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SUPERSYMMETRIC FUNCTIONAL INTEGRATION MEASURES^{*)}

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ABSTRACT

Supersymmetric theories can be directly characterized in terms of the associated bosonic functional integration measures where fermions have been integrated out. This method and its various recent developments are reviewed.

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1. INTRODUCTION

Supersymmetric theories (for reviews and further references, see e.g.[1]) are usually characterized by their invariance properties with respect to transformations that involve anticommuting parameters. The latter play an essential role in the formulation of supersymmetric theories and their use sometimes facilitates complex calculations, for example in perturbation theory. On the other hand, they are not as wieldy as ordinary numbers and many arguments based on their use remain formal. It is therefore of interest that there exists an alternative characterization of (a subclass of) supersymmetric theories directly in terms of the functional integration measure by means of which expectation values in a supersymmetric theory are defined [2] . This result, which enables us to reconstruct supersymmetric theories and to study their properties without recourse to anticommuting quantities, is essentially

a consequence of the fact that, in (flat space) supersymmetric theories, the vacuum energy vanishes if the groundstate is supersymmetric [3]. In the absence of interactions, this just means that fermionic and bosonic degrees of freedom cancel (fermions being counted as negative). For interacting supersymmetric theories, the vanishing of the vacuum energy density is expressed by

$$\langle 0 | H(\lambda_1, \dots, \lambda_n) | 0 \rangle = 0 \quad (1.1)$$

for arbitrary interaction parameters $\lambda_1, \dots, \lambda_n$. By virtue of the well known relation between the vacuum energy and the (Euclidean) vacuum functional, (1.1) implies that

$$Z(\lambda_1, \dots, \lambda_n) = \int d\phi \exp[-S(\lambda_i, \phi)] = \text{const.} \quad (1.2)$$

as a function of the parameters $\lambda_1, \dots, \lambda_n$ (we will see later on that $Z(\lambda)$ may exhibit jump discontinuities on certain "critical surfaces" in parameter space). For an interacting theory, the vacuum functional (1.2) is given by a non-trivial integral and the fact that it is piecewise constant as a function of $\lambda_1, \dots, \lambda_n$ puts stringent constraints on the form of the interaction in (1.2). One may therefore expect that supersymmetry can be alternatively characterized as a property of the associated functional integration measure.

To set up the necessary conventions and notation, let us assume that the supersymmetric model under consideration is based on one (or several) supermultiplet(s) consisting of one (or several) bosonic field(s) $A(x)$ together with their fermionic partner(s) $\psi(x)$ which are usually Majorana spinors. In sections 2 and 3 of these lectures, it will furthermore be assumed that the maximum spin is 1/2; supersymmetric gauge theories are treated in section 4. The number of dimensions is left unspecified for the moment. More significantly, all models to be discussed will be used in their "on-shell" version: it appears to be one of the advantages of the present approach that it works independently of the existence or non-existence of an "off-shell" formulation. A central input is the fact that the fermions can be "integrated out" [4]. To do so, one decomposes the action into a bosonic and a fermionic part, i.e.

$$S(\lambda, \phi) = S_B(\lambda, A) + \frac{1}{2} \bar{\psi} M(\lambda, A) \psi \quad (1.3)$$

with the fermionic part

$$\begin{aligned} & \frac{1}{2} \bar{\psi} M(\lambda, A) \psi \equiv \\ & \equiv \frac{1}{2} \int d^4x d^4y \bar{\psi}(x) M(x, y; \lambda, A) \psi(y) \end{aligned} \quad (1.4)$$

Since the expression (1.4) is quadratic in the fermions, its exponential can be integrated in closed form with the result

$$\int d\psi \exp \left[-\frac{1}{2} \bar{\psi} M(\lambda, A) \psi \right] = \left\{ \det M(\lambda, A) \right\}_{(1.5)}^{1/2} = \\ \equiv \left\{ \det M(\lambda = 0, A) \right\}^{1/2} \mathcal{D}(\lambda, A)$$

for Majorana fermions (which obey $\bar{\psi}_\alpha = \psi_\beta \mathcal{C}_{\beta\alpha}$). $\mathcal{D}(\lambda, A)$ is the Matthews Salam determinant [4]. The restriction to actions, which are quadratic in the fermions is mainly for practical reasons, although it remains to be explored which of the results presented in these lectures would survive if this restriction were dropped. In the same vein as (1.5), the fermionic propagator in a scalar background may be obtained by explicit integration. It is given by the formula (I omit Dirac and possible internal indices)

$$\underbrace{\psi(x) \bar{\psi}(y)} \equiv M^{-1}(x, y; \lambda, A) = \\ = (\det M)^{-1/2} \int d\psi \psi(x) \bar{\psi}(y) \exp \left[-\frac{1}{2} \bar{\psi} M \psi \right] \quad (1.6)$$

Analogous formulas hold for all n-point fermionic expectation values ("Wick's theorem"). Eliminating the fermionic variables by means of (1.5) or (1.6), one can transform the functional integration measure (1.2) into a purely bosonic functional integration measure

$$d\mu(\lambda, A) \equiv \exp \left[-S_B(\lambda, A) \right] \det M(\lambda, A)^{1/2} dA \quad (1.7)$$

The central result to be elaborated in these lectures is summarized in the following

Theorem [2]: Supersymmetric theories are characterized by the existence of a (generally non-linear and non-local) transformation T of the bosonic fields

$$T_\lambda : A(x) \rightarrow A'(x, \lambda, A) \quad (1.8)$$

with the following properties

- (i) T_λ is invertible in the sense of formal power series at least.
- (ii) $S(\lambda; A) = S_0(A'(\lambda, A))$ where S is the full bosonic part of the action and S_0 its quadratic part (including mass terms)
- (iii) The Jacobi determinant of the transformation T equals the Matthews-Salam determinant, i.e.

$$\det \frac{\delta A'(x, \lambda, A)}{\delta A(y)} = D(\lambda, A) \quad (1.9)$$

For supersymmetric gauge theories, the presence of a gauge fixing term which violates supersymmetry requires some modifications of these statements which will be extensively discussed in section 4.

As a prelude, it is instructive to see how the theorem works for supersymmetric quantum mechanics ([5], see also [6]) where the transformation (1.8) can be easily constructed in closed form [7,8]. This example may serve as the prototype for the more complicated models to be discussed later since it displays all the relevant features. The relevant variables in a Lagrangian formalism are $q(t) \in \mathbb{R}$ and two anticommuting variables $\psi(t)$ and $\bar{\psi}(t)$. The (Euclidean) Lagrangian reads

$$\mathcal{L} = \frac{1}{2} \dot{q}^2 + \frac{1}{2} V(q)^2 + \bar{\psi} \left(\frac{d}{dt} + V'(q) \right) \psi \quad (1.10)$$

The Matthews-Salam determinant for this model is

$$\begin{aligned} M &= \det \left[\left(\frac{d}{dt} + V'(q(t)) \right) \delta(t - t') \right] = \\ &= D(V(q)) \det \frac{d}{dt} \delta(t - t') \end{aligned} \quad (1.11)$$

The variable substitution which makes the bosonic part of the action Gaussian and whose Jacobian equals D , is given by

$$(\tau q)(t) = q(t) \pm \int \theta(t - t') V(q(t')) dt' \quad (1.12)$$

where $\theta(t - t')$ is the "free fermionic propagator", i.e.

$$\frac{d}{dt} \theta(t - t') = \delta(t - t') \quad (1.13)$$

Note that (1.12) is usually presented in the equivalent form [7,8]

$$\begin{aligned} \xi(t) &= \frac{d}{dt} (\tau q)(t) = \\ &= \dot{q}(t) \pm V(q(t)) \end{aligned} \quad (1.14)$$

The sign ambiguity in (1.12) and (1.14) arises from the fact that the only term which is sensitive to it is the topological invariant

$$\int_{-\infty}^{+\infty} \dot{q} V(q(t)) dt = W(\infty) - W(-\infty) \quad (1.15)$$

where $W(q)$, $V(q) = W'(q)$, is the "superpotential"; in fact, (1.15) is just the instanton number of the configuration $q(t)$ [8]. (1.14) is a stochastic differential equation to which well-developed techniques may be applied [9] (the connection between supersymmetry and stochastic differential equations was first pointed out in [10]). However, the mapping (1.8) is non-local in general, and therefore the stochastic structure is lost unless one admits the possibility of "stochastic integro-differential equations". Nonetheless, there are some models for which (1.8) becomes local just as for (1.14), namely the $N=2$ Wess-Zumino model in two dimensions [7,10] and the $N=1$ Yang-Mills theory in four dimensions, see refs. [11-14] and section 4 of these lectures. Such theories are sometimes called "stochastic".

In these lectures, I will not be overly concerned with questions of mathematical rigour although many of the results described can probably be made more rigorous (e.g. for supersymmetric quantum mechanics, see [15]), and the existence of the mapping (1.8) may even facilitate a more rigorous investigation of supersymmetric theories (for instance, one may possibly employ (1.8) to regularize supersymmetric theories [16]). A Euclidean signature will be used throughout these lectures. Although this is a common practice, one here has to deal with Euclidean Majorana spinors. These are sometimes professed not to exist, but in fact do, if one renounces the usual hermiticity properties [17]. This causes no harm since in Euclidean field theory, the relevant requirement is Osterwalder Schrader positivity [18] not hermiticity. For the construction of the transformation (1.8), I will mainly rely on the recently developed method of ref. [19], which is much more transparent than the original one of [2]. Moreover, as a new result, a simple proof of (1.9) is given; this closes a gap that had occurred in the original proof [20]. An alternative method is based on the use of "stochastic identities" [12]. The idea is to determine the combinations of bosonic fields that satisfy free field vacuum expectation values by a careful analysis of the supersymmetry transformation rules. This approach is not discussed in detail here since it has been reviewed in a recent paper [14]. Finally, although it is somewhat contrary in spirit to the approach expounded in these lectures, it should be noted that the theorem has a superspace analog: the Jacobian for the change of variables in an integral over unconstrained superfields is always equal to one [21].

2. PROOF OF MAIN THEOREM

For the sake of definiteness and notational convenience, only the N=1 Wess-Zumino model in two dimensions will be considered in this section as the extension of the proof given below to other cases without additional gauge invariances is immediate. In two space-time dimensions, the minimal supermultiplet contains a scalar A and a two component Majorana spinor ψ_α [22]. The most general Lagrangian reads

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (m A + \lambda p(A))^2 + \\ & + \frac{1}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{1}{2} (m + \lambda p'(A)) \bar{\psi} \psi \end{aligned} \quad (2.1)$$

where auxiliary fields have been eliminated and the interaction part has been split off. Thus, $p(A)$ is an arbitrary polynomial of degree ≥ 2 . (2.1) transforms as a total divergence under the supersymmetry transformations

$$\delta \psi_\beta = \varepsilon_\alpha \delta_\alpha \psi_\beta, \quad \delta A = \varepsilon_\alpha \delta_\alpha A \quad (2.2)$$

where

$$\begin{aligned} \delta_\alpha A &= e_{\alpha\beta} \psi_\beta = -\bar{\psi}_\alpha \\ \delta_\alpha \psi_\beta &= \gamma^\mu_{\beta\alpha} \partial_\mu A - \delta_{\alpha\beta} (m A + \lambda p(A)) \end{aligned} \quad (2.3)$$

with the two dimensional charge conjugation matrix $e_{\alpha\beta}$.

The transformation (1.8) can be constructed iteratively by exploiting [19] the fact that the derivative of $\mathcal{L}$ with respect to any of its coupling parameters can be written as a supervariation [3]. The relevant result in our case reads

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\delta_\alpha \hat{\Delta}_\alpha \quad \text{with} \quad \hat{\Delta}_\alpha \equiv \frac{1}{2} \psi_\alpha p(A) \quad (2.4)$$

We also need the integrated operator insertion

$$\Delta_\alpha \equiv \int \hat{\Delta}_\alpha(x) d^2x \quad (2.5)$$

The validity of (2.4) may be verified directly from (2.1) and (2.3). The vanishing of the vacuum energy is a consequence of

$$\frac{\partial}{\partial \lambda} \langle 1 \rangle = \langle \delta_\alpha \Delta_\alpha \rangle \quad (2.6)$$

and the supersymmetry Ward identity which states that

$$\langle \delta_\alpha Y \rangle = 0 \quad (2.7)$$

for an arbitrary string of operators Y . If X is some (not necessarily local) product of bosonic operators, one gets in a similar manner

$$\begin{aligned} \frac{\partial}{\partial \lambda} \langle X \rangle &= \left\langle \frac{\partial X}{\partial \lambda} \right\rangle + \langle X \delta_\alpha \Delta_\alpha \rangle = \\ &= \left\langle \frac{\partial X}{\partial \lambda} \right\rangle - \langle \delta_\alpha X \cdot \Delta_\alpha \rangle \end{aligned} \quad (2.8)$$

where the Ward identity (2.7) has been used to "integrate by parts". The expectation values in (2.6), (2.7), and (2.8) signify integration over both fermions and bosons. Since the second term on the right side of (2.8) contains two fermion operators, we can make use of (1.6) to integrate out the fermions before integrating over bosons. This motivates the definition of the "R-operator" [19,13]

$$RX \equiv \frac{\partial X}{\partial \lambda} - \delta_\alpha \underbrace{X \cdot \Delta_\alpha} \quad (2.9)$$

which is now a purely bosonic expression and defines a nonlocal and nonlinear mapping of the functional $X[A]$ onto another functional $RX[A]$ (it is, in some sense, the generator of the transformation to be constructed). The fermion propagator occurring in (2.9) is defined by

$$\left[\not{\partial} + m + \lambda p'(A(x)) \right]_{\alpha\gamma} \underbrace{\psi_\gamma(x) \bar{\psi}_\beta(y)} = \delta_{\alpha\beta} \delta(x-y) \quad (2.10)$$

or, equivalently,

$$\begin{aligned} \underbrace{\psi_\alpha(x) \bar{\psi}_\gamma(y)} \left[-\overleftarrow{\not{\partial}} + m + \lambda p'(A(y)) \right]_{\gamma\beta} &= \\ &= \delta_{\alpha\beta} \delta(x-y) \end{aligned} \quad (2.11)$$

Both terms in (2.9) act like derivatives, i.e.

$$R(X_1, X_2) = RX_1 \cdot X_2 + X_1, RX_2, \text{ etc.} \quad (2.12)$$

Since the fermionic integration has already been performed in (2.9), the expectation value can now be represented as a purely bosonic expectation value. One then takes higher order derivatives of

$\langle X \rangle$ with respect to λ . For the iteration of this procedure, it is crucial that the expectation value (2.9) can be computed by "undoing" the fermionic integration for the Matthews-Salam determinant but not for the fermionic propagator in (2.9) before applying the R operator again. Thus,

$$\begin{aligned} \langle X \rangle &= \langle X \rangle_{\lambda=0} + \sum_{s=1}^{\infty} \frac{\lambda^s}{s!} \frac{\partial^s}{\partial \lambda^s} \langle X \rangle \big|_{\lambda=0} = \\ &= \int d\mu(\lambda=0, A) \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} R^s X \big|_{\lambda=0} \end{aligned} \quad (2.13)$$

The bosonic measure is defined as in (1.7) with $S_B(\lambda, A)$ from (2.1). From (2.12) and (2.13), it is obvious that

$$\begin{aligned} \langle \prod_n A(x_n) \rangle &= \int d\mu(\lambda, A) \prod_n A(x_n) = \\ &= \int d\mu(\lambda=0, A') \prod_n A(x_n, \lambda, A') \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} A(x, \lambda, A') &= (T_{\lambda}^{-1} A')(x) \equiv \\ &\equiv \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} (R^s A')(x) \big|_{\lambda=0} \end{aligned} \quad (2.15)$$

Consequently, the prescription (2.15) enables one to rewrite the vacuum expectation values of the interacting theory as vacuum expectation values of transformed fields in a free theory. As (2.14) is true for arbitrary monomials of the field operators, it follows that

$$d\mu(\lambda, A) = d\mu(\lambda=0, A'(\lambda, A)) \quad (2.16)$$

To prove parts (ii) and (iii) of the main theorem, it is sufficient to show that

$$S_B(\lambda, A) = S_B(\lambda = 0, A'(\lambda, A)) \quad (2.17)$$

since the equality of determinants (part (iii)) then follows by (2.16). Equation (2.17) means that the bosonic action is a fixed point of the transformation (2.15) and follows if one can show that

$$R S_B(\lambda, A) = 0 \quad (2.18)$$

Writing out (2.18) explicitly, one gets two terms according to (2.8), namely

$$\begin{aligned} - \underbrace{\delta_\alpha S_B \cdot \Delta_\alpha} &= + \underbrace{\Delta_\alpha \delta_\alpha S_B} = \\ &= \frac{1}{2} \int dx p(A) \underbrace{\psi_\alpha \cdot \int dy \bar{\psi}_\alpha \square A} - \\ &\quad - \frac{1}{2} \int dx p(A) \underbrace{\psi_\alpha \cdot \int dy \bar{\psi}_\alpha} \cdot [m A^2 + \\ &\quad + m \lambda (p(A) + A p'(A)) + \lambda^2 p(A) p'(A)] \end{aligned} \quad (2.19)$$

and

$$\begin{aligned} \frac{\partial S_B}{\partial \lambda} &= \int dx (m A p(A) + \lambda p^2(A)) = \\ &= \frac{1}{2} \delta_{\alpha\alpha} \int dx dy p(A(x)) \delta(x-y) [m A(y) + \lambda p(A(y))] = \\ &= \frac{1}{2} \int dx p(A) \underbrace{\psi_\alpha \cdot \int dy \bar{\psi}_\beta} \cdot \left[-\overleftrightarrow{\partial} + m + \right. \\ &\quad \left. + \lambda p'(A) \right]_{\beta\alpha} (m A + \lambda p(A)) \end{aligned} \quad (2.20)$$

where (2.11) has been used. Clearly, the terms without derivative in (2.20) cancel the second term in (2.19). To take care of the remainder, one integrates the derivative term in (2.20) by parts, using (2.10). This yields

$$\begin{aligned}
 & \frac{1}{2} \int dx \, p(A) \gamma_{\alpha\beta}^{\mu} \psi_{\beta} \underbrace{\int dy \, \bar{\psi}_{\alpha} \partial_{\mu} A}_{\text{}} (m + \lambda p'(A)) = \\
 & = \frac{1}{2} \int dx dy \, p(A(x)) \gamma_{\alpha\beta}^{\mu} \partial_{\mu} A(y) \left[\delta_{\alpha\beta} \delta(x-y) + \right. \\
 & \quad \left. + \underbrace{\psi_{\beta}(x) \partial^{\nu} \bar{\psi}_{\gamma}(y) \gamma_{\gamma\alpha}^{\nu}}_{\text{}} \right] = \quad (2.21) \\
 & = - \frac{1}{2} \int dx \, p(A) \underbrace{\psi_{\alpha}}_{\text{}} \cdot \int dy \, \bar{\psi}_{\alpha} \square A
 \end{aligned}$$

Adding (2.21) and the uncanceled term in (2.19), we arrive at the desired result (2.18).

As an example, we take $p(A) = A^3$. Then, it is not difficult to show that

$$\begin{aligned}
 (RA)(x) |_{\lambda=0} &= -m \int C(x-y) A^3(y) dy \\
 (R^2 A)(x) |_{\lambda=0} &= 6m^2 \int C(x-y) A^2(y) C(y-z) A^3(z) dy dz \\
 &+ 3 \int \partial_{\mu} C(x-y) A^2(y) \partial_{\mu} C(y-z) A^3(z) dy dz \quad (2.22)
 \end{aligned}$$

with the free field propagator

$$C(x) = \int \frac{e^{ikx}}{k^2 + m^2} \frac{d^2k}{(2\pi)^2} \quad (2.23)$$

Substituting (2.22) into (2.15) and inverting, one obtains

$$\begin{aligned} (T_\lambda A)(x) &= A'(x, \lambda, A) = \\ &= A(x) + m\lambda \int C(x-y) A^3(y) dy - \\ &\quad - \frac{3}{2} \lambda^2 \int \partial_\mu C(x-y) A^2(y) \partial_\mu C(y-z) A^3(z) dy dz \\ &\quad + O(\lambda^3) \end{aligned} \quad (2.24)$$

which coincides with the result given in [2], eq.(27).

Remarkably, the expansion (2.15) admits a graphical representation [19]. For instance, (2.24) corresponds to

$$\begin{aligned} T_\lambda A &= \text{wavy line} + \lambda \text{ (wavy line) } \cdot \text{ (wavy line) } \\ &\quad + \lambda^2 \text{ (wavy line) } \cdot \text{ (wavy line) } \cdot \text{ (wavy line) } + O(\lambda^3) \end{aligned} \quad (2.25)$$

with self-explanatory notation: wavy lines correspond to scalar fields and straight lines represent free spinor propagators which are contracted over their spinor indices. The action of R is schematically represented by two "insertions"

$$R \left(\text{wavy line} \right) = \text{wavy line} \cdot \text{wavy line} \quad (2.26)$$

and

which are the graphical analogues of the two terms in the definition of R , see (2.9). A most important consequence of (2.26) is that only tree graphs occur in the expansion (2.15). Since the number of tree graphs with n vertices is bounded by $n! K^n$, where K is proportional to the number of wavy lines running into a vertex, it follows that the expansion (2.15) has a non-vanishing radius of convergence. This can be intuitively understood as the "combinatorial divergences" of ordinary perturbation theory (i.e. uncanceled factors of $n!$) are produced by functional integration, i.e. Wick's theorem, and the expansion (2.15) is done before integrating over the boson fields. Of course, the perturbation expansion (2.15) with subsequent integration over the boson fields as in (2.14) is entirely equivalent to the usual perturbation expansion. The problem of ultraviolet divergences appears in a somewhat different guise. In expressions such as (2.24), one has to deal with products of fields at the same space-time point. Since the latter are distributions (or possibly even more singular mathematical entities) such products are ill-defined unless a regularization is introduced, e.g. by smearing the fields with C^∞ -functions of compact support. The existence of the renormalized theory is then equivalent to the existence of a suitably "renormalized" version of (2.15) as a non-linear and non-local map of a distribution space into itself. Sometimes a finite volume cutoff is required which is effected by switching on the interaction only in a finite region $\Lambda \subset \mathbb{R}^d$. Since supersymmetry is then broken by surface terms all statements have to be modified. It is not difficult to show that the main theorem remains valid in this case up to terms which have support on the boundary $\partial\Lambda$ of the region Λ (for a discussion of these matters in the context of the Wess-Zumino model, see [23]).

Furthermore, the divergences which can appear can be read off from the expansion (2.25); performing the bosonic integration is equivalent to connecting wavy lines in all possible ways and summing over all these diagrams*. As an exercise the reader may verify that the Wess-Zumino model in four dimensions is only logarithmically divergent. Finally, "stochastic" models are naturally recovered in the present framework: for these, the expansion (2.14) turns out to be a geometric series whose inverse is local in the sense of e.g. eq. (1.14) [24]. Stochastic variables are nothing but local fixed points of (2.15).

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*This approach to perturbation theory was first suggested in [7] and has been termed "infradiagrammar".

3. GLOBAL ASPECTS

The piecewise constancy of the vacuum functional (1.2) in a supersymmetric theory is evidently a consequence of the equivalence of the functional integration measure (1.7) to that of a free theory. However, this equivalence is valid locally in the space of bosonic field configurations, and, when integrating over this space, global properties will become significant. Substituting formula (2.16) into (1.2), one must take into account how many times the image space is covered when $A(x)$ assumes all possible values. Hence,

$$\begin{aligned} Z(\lambda) &= \int d\mu(\lambda, A) = \int d\mu(\lambda=0, A'(\lambda, A)) = \\ &= \text{winding number of } T_\lambda \end{aligned} \quad (3.1)$$

This shows that $Z(\lambda)$ is a topological invariant but of a different type from the one usually encountered in field theories since it characterizes mappings of infinite-dimensional spaces. It was first recognized by the authors of [7], that (3.1) can be identified with the Witten index [25],

$$Z(\lambda) = n_B(E=0) - n_F(E=0) \quad (3.2)$$

Since winding numbers in infinite dimensional spaces are rather awkward to deal with, it is desirable to have a simplified prescription which allows one to compute (3.1) efficiently. It was argued in [26] that it is sufficient to determine the winding number in the ultralocal limit where kinetic terms are neglected*. This amounts to introducing a factor Z in front of all kinetic terms and a factor of Z^{-1} in front of the scalar field potential and then taking the limit $Z \rightarrow 0$. E.g., for the Wess-Zumino model in four dimensions [27], the transformation T_λ reads

$$\begin{aligned} A'(x, \lambda, A) &= A(x) + \\ &+ m\lambda \int C(x-y) A^2(y) dy + O(\lambda^2) \end{aligned} \quad (3.3)$$

*I.e., (3.1) is equal to an ordinary ("zero-dimensional") integral such as (3.10) below, or, equivalently, the winding number is continuous in the wave function renormalization parameter Z at $Z=0$. To be sure, a rigorous justification for this recipe remains to be given, and the reader should keep in mind the somewhat heuristic character of the arguments in this chapter.

with a complex field $\mathcal{A} = A + iB$ and the free (now $\mathbf{z}$ -dependent) propagator

$$C(x) = \int \frac{e^{ikx}}{\mathbf{z}^2 k^2 + m} \frac{d^4 k}{(2\pi)^4} \quad (3.4)$$

Making use of $\lim_{\mathbf{z} \rightarrow 0} C(x) = m^{-2} \delta(x)$ and the fact that all higher order terms in (3.3) contain at least one factor of $\mathbf{z}$ and therefore disappear in the limit $\mathbf{z} \rightarrow 0$, one gets

$$\lim_{\mathbf{z} \rightarrow 0} \mathcal{A}'(x, \lambda, \mathcal{A}) = \mathcal{A}(x) + \frac{\lambda}{m} \mathcal{A}^2(x) \quad (3.5)$$

Since $\mathcal{A}$ is complex, the winding number of (3.5) is two. Assuming continuity of the winding number at $\mathbf{z} = 0$, one arrives at the same conclusion for the full nonlocal transformation (3.3). Evidently, the mapping (3.5), although being locally invertible almost everywhere, is no longer globally invertible. This result is in agreement with the existence of two supersymmetric ground states in the Wess-Zumino model and the fact that the Witten index equals two in this case.

The above considerations are also relevant when supersymmetry is spontaneously broken. In this case, the vacuum energy no longer vanishes but (2.4) remains true, and therefore the construction (2.15) still makes sense. As an example, one may consider the O'Raifeartaigh model which exhibits spontaneous breakdown of supersymmetry [28]. By the previous arguments it is sufficient to study this model in the ultralocal limit. Moreover, I introduce an artificial parameter μ such that supersymmetry is restored for $\mu \neq 0$ whereas the original model is recovered by putting $\mu = 0$. The model contains three complex scalar fields $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$; the relevant quantity is the superpotential

$$W(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = f \mathcal{A}_1 + m \mathcal{A}_2 \mathcal{A}_3 + g \mathcal{A}_1 \mathcal{A}_3^2 + \mu \mathcal{A}_1^2 \quad (3.6)$$

(I assume f, m and g to be non-zero), from which the ordinary potential can be derived in the usual fashion [1]. It is equal to the sum $\sum_{i=1,2,3} |\mathcal{A}'_i|^2$ where

$$\begin{aligned}
\mathcal{A}'_1 &= \frac{\partial W}{\partial \mathcal{A}_1} = f + 2\mu \mathcal{A}_1 + g \mathcal{A}_3^2 \\
\mathcal{A}'_2 &= \frac{\partial W}{\partial \mathcal{A}_2} = m \mathcal{A}_3 \\
\mathcal{A}'_3 &= \frac{\partial W}{\partial \mathcal{A}_3} = m \mathcal{A}_2 + 2g \mathcal{A}_1 \mathcal{A}_3
\end{aligned} \tag{3.7}$$

Supersymmetry is spontaneously broken if there is no value of $\mathcal{A}_i^{(0)}$ for which $\mathcal{A}'_i(\mathcal{A}^{(0)}) = 0$. For $\mu \neq 0$, one sees that $\mathcal{A}'_i = 0$ if

$$\mathcal{A}_i^{(0)} = (-f/2\mu, 0, 0) \tag{3.8}$$

so the ground state is supersymmetric for any $\mu \neq 0$. For $\mu = 0$, however, the minimum (3.8) disappears and supersymmetry is spontaneously broken. To construct the transformation T, we first take $\mu \neq 0$, apply the procedure of the foregoing section and take the limit $\mu \rightarrow 0$ only afterwards. Again it can be shown that, in the ultralocal limit, higher order terms do not contribute and the transformation is reduced to the local mapping (3.7). It is quite instructive to exhibit the discontinuity at $\mu = 0$ explicitly by deriving the "zero-dimensional" vacuum functional from (3.7). In the ultralocal limit, the fermionic determinant is found to be (after a little algebra)

$$\det M(\mathcal{A})^{1/2} = \left| \det \frac{\partial \mathcal{A}'_i}{\partial \mathcal{A}_j} \right|^2 = 4m^4 \mu^2 \tag{3.9}$$

The "zero-dimensional" vacuum functional is therefore

$$\mathbb{Z}(f, \mu, m, g) = \int d\mathcal{A} d\bar{\mathcal{A}} \prod_{i=1}^3 \delta(\mathcal{A}'_i(\mathcal{A})) \left| \det \frac{\partial \mathcal{A}'_i}{\partial \mathcal{A}_j} \right|^2 \tag{3.10}$$

(the δ -function comes from the factor $\mathbb{Z}^{-1}$ in front of the potential in the limit $\mathbb{Z} \rightarrow 0$) and, by a well-known formula, one

obtains

$$\begin{aligned} \mathbb{Z}(f, \mu, m, g) &= \int_{\mathbb{C}^3} d\mathcal{A}' d\bar{\mathcal{A}}' \sum_{\mathcal{A}'(\mathcal{A})=0} \prod_i \delta(\mathcal{A}'_i) = \\ &= \begin{cases} 1 & \text{for } \mu \neq 0 \\ 0 & \text{for } \mu = 0 \end{cases} \end{aligned} \quad (3.11)$$

Note that (3.11) stays well defined for $\mu=0$ although the determinant (3.9) vanishes there. Clearly, the derivative of $\mathbb{Z}$ with respect to any of its parameters vanishes, so $\mathbb{Z}$ is indeed piecewise constant, but discontinuous on the "critical surface" $\mu=0$. (for $f, m, g \neq 0$). The analysis of winding numbers leads to the same conclusion. For $\mu \neq 0$, the mapping (3.7) is one-to-one as one may easily verify, whereas, for $\mu=0$, (3.7) maps $\mathbb{C}^3$ into a lower-dimensional submanifold of $\mathbb{C}^3$; since the image of $\mathbb{C}^3$ has measure zero in this case, the winding number equals zero for $\mu=0$ in agreement with (3.11).

Quite generally, one might expect the cases where supersymmetry breaking occurs to be in one-to-one correspondence with the cases where the image space of the transformation (1.8) has lower dimension (although this "façon de parler" begs for some refinement if the relevant spaces are infinite dimensional), but this is not quite true. I have so far glossed over the fact that it is not the Jacobian but rather its modulus that appears in the textbook formula for the change of variables in an integral. In many cases of interest, one can actually prove that the Matthews Salam determinant is positive for arbitrary values of the scalar (or spin-1) fields*. If it is not, then another kind of global obstruction may arise in doing the integral (3.1). To have an example in this case as well, let us have a look at the ultralocal limit of supersymmetric quantum mechanics (1.10) [7,8]. The analogue of (3.10)

*E.g. (3.9) is manifestly positive as it is for all Wess-Zumino type models in the ultralocal limit.

now reads

$$Z = \int_{-\infty}^{+\infty} \delta(V(q)) V'(q) dq \quad (3.12)$$

To evaluate (3.12), one must distinguish between two possibilities. If V runs from $-\infty$ to $+\infty$, the number of points q with $V(q) = 0$ is odd, and, for simplicity, I assume that $V(q)$ is monotonous. (3.12) may thus be replaced by*

$$Z = \pm \int_{-\infty}^{+\infty} \delta(V(q)) |V'(q)| dq = \pm 1 \quad (3.13)$$

In this case supersymmetry breaking is impossible. If, on the other hand, $V(q)$ covers a subset of $\mathbb{R}$ only, say the interval $[\xi_0, \infty]$, then

$$\begin{aligned} Z &= \int_{-\infty}^{+\infty} \delta(V(q)) \operatorname{sgn} V'(q) |V'(q)| dq = \\ &= \int_{\xi_0}^{\infty} \delta(\xi) [1 - 1] d\xi = 0 \end{aligned} \quad (3.14)$$

and supersymmetry breaking is possible. (3.13) and (3.14) correspond to convergence or non-convergence of the integral

$\int_{-\infty}^{+\infty} \exp[\pm W(q)] dq$ where $W(q)$ is the superpotential; this is just the criterion given in [25]. As the foregoing discussion shows, Z may vanish although the image of (1.8) is not lower dimensional. This phenomenon is, of course, due to the change in sign of the determinant, but I do not know whether a similar effect can occur for more interesting models in higher dimensions.

4. SUPERSYMMETRIC GAUGE THEORIES

Supersymmetric gauge theories [29] are more interesting but also more difficult to deal with in the present framework than the models considered so far. The main source of difficulties is, of

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*The occurrence of a negative winding number is evidently related to the orientation of the real axis and somewhat artificial. It is, however, not relevant for this argument.

course, the additional gauge symmetry which necessitates a gauge fixing procedure [30] that either explicitly violates supersymmetry [31] or, through additional ghost supermultiplets, renders the theory considerably more complicated [32]. Nonetheless, the vacuum energy still vanishes for supersymmetric ground states, and one therefore expects some version of the main theorem to remain true. Before stating the revised version of the theorem, however, some more notation must be introduced.

For the sake of clarity, only the pure $N=1$ Yang Mills theory in four dimensions will be considered in the following. For the same reason, I will adopt a non-supersymmetric gauge fixing procedure. The model is based on a supermultiplet containing one spin-1 field $A_\mu^a(x)$ and a Majorana spinor $\lambda^a(x)$, both in the adjoint representation of some gauge group. As before, auxiliary fields are assumed to have been eliminated since the question of off-shell versus on-shell representations is not relevant in the present context. In Euclidean signature, the invariant Lagrangian is

$$\mathcal{L}_0 = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} \bar{\lambda}^a \gamma^\mu D_\mu \lambda^a \quad (4.1)$$

with

$$\begin{aligned} F_{\mu\nu}^a &\equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \\ D_\mu \lambda^a &\equiv \partial_\mu \lambda^a + g f^{abc} A_\mu^b \lambda^c \end{aligned} \quad (4.2)$$

(4.1) is invariant up to a total derivative under the supersymmetry transformations

$$\delta A_\mu^a = \varepsilon_\alpha \delta_\alpha A_\mu^a, \quad \delta \lambda_\beta^a = \varepsilon_\alpha \delta_\alpha \lambda_\beta^a \quad (4.3)$$

where

$$\begin{aligned} \delta_\alpha A_\mu^a &= -(\bar{\lambda}^a \gamma_\mu)_\alpha \\ \delta_\alpha \lambda_\beta^a &= (\sigma^{\mu\nu})_{\beta\alpha} F_{\mu\nu}^a \end{aligned} \quad (4.4)$$

The Matthews Salam determinant $D(g, A_\mu^a)$ is defined from (4.1) as in (1.5)

$$\det M(g, A)^{1/2} = \det M(0, A)^{1/2} D(g, A) \quad (4.5)$$

where

$$\begin{aligned} M_{\alpha\beta}^{ac}(x, y; g, A) &= \\ &= [\delta^{ac} \gamma_{\alpha\beta}^\mu \partial_\mu + g f^{abc} \gamma_{\alpha\beta}^\mu A_\mu^b(x)] \delta(x-y) \end{aligned} \quad (4.6)$$

The superfluous gauge degrees of freedom in (4.1) are eliminated by adding a gauge fixing term $\frac{1}{2\alpha} G^a[A_\mu]^2$ to the Lagrangian (4.1). The gauge fixing function may be taken as, for example,

$$G^a[A_\mu] = \partial^\mu A_\mu^a \quad (4.7)$$

Other choices such as the axial gauge are possible, but in any case, I will impose the requirement

$$\frac{\partial}{\partial g} G^a[A_\mu] = 0 \quad (4.8)$$

The explicit breaking of gauge invariance must be compensated for by weighting the functional integration measure with the Fadeev-Popov determinant [30]

$$\Delta_{F.P.}(g, A_\mu) = \det \frac{\delta G^a[A_\mu]}{\delta A_\mu^b} D_\mu \delta(x-y) \quad (4.9)$$

(In the following I will not explicitly distinguish between $\Delta_{F.P.}(g, A)$ and $\Delta_{F.P.}(g, A) \cdot \Delta_{F.P.}^{-1}(0, A)$). Rewriting (4.9) by means of the Fadeev-Popov ghosts $C^a, \bar{C}^a$ in the usual way, the gauge fixing procedure results in the additional contribution to the Lagrangian (4.1)

$$\mathcal{L}_1 = \frac{1}{2\alpha} G^a[A_\mu]^2 + \bar{C}^a \frac{\delta G^a}{\delta A_\mu^b} D_\mu C^b \quad (4.10)$$

The combined Lagrangian $\mathcal{L}_0 + \mathcal{L}_1$ is then invariant under the BRS-transformations [33]

$$\begin{aligned} s A_\mu^a &= D_\mu C^a, \quad s \lambda^a = -g f^{abc} C^b \lambda^c \\ s C^a &= -\frac{g}{2} f^{abc} C^b C^c, \quad s \bar{C}^a = -\frac{1}{\alpha} G^a[A_\mu] \end{aligned} \quad (4.11)$$

The full bosonic integration measure of the theory is obtained by integrating out the fermionic variables, i.e. the Majorana spinors and the ghosts; it is

$$d\mu(g, A_\mu) = \exp \left[- \int dx \left(\frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2\alpha} G^a[A_\mu]^2 \right) \right] \quad (4.12)$$

$$D(g, A_\mu) \Delta_{F.P.}(g, A) dA$$

The revised version of the theorem can then be stated as follows:

Theorem: Supersymmetric gauge theories are characterized by the existence of a transformation T_g of the Yang-Mills fields

$$T_g : A_\mu^a(x) \rightarrow A_\mu'^a(x, g, A_\mu) \quad (4.13)$$

which is invertible at least in the sense of formal power series such that

- (i) The Yang-Mills action without gauge-fixing term is mapped into a bosonic (i.e. abelian) action;
- (ii) The Jacobi determinant of T_g is equal to the product of the Matthews-Salam and Fadeev-Popov determinants, i.e.

$$\det \frac{\delta A_\mu'^a(x, g, A)}{\delta A_\nu^b(y)} = \Delta_{F.P.}(g, A) D(g, A) \quad (4.14)$$

at least order by order in perturbation theory;

- (iii) The gauge fixing function $G^a[A_\mu]$ is a fixed point of T_g .

For the proof, we again follow the prescription of ref. [19] *). One first notes that the supersymmetric Ward identity is no longer valid in the form (2.7) but rather, owing to the non-invariance of (4.10) under (4.4),

$$\langle \delta_\alpha Y \rangle = \langle \delta_\alpha S_1 \cdot Y \rangle \quad (4.15)$$

where $S_1 = \int \mathcal{L}_1 dx$ and the brackets stand for integration over all fields including fermions and ghosts. After some rearrangement, (4.15) can be written in the form

$$\langle \delta_\alpha Y \rangle = - \langle \int dx \bar{C}^a \delta_\alpha G^a[A_\mu] s(Y) \rangle \quad (4.16)$$

As in section 2, one is interested in constructing a suitable "R-operator". This can be derived by acting with the derivative on the expectation value of an arbitrary string of bosonic fields. There are two terms from the exponential of the full action, namely

$$\begin{aligned} \frac{\partial}{\partial g} S_0 &= \frac{1}{2} f^{abc} \int dx (F_{\mu\nu}^a A_\mu^b A_\nu^c + \bar{\lambda}^a \gamma^\mu A_\mu^b \lambda^c) = \\ &= \delta_\alpha \Delta_\alpha - \frac{1}{4} f^{abc} \int dx \bar{\lambda}^a \gamma^\mu A_\mu^b \lambda^c \end{aligned} \quad (4.17)$$

with the "operator insertion"

$$\Delta_\alpha \equiv - \frac{1}{4} f^{abc} \int dx (\sigma^{\mu\nu} \lambda^a)_\alpha A_\mu^b A_\nu^c \quad (4.18)$$

and

$$\frac{\partial}{\partial g} S_1 = \int dx \bar{C}^a \frac{\delta G^a}{\delta A_\mu^b} f^{bcd} A_\mu^c C^d \quad (4.19)$$

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*I am indebted to O. Lechtenfeld for informing me about the results of [24], on which this derivation up to (4.20) is based, prior to publication.

where (4.8) has been taken into account. After a somewhat lengthy calculation, the correct form of the R-operator is found to be [13, 19, 24]

$$RX = \frac{\partial X}{\partial g} + \underbrace{\delta_\alpha X \cdot \Delta_\alpha} + \int dx \underbrace{\bar{C}^a \delta_\alpha G^a[A_\mu] \Delta_\alpha S(X)} \quad (4.20)$$

In complete analogy with (2.9), this is again a purely bosonic expression. The fermion and ghost propagators appearing in (4.20) are given by, respectively,

$$\underbrace{(\not{D}\lambda)_\alpha^a(x)} \bar{\lambda}_\beta^b(y) = \delta_{\alpha\beta} \delta^{ab} \delta(x-y) \quad (4.21)$$

$$\underbrace{(\bar{\lambda} \overleftarrow{\not{D}})_\alpha^a(x)} \lambda_\beta^b(y) = \delta_{\alpha\beta} \delta^{ab} \delta(x-y)$$

and

$$\frac{\delta G^a}{\delta A_\mu^b} D_\mu \underbrace{C^b(x)} \bar{C}^c(y) = \delta^{ac} \delta(x-y) \quad (4.22)$$

$$D_\mu \left(\frac{\delta G^a}{\delta A_\mu^b} \underbrace{\bar{C}^b(x)} \right) C^c(y) = -\delta^{ac} \delta(x-y)$$

The crucial difference between (2.9) and (4.20) is the third term on the right side of (4.20) that disappears only for gauge invariant operators X which obey $S(X) = 0$.

The next step is to apply (4.20) to prove parts (i) and (iii) of the revised theorem; (ii) then follows as before. Let us begin with $X_0 = \frac{1}{4} \int F_{\mu\nu}^a F_{\mu\nu}^a$; the relevant expressions are

$$\begin{aligned} \underbrace{\delta_\alpha X_0 \cdot \Delta_\alpha} &= \\ &= \frac{1}{4} \int dx F_{\mu\nu}^a \underbrace{(D_\mu \bar{\lambda}^a \gamma_\nu)_\alpha f^{bcd} \int dy (\sigma^{\rho\sigma} \lambda^b)_\alpha A_\rho^c A_\sigma^d} \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} \frac{\partial}{\partial g} X_0 &= \frac{1}{2} f^{abc} \int dx F_{\mu\nu}^a A_\mu^b A_\nu^c = \\ &= \frac{1}{2} \delta^{ab} \delta^{\rho\sigma} f^{bcd} \int dx dy F_{\mu\nu}^a(x) \delta(x-y) A_\rho^c(y) A_\sigma^d(y) \end{aligned} \quad (4.24)$$

This can be reexpressed by writing (4.21) in the form

$$D_\tau \underbrace{\bar{\lambda}^a(x) \gamma^\tau \sigma^{\mu\nu} \sigma_{\rho\sigma} \lambda^b(y)} = -2 \delta^{ab} \delta^{\rho\sigma} \delta(x-y) \quad (4.25)$$

and using the identity

$$\gamma^\tau \sigma^{\mu\nu} = \delta^{\tau[\mu} \gamma^{\nu]} + \gamma^{[\tau} \sigma^{\mu\nu]} \quad (4.26)$$

In this way, (4.24) becomes, after partial integration,

$$\begin{aligned} \frac{\partial}{\partial g} X_0 &= -\frac{1}{4} \int dx F_{\mu\nu}^a \underbrace{(D_\mu \bar{\lambda}^a \gamma_\nu)_\alpha f^{bcd} \int dy (\sigma^{\rho\sigma} \lambda^b)_\alpha A_\rho^c A_\sigma^d} \\ &+ \frac{1}{8} f^{bcd} \int dx D_{[\tau} F_{\mu\nu]}^a \underbrace{\bar{\lambda}^a \gamma^\tau \sigma^{\mu\nu} \int dy \sigma^{\rho\sigma} \lambda^b A_\rho^c A_\sigma^d} \end{aligned} \quad (4.27)$$

The first term in (4.27) cancels (4.23) while the second term vanishes by the Bianchi identity. Finally, there is no contribution from the third term in (4.20) as X_0 is a gauge invariant operator. Altogether, it has been shown that

$$R X_0 = R \left(\frac{1}{4} \int F_{\mu\nu}^a F_{\mu\nu}^a dx \right) = 0 \quad (4.28)$$

which proves part (i) of the theorem. Part (iii) can be shown in a similar fashion; this time the first term in (4.20) does not contribute because of (4.8). The second term gives

$$\begin{aligned} \underbrace{\delta_\alpha G^a[A_\mu(x)] \Delta_\alpha} &= \\ &= \frac{1}{4} \frac{\delta G^a}{\delta A_\mu^b} \underbrace{(\bar{\lambda}^b(x) \gamma_\mu)_\alpha f^{cde} \int dy (\sigma^{\nu\rho} \lambda^c)_\alpha A_\nu^d A_\rho^e} \end{aligned} \quad (4.29)$$

while the third one gives

$$\begin{aligned}
 & \int dy \underbrace{\bar{C}^b \delta_\alpha G^b \Delta_\alpha}_{} s(G^a[A_\mu(x)]) = \\
 & = \frac{1}{4} \int dy \underbrace{\bar{C}^f \frac{\delta G^f}{\delta A_\nu^g} (\bar{\lambda}^g \gamma_\nu)_\alpha}_{} f^{cde} \cdot \\
 & \quad \cdot \underbrace{\int dz (\sigma^{\rho\sigma} \lambda^c)_\alpha A_\rho^d A_\sigma^e \cdot \frac{\delta G^a}{\delta A_\mu^b} D_\mu C^b(x)}_{(4.30)} \\
 & = -\frac{1}{4} \frac{\delta G^a}{\delta A_\mu^b} \underbrace{(\bar{\lambda}^b(x) \gamma_\mu)_\alpha f^{cde} \int dz (\sigma^{\rho\sigma} \lambda^c)_\alpha A_\rho^d A_\sigma^e}_{(4.30)}
 \end{aligned}$$

where, in the last line, the definition (4.22) of the ghost propagator has been used. Combining (4.29) and (4.30), we get

$$R G^a[A_\mu(x)] = 0 \quad (4.31)$$

which is part (iii) of the theorem. Note that (4.31) is a local statement whereas (4.28) expresses the fixed point property of an integrated quantity (in the second case, the integral was needed for the partial integration in (4.27)). The importance of (4.31) resides in the fact that it allows us to choose the gauge fixing parameter α in (4.10) arbitrarily and to take the limit $\alpha \rightarrow 0$; all results remain valid in the singular gauge where the functional integration measure acquires a factor $\prod \delta(G^a[A_\mu(x)])$. Thus, the gauge fixing function $G^a[A_\mu]$ may be considered as a stochastic variable for arbitrary gauges.

It is possible to obtain another stochastic variable by slightly modifying the prescription (4.20) [13]. To do so, one defines

$$\begin{aligned}
 R_\pm X & \equiv \frac{\partial X}{\partial g} + 2 \delta_\alpha X \cdot \Delta_\alpha^\pm + \\
 & \quad + 2 \int dx \underbrace{\bar{C}^a \delta_\alpha G^a[A_\mu] \Delta_\alpha^\pm}_{} s(X)
 \end{aligned} \quad (4.32)$$

with the chirally projected insertions

$$\Delta_{\alpha}^{\pm} \equiv \frac{1}{2} (1 \pm \gamma^5)_{\alpha\beta} \Delta_{\beta} \quad (4.33)$$

Let us apply (4.32) to $F_{\mu\nu}^a(\pm) \equiv \frac{1}{2} F_{\mu\nu}^a \pm \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}^a$ and fix the sign for definiteness. As in (4.23-27), the first two terms in (4.32) give

$$\begin{aligned} & 2 \delta_{\alpha} \underbrace{F_{\mu\nu}^a}^{+} \cdot \Delta_{\alpha}^{-} + \frac{\partial}{\partial g} F_{\mu\nu}^a = \\ &= \frac{1}{4} \left(D_{[\mu} \bar{\lambda}^a_{\nu]} + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} D_{\alpha} \bar{\lambda}_{\beta}^a \right) f^{bcd} \int dy \, \sigma^{\rho\sigma} \lambda_{-}^b A_{\rho}^c A_{\sigma}^d \\ & - \frac{1}{8} D_{\tau} \bar{\lambda}^a \left(\gamma^{\tau\mu\nu} + \frac{1}{2} \gamma^{\tau\alpha\beta} \epsilon_{\alpha\beta\mu\nu} \right) f^{bcd} \int dy \, \sigma^{\rho\sigma} \lambda_{-}^b A_{\rho}^c A_{\sigma}^d \end{aligned} \quad (4.34)$$

where, of course, $\lambda_{-} \equiv \frac{1}{2}(1 - \gamma^5)\lambda$. Now, in contrast to (4.27), the last term in (4.34) obviously cannot be eliminated by partial integration. Instead, one must use the identity

$$\gamma^{\tau\mu\nu} + \frac{1}{2} \gamma^{\tau\alpha\beta} \epsilon_{\alpha\beta\mu\nu} = -2 \left(\delta_{\mu\nu}^{\tau\lambda} + \frac{1}{2} \epsilon_{\mu\nu\tau\lambda} \right) \gamma^{\lambda} \gamma^5 \quad (4.35)$$

together with $\gamma^5 \lambda_{-} = -\lambda_{-}$, to see that (4.34) vanishes (note the factors of 2 in (4.32)). The third term in (4.32) produces a gauge rotation on $F_{\mu\nu}^a$ which vanishes on the hypersurface

$G[A_{\mu}] = 0$ [13]. Altogether, this implies that the (anti)self-dual field strengths

$$F_i^a(\pm) \equiv F_{0i}^a \pm \frac{1}{2} \epsilon_{ijk} F_{jk}^a, \quad i, j, k = 1, 2, 3 \quad (4.36)$$

constitute fixed points of R_{-} and R_{+} , respectively.

It was already pointed out in [11] that the stochastic variables (4.36) may be employed to "take the square-root of the Yang-Mills action" because

$$\frac{1}{4} \int F_{\mu\nu}^a F_{\mu\nu}^a dx = \frac{1}{2} \int F_i^a F_i^a dx \pm Q(A) \quad (4.37)$$

with the topological charge $Q(A)$. (The analogy of (4.37) with (1.10 - 15) is quite striking). It has furthermore been shown by exploiting "stochastic identities" that either $F_i^{a(+)}$ or $F_i^{a(-)}$ satisfy free field vacuum expectation values and therefore are indeed the Gaussian variables we have been looking for [12]. In fact, the equality (4.14) may be demonstrated directly in Minkowski space by cleverly choosing a light-cone gauge [12,14]; however, the re-

levant "Jacobian" there is complex and not obviously interpretable as an ordinary Jacobian. There is another interesting aspect of (4.37) and the present approach in general [11]. If one regards the bosonic action $S_B(\lambda, A)$ or (4.37) as a "Morse function" [34] on the manifold of bosonic field configurations it follows from the main theorem that, among all possible Morse transformations, there exists one and only one whose Jacobian equals a Matthews-Salam determinant (or (4.14)) if the bosonic action can be "supersymmetrized". Ordinarily, in Morse theory, such Morse functions are used to classify the topological properties of certain (finite-dimensional) manifolds [34]. In contradistinction to the case of scalar field theories considered in section two, for which the relevant "manifolds" are topologically trivial, this aspect may deserve some study for gauge theories. There, the relevant manifold $\mathcal{A}/\mathcal{G}$, i.e. the space of gauge-equivalent gauge field configurations, does exhibit non-trivial topology, and (4.36) may be regarded as the Morse transformation which renders the "Morse-function" (4.37) Gaussian.

To conclude this section, I give the explicit form of the transformation to second order in the gauge (4.7). It is [2]

$$\begin{aligned}
 A_\mu^{'a}(x, g, A) = & A_\mu^a(x) + g f^{abc} \int dy \partial_\nu C(x-y) A_\mu^b(y) A_\nu^c(y) + \\
 & + \frac{1}{2} g^2 f^{abc} f^{bde} \int dy dz \partial_\rho C(x-y) A_\lambda^c(y) \cdot \\
 & \cdot \left[2 \partial_{[\rho} C(y-z) A_{\mu]}^d(z) A_\lambda^e(z) + \right. \\
 & \left. + \partial_\lambda C(y-z) A_\rho^d(z) A_\mu^e(z) \right] + O(g^3)
 \end{aligned} \tag{4.38}$$

from which the reader may verify the theorem directly up to second order. In particular, it is easily seen that

$$\partial^\mu A_\mu^{'a}(x) = \partial^\mu A_\mu^a(x) + O(g^3) \tag{4.39}$$

From the structure of the perturbative construction, of course nothing can be said about possible non-perturbative aspects, and it would be interesting to see what might happen in presence of anomalies of the type encountered in [35] or other anomalies. Nevertheless, the comments made at the end of section two are also relevant here.

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