behaviour of the material assumed, the time varying quantities can be replaced by complex functions:

$$\mathbf{H} = \mathbf{H}_0 e^{j\omega t} \quad \frac{\partial}{\partial t} = j\omega.$$

This gives time independent starting equations having complex coefficients [1]. The above methods can be applied by adapting them to this formulation.

In this context, it has recently been proposed to define a new vector potential

$$\mathbf{A}^* = \mathbf{A} + \int \text{grad } \phi \, dt$$

inside conducting regions. This leads to a simplified equation

$$\text{curl} \frac{1}{\mu} \text{curl } \mathbf{A}^* = -\sigma \frac{\partial \mathbf{A}^*}{\partial t}.$$

With an implied gauge of

$$\text{div } \sigma \mathbf{A}^* = 0,$$

this equation has a unique solution. Outside the conducting region, the magnetic scalar potential is used. The two regions are coupled on the interface by the continuity condition [31].

The T- Ω method has its origin in the defining equation

$$\mathbf{H} = \mathbf{T} - \text{grad } \Omega,$$

where $\mathbf{T}$ and Ω are the electric vector and magnetic scalar potentials, respectively. The curl of $\mathbf{H}$ equals that of $\mathbf{T}$, thus

$$\text{curl } \mathbf{H} = \text{curl } \mathbf{T} = \mathbf{j}.$$

$\mathbf{T}$ can be regarded as a variation of the $\mathbf{H}$ vector since it has the same curl but not the same divergence. $\mathbf{T}$ is the electric equivalent of the magnetic vector potential $\mathbf{A}$. With the corresponding Maxwell's equations, the system equation to be solved is:

$$\text{div} (\mu \text{grad } \Omega) = \text{div} (\mu \mathbf{T})$$

and

$$\text{curl} \left(\frac{1}{\sigma} \mathbf{T} \right) + \frac{\partial}{\partial t} (\mu \mathbf{T}) = \frac{\partial}{\partial t} (\mu \text{grad } \Omega).$$

For these, one applies the finite element method by establishing the energy functional in order to minimize it numerically. It consists of the stored energy, the eddy current losses and the Poynting vector, hence the energy transfer to the outside. The variables $\mathbf{H}$, $\mathbf{E}$ and $\mathbf{j}$ are replaced by $\mathbf{T}$ and Ω . At the material interfaces, $\mathbf{T}$ is discontinuous and leads to further terms in the energy functional, which is finally minimized with respect to the magnetic scalar potential [32].

The equation matrix obtained after discretization of the problem is asymmetric and complex. It may be solved by means of Gauss' method of elimination. For the general three-dimensional case, four unknowns in conducting materials, i.e. the magnetic scalar potential Ω and the three components of the electric vector potential $\mathbf{T}$, have to be handled. One profits, however, from the fact that outside conducting domains, $\mathbf{T}$ can be set to zero. In addition, any reduction in dimensions is well exploited, as for example in cases with laminated iron or when taking only one magnetic field component.

PROGRAMS

It is not intended to give here an exhaustive list of available programs for magnetic field calculations. This has been done by other authors in recent papers [33,34].

A few packages, which are known to the author, some of them by dint of practical experience, are enumerated here together with some fragmentary comments about their methods of solution and their application.

- MARE-C (R. Perin, S. van der Meer, S. Caeymaex - CERN)

This two-dimensional program relies on finite differences [35]. It is based on a program developed in 1962-63 by J.S. Hornsby for the solution of elliptical partial differential equations and applies SOR. MARE-C allows to obtain high-precision results, even for non-linear materials, and has been successfully applied, especially to accelerator and even superconducting magnets. Because of limitations in its concept, such as only one iron region and little possibility for exploiting symmetries, its application remained restricted to magnets designed for the handling of particle beams.

- MAGNET (Ch. Iselin - CERN)

This program is based on the same theoretical concept as MARE-C, but it includes numerous improvements. It is well proven and has been fairly extensively employed.

- POISSON (K. Halbach, R. Holsinger - LBL)

Being a very versatile two-dimensional program, it is perhaps the most widely distributed. It is based on a paper by A.M. Winslow (1967, TRIM) and makes use of finite elements [36]. Mesh generation, problem solving and interpretation of results are well organized. It permits almost any geometry and accepts materials with different magnetic properties. In its earlier version, the number of mesh points is limited to 6000. A newer, upgraded one (Iselin, Holsinger) avoids this restriction (40'000 points) [37].

- GFUN (C.W. Trowbridge et al., RAL)

Its predominance for several years, the method used, and its limitations, have already been mentioned. The data input, which usually in three dimensions is a tedious task, is organized by means of interactive graphics. Only after having successfully completed the creation of data, and fully verified it by visual inspection, can the program for the analysis be launched [16]. This technique also depends on the type of computer used. Batch versions of this program also exist.

- PROFI (W. Müller, TH Darmstadt)

Until recently, PROFI, which also solves the three-dimensional problem, applied exclusively finite differences and SOR, but now it also has an option for finite elements [38]. It has been written for the

development of electrical machinery. Problems which can be accommodated within the structure foreseen for geometry approximation can be computed to a high degree of precision. It has a field of application which extends beyond magnetic field computations, and is also capable of eddy-current computations.

- TOSCA (C.W. Trowbridge et al., RAL)

A three-dimensional program which can be regarded as the successor program of GFUN. Featuring powerful graphical interactive pre-processing for data input and post-processing for result interpretation, this program uses the method of finite elements [39]. It permits selection between 8- or 20-point space elements. For the numerical solution, the Galerkin method is applied. The magnetic scalar potential is used, split into a total one and a reduced one according to the presence of current carrying regions [40].

- PE2D (C.W. Trowbridge, J. Simkin - RAL)

PE2D is the two-dimensional version of TOSCA, and is organized in a similar fashion. It can be used to compute eddy currents and transient problems [27].

- BIM2D (C.W. Trowbridge et al. RAL)

In contrast to the above-mentioned programs, all of which accept variable permeability, this package is conceived for linear, i.e. constant μ , problems only. However, it is extremely fast and in many cases still gives good results. As its name indicates, it applies the boundary integral method. Since only the interfaces have to be discretized, its input is rather simple [41].

- FLUX (J.C. Sabonnadière et al., ENSIEG, Saint-Martin-d'Hères, France)

This two-dimensional program works for vector and scalar potentials and applies the finite element method. It solves the system of equations, classically by Newton-Raphson or Newton-Kantorovitch. Interactive pre- and postprocessing are available [42].

- MAGNET 11 (D.A. Lowther, P.P. Silvester, McGill University, Montreal)

In this two-dimensional package, vector potential and finite elements are used. For solving the system of equations, the incomplete Cholesky-conjugate gradient method is taken. The software has been written to run interactively on a dedicated minicomputer. Some effort has been made to implement even a three-dimensional version on a minicomputer. Here, the scalar potential is taken and the preconditioned conjugate gradient method is employed in conjunction with a frontal solution system limiting the data in the core memory to those belonging to the point actually treated and the nodes connected to it [43].

- AOS/MAGNETIC (A.O. Smith Corp., U.S.A.)

A two-dimensional finite element program commercialized mainly for the design of electrical machinery.

Many more programs exist or are under development. Work in this field is frequently done in university or other research institutes, inside the large electrotechnical companies [44,45,24] and by specialized software firms which commercialize packages they develop [19]. It is impossible to list all of them in the framework of this report. The mentioned programs should thus only be taken as illustrative examples.

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