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MONTE CARLO RENORMALIZATION GROUP METHODS AND RESULTS IN QCD^{*)}

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A B S T R A C T

The way quantum field theory is defined and working beyond perturbation theory - this is the main topic of these lectures. The first part is a general introduction to renormalization group ideas, while the specific methods and results are discussed in the second part.

- I.
 - 1. Introduction
 - 2. Renormalization group
 - 3. "Good" and "bad" regularizations, gauge theory on the lattice
 - 4. The Gaussian fixed point in a Yang-Mills theory
 - 5. Renormalizability
 - 6. The β -function and $\Delta\beta(\beta)$
- II.
 - 1. Monte Carlo renormalization group methods; optimization
 - 2. Results

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- 2. Renormalization group
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I.1. INTRODUCTION

The typical distance over which events are correlated in strong interaction physics is ~ 1 fermi - a very small distance on human scales. On the other hand, if this physics is described by a local field theory, 1 fermi should be considered as a very large distance.

In field theory, the variables are associated with space-time points. These variables have some kind of self-interaction, while the a priori ("elementary") interaction between different degrees of freedom is over infinitesimal distances as expressed by derivatives. It requires a highly non-trivial collective behaviour to produce correlations over finite distances starting from elementary interactions which are defined over infinitesimal distances.

Really, the interaction between two variables associated with different points comes from derivatives and has typically the form: $K\phi(x)\phi(x+\delta)$, where δ is infinitesimal. (Here and in the following, we consider field theories defined in Euclidean space via path integrals.) The a priori elementary interaction extends over infinitesimal distances and for a generic value of K (and of the other parameters, couplings of the model) the correlation length of the full theory remains infinitesimal also. In general, it requires a careful tuning of the parameters to get a non-trivial, collective behaviour, where events are correlated over finite distances in spite of the fact that the interaction is local. The difficulty of field theories lies in creating and studying this collective behaviour.

In Gaussian field theories, where the action is quadratic in the fields and describes free particles, the procedure is trivial. In interacting theories in perturbation theory, the problem shows up already, although its real nature is hidden somewhat behind the formal machinery of perturbative renormalization theory. As is well known from these perturbative studies, for a generic (i.e., not carefully chosen) value of the starting parameters (called bare couplings and masses) the calculated masses are proportional to the cut-off, therefore the correlations remain on the infinitesimal level of the a priori interactions. No collective behaviour occurs unless these parameters are tuned carefully. Unfortunately, in perturbation theory, this problem occurs mainly as cancelling divergences in a Feynman graph expansion, creating the atmosphere of sinful manipulations.

Defining a quantum field theory is very similar to the study of large-scale collective behaviour in statistical systems, and the tuning of parameters to achieve this collective behaviour is no more sinful than, say, tuning the temperature towards the Curie point in a ferromagnetic system.

Actually there is a deep similarity between the problems of QFT (defined via path integrals in Euclidean space) and those of critical phenomena¹. For illustration, consider a large piece of magnet. The magnet is built of atoms forming a lattice structure with a lattice distance $a \sim O(1\text{\AA})$. At a given generic temperature $T > T_c$, the distance over which the magnetic spins are correlated (correlation length) is of the order of the lattice distance (Fig. 1). Let us tune the temperature towards the critical temperature where spontaneous magnetization sets in. In approaching T_c , the correlation length is getting larger and larger, several hundreds of lattice units (hundreds of $\text{\AA}$'s), and beyond. A field theorist looks at this problem through his special glasses. These are reducing glasses. Through his glasses, the original lattice unit and correlation length is $O(1\text{ fm})$. This is a trivial rescaling. The most important thing is that the field theorist is

increasing the reducing factor of his glasses in such a way that through his glasses, the correlation length stays of the order of 1 fermi even when the temperature is getting close to the Curie point (Fig. 1). For the solid state physicist, the correlation length approaches macroscopical distances, while looking through the increasingly reducing glasses of the field theorist, the correlation length remains the same, but the lattice becomes increasingly fine grained. At the Curie point, the field theorist obtains a continuum field theory ($a \rightarrow 0$) with a finite $O(1 \text{ fm})$ correlation length. His colleague has a lattice with fixed lattice unit ($a \sim 1\text{\AA}$) and macroscopical correlation length. It is clear, however, that the main difficulty - obtaining and controlling the large scale collective behaviour - is a common problem for them.

As this example shows, the definition of a quantum field theory involves some kind of limiting procedure, the need for which is closely related to the a priori local interactions expressed by derivatives. Just as in the classical theory, the definition of a derivative requires the temporary introduction of a finite increment (of the argument of the function considered) which disappears at the end by some limiting procedure, the very definition of a field theory requires the temporary introduction of a defining framework called regularization, which disappears from the theory via a limiting procedure.

The close analogy between QFT and statistical physics suggests a natural regularization: the regularization by discretizing the spacetime points, by introducing a lattice. This is the framework we shall use in the following. In this framework, the general problem of defining a QFT is often referred as the continuum limit of lattice field theories.

The most powerful theoretical tool for the study of this problem is called the renormalization group (RG)^{2,3-9}. RG ideas govern almost everything we know about the continuum limit of lattice field theories. The renormalization group methods are powerful technical tools also as I will try to illustrate by discussing recent results in QCD.

A question is raised naturally: if these methods are beautiful and powerful, what delayed their application in $d = 4$ gauge theories? The answer is, in short: these techniques are difficult to implement. Let me quote a few sentences written by K. Wilson almost ten years ago⁶:

"It is difficult to formulate the RG approach for a particular problem and additionally, one is likely to have to carry out a complicated computer calculation, which makes most theoretical physicists cringe. Especially in the case of strong interactions of elementary particles, most theorists hope to solve the problem

without turning to RG methods and computers. It will probably require several years of stagnation before theorists will accept the inevitability of (these techniques)."

I am sure that these techniques will be increasingly with us in the near future. Not only in the study of QCD, but also (or, even the more so) in the study of non-asymptotically free theories.

I.2. RENORMALIZATION GROUP

As we discussed before, looking at a critical system through the highly reducing glasses of the field theorist, anything which spreads over finite distances (~physics) will involve a very large number (infinite in the limit) of lattice units. Continuum physics is recovered in the long-distance behaviour of the lattice problem approaching a phase transition point.

Close to the phase transition point, a large number of variables are coupled in an essential way. It is very difficult to control and solve this collective behaviour. One might try to reduce the number of essentially coupled degrees of freedom systematically, step by step, by thinning out the variables in configuration or momentum space¹⁰ in such a way that the long-distance properties remain unchanged. This transformation is called renormalization group (RG) transformation.

RG transformations in a gauge theory are complicated by the requirement of gauge invariance. In order to avoid this problem at the start, consider a scalar model. The basic variables are $\phi(n)$, $n = (n_1, \dots, n_d)$ and the local action S is expressed in terms of these variables. The partition function is given by:

$$Z = \prod_n \int d\phi(n) e^{-S(\phi)} \quad (1)$$

Perform a RG transformation by averaging out the short-distance components of the model. Let us work in configuration space¹¹ and define blocks of size 2^d as illustrated in Fig. 2. We define a block field variable $\phi'(n')$ which is associated with the cell n' . $\phi'(n')$ is some kind of average of the original 2^d field variables in the block. A possibility is a simple averaging:

$$\phi'(n') = b \sum_{n \in \text{block } n'} \phi(n) \quad (2)$$

where b is a constant. There are many other possibilities. Keeping the block variables $\phi'(n')$ fixed, we integrate over the original variables $\phi(n)$:

$$\prod_n \int d\phi(n) \prod_{\text{blocks } n'} f(\phi'(n'); \phi(n), n \in \text{block } n') e^{-S(\phi)} = e^{-S'(\phi')} \quad (3)$$

Here the function f describes the way the block variable $\phi'(n')$ is constructed from the variables $\phi(n)$, $n \in \text{block } n'$. In the above example [Eq. (2)], $f = \delta(\phi'(n') - b \sum_{n \in \text{block } n'} \phi(n))$. The new action $S'(\phi')$ represents a step towards our goal: we got rid of part of the short-distance variables, S' is defined on a lattice $a' = 2a$.

Let us make a few remarks here.

- i) The block size (=2 above) is largely arbitrary (belongs to an infinite set of discrete numbers). If the RG transformation is performed in Fourier space (by integrating over momenta $\Lambda^{\text{cut-off}} > k > \Lambda^{\text{cut-off}}/\lambda$) the scale factor is continuous and can be even infinitesimally close to 1.
- ii) The precise way the new variable is constructed from the old ones (as specified by the function f above) is largely arbitrary. There are a few constraints, however:

Normalization condition:

$$\int d\phi'(n') f(\phi'(n'); \phi(n), n \in \text{block } n') = 1 \quad (4)$$

which gives $\ln Z = \ln Z'$, the free energy and thermodynamics remain unchanged.

Symmetry requirements

If we want to preserve the symmetries of S , f should conform with these symmetries. Let us assume that the measure and the action are invariant under some symmetry transformation R : $S(R\phi) = S(\phi)$. If the block transformation satisfies the condition

$$f(R\phi'(n'); R\phi(n)) = f(\phi'(n'); \phi(n)) \quad (5)$$

then $S'(R\phi') = S'(\phi')$ holds also, as easily seen by changing integration variables in Eq. (3). Additionally, for the RG transformation to be useful, f should conform with the intuitive goal of a RG transformation. For instance, $f = \delta(\phi'(n') - \text{const.})$ is clearly not a useful transformation.

- iii) The long-distance properties, like the dimensionful correlation length, are unchanged by the RG transformation. Therefore the dimensionless correlation length ξ which is measured in the actual lattice unit is decreased by a factor of 2:

$$\xi' = \frac{1}{2} \xi \quad (6)$$

- iv) In general, $S'(\phi')$ will contain all kinds of different interactions even if $S(\phi)$ had a simple form.

As we want to repeat the RG transformation, the remark in iv) suggests to start with a general action

$$-S(\phi) = \sum_{\alpha} K_{\alpha} \theta_{\alpha}(\phi) \quad (7)$$

where θ_{α} ($\alpha = 1, 2, \dots$) are different interactions conforming with the symmetries of the lattice theory. In Eq. (7) and in the following, dimensionless couplings and fields are used. This can be arranged by absorbing their engineering dimension by appropriate powers of "a". The interactions θ_{α} might include higher derivatives and powers of the field ϕ , like $\sum_{\mu} \Delta_{\mu} \Delta_{\nu} \phi \Delta_{\mu} \Delta_{\nu} \phi, \dots, \phi^6, \phi^8, \dots, \sum_{\mu} \Delta_{\mu} \Delta_{\mu} \phi \Delta_{\mu} \Delta_{\mu} \phi, \dots$, where $\Delta_{\mu} \phi(n) = \phi(n+\hat{\mu}) - \phi(n)$. The set $\{K_{\alpha}\}$ forms a multidimensional coupling constant space. The RG transformation moves the system in this space:

$$\{K_{\alpha}^{(1)}\} \xrightarrow{RG} \{K_{\alpha}^{(2)}\} \xrightarrow{RG} \{K_{\alpha}^{(3)}\} \rightarrow \dots \quad (8)$$

An interesting possibility is that this sequence has a fixed point:

$$\{K_{\alpha}^*\} \xrightarrow{RG} \{K_{\alpha}^*\} \quad (9)$$

For the fixed point action $S^*(\phi) = \sum_{\alpha} K_{\alpha}^* \theta_{\alpha}(\phi)$ the consistency of Eqs. (6) and (9) requires the correlation length ξ being either zero or infinite. We are interested in those fixed points where $\xi = \infty$. At this point, the correlations extend over infinitely many lattice units. This is the collective behaviour we are looking for, where a statistical physicist studies critical phenomena, while the field theorist (through his reducing glasses) attempts to define a continuum field theory.

Critical behaviour is a result of fluctuations on all scales from the cut-off to small momenta. As the transformation $\{K_{\alpha}^{(n)}\} \rightarrow \{K_{\alpha}^{(n+1)}\}$ involves only one scale, it is expected that the critical singularities come from the repeated application of the RG transformation and not from the functional relation $K_{\alpha}^{(n+1)}(K_{\beta}^{(n)})$ itself. Let us expand this relation around the fixed point:

$$K_{\alpha}^{(n+1)}(K_{\beta}^{(n)}) = K_{\alpha}^{(n+1)}(K_{\beta}^*) + (K_{\beta}^{(n)} - K_{\beta}^*) \frac{\partial K_{\alpha}^{(n+1)}}{\partial K_{\beta}^{(n)}} \Big|_{K^*} + \dots \quad (10)$$

By keeping the leading terms only and using the relation $K_{\alpha}^{(n+1)}(K_{\beta}^*) = K_{\alpha}^*$ one obtains a linearized equation for the difference $\Delta K_{\alpha} = K_{\alpha} - K_{\alpha}^*$:

$$\Delta K_{\alpha}^{(n+1)} = T_{\alpha\beta} \Delta K_{\beta}^{(n)} \quad (11)$$

Let us denote the eigenvectors and eigenvalues of the matrix T by h^a and λ^a respectively ($a = 1, 2, \dots$). Denote the corresponding interactions by $h^a(\phi)$: $h^a(\phi) = \sum_{\alpha} h_{\alpha}^a \theta_{\alpha}(\phi)$. In the neighbourhood of

the fixed point interaction, we can write¹²:

$$S(\phi) = S^*(\phi) + \sum_a c^a h^a(\phi) \xrightarrow{RG} S^*(\phi) + \sum_a \lambda^a c^a h^a(\phi) \rightarrow \dots \xrightarrow{n^{th} \text{ step}} S^*(\phi) + \sum_a (\lambda^a)^n c^a h^a(\phi) \quad (12)$$

Those interactions which have $\lambda^a < 1$ are suppressed after a few RG steps. These are called irrelevant interactions. On the other hand, the relevant interactions ($\lambda^a > 1$) are growing, while the fate of the marginal interactions ($\lambda^a = 1$) can be decided only by considering corrections to the linearized RG equations.

As an example, consider two couplings only and assume that one of the eigenoperators is relevant, the other one is irrelevant. The flow lines (along which a given action moves under repeated RG transformations) are illustrated in Fig. 3. The line starting along the 1 axes (the relevant direction) is called renormalized trajectory (RT), while the line starting along 2 is the line of criticality (in general there is a surface of criticality spanned by the irrelevant variables). The points of the critical surface end up in the critical point under the repeated application of the RG transformation. In every point of this surface, $\xi = 0$, the system is critical. Starting from a point outside the critical surface the flow line moves away from the surface approaching the RT after a few RG steps.

In general, there might be n relevant operators at a given fixed point denoted by $FP^{(n)}$. In this case it requires the tuning of n parameters to get to the critical surface¹³.

Example 1

The Curie point of ferromagnetic systems is expected to be described by an $FP^{(1)}$ (in the absence of symmetry breaking operators). Really, these systems can be driven to criticality by tuning one parameter (the temperature) only. To be more explicit, consider a spin model where the spins S_n can take the value ± 1 . Consider the general action

$$\text{action} = K_1 (\text{nearest-neighbour int.}) + K_2 (\text{next-to-nearest neighbour int.}) + K_3 (\text{diagonal int.}) + \dots \quad (13)$$

It is expected that a given RG transformation will have a fixed point $FP^{(1)}$ in the multiparameter space $(K_1, K_2, \dots)$ (Fig. 4). The critical surface (where $\xi = \infty$) is $N-1$ dimensional if N couplings are considered. In the neighbourhood of the critical point $(K_1^*, K_2^*, \dots)$ it is enough to tune one parameter (say K_1) keeping the other parameters fixed to get to the critical surface. We might consider, for instance $K_2 = K_3 = \dots = 0$ (Ising model). The criti-

cal surface is crossed by this line at $K_1 = K_1^C$, where K_1^C is the usual critical point of the Ising model.

Example 2^{3,4,7,14}

Let us consider an example now in some detail. This example is very simple but reflects many of the important properties of the RG transformations. Consider a Gaussian scalar theory, i.e., an action which is quadratic in the field:

$$S = \frac{1}{2} \int_0^{\Lambda^{\text{cut}}} d^d k \tilde{\phi}(k) \tilde{\phi}(-k) \tilde{g}(k) \quad (14)$$

where

$$\tilde{g}(k) = \tilde{r}_0 + \tilde{\alpha} \sum_{\mu} k_{\mu} k_{\mu} + \tilde{\beta} \left(\sum_{\mu} k_{\mu} k_{\mu} \right)^2 + \tilde{\gamma} \sum_{\mu} k_{\mu} k_{\mu} k_{\mu} k_{\mu} + \dots, \quad d k \equiv \frac{d k_1 \dots d k_d}{(2\pi)^d} \quad (15)$$

In $\tilde{g}(k)$ we included non-rotational invariant (but cubic invariant) terms also in anticipating their unavoidable presence on a hyper-cubic lattice. By introducing the dimensionless variables

$$\begin{aligned} \phi(q) &= (\Lambda^{\text{cut}})^{\frac{d+2}{2}} \tilde{\phi}(k) \\ g(q) &= (\Lambda^{\text{cut}})^{-2} \tilde{g}(k) \\ q_{\mu} &= (\Lambda^{\text{cut}})^{-1} k_{\mu} \end{aligned} \quad (16)$$

one obtains

$$S = \frac{1}{2} \int_0^1 d^d q \phi(q) \phi(-q) g(q) \quad (17)$$

where $\rho(q)$ is expressed in terms of the dimensionless parameters $r_0 = (\Lambda^{\text{cut}})^{-2} \tilde{r}_0$, $\alpha = \tilde{\alpha}$, $\beta = (\Lambda^{\text{cut}})^2 \tilde{\beta}$, ... By changing integration variables in the functional integral $\phi(q) \rightarrow \sqrt{\alpha} \phi(q)$, the coefficient of the $\sum_{\mu} q_{\mu} q_{\mu}$ term can be put to 1 and we get:

$$g(q) = r_0 + \sum_{\mu} q_{\mu} q_{\mu} + \beta \left(\sum_{\mu} q_{\mu} q_{\mu} \right)^2 + \gamma \sum_{\mu} q_{\mu} q_{\mu} q_{\mu} q_{\mu} + \dots \quad (18)$$

Consider the RG transformation, where in one step we integrate over the variables $\phi(q)$, $\frac{1}{2} < q < 1$. In a quadratic action like that in Eq. (17), the high and low momentum variables are not coupled and one obtains:

$$\int_{0 < q < 1} D[\phi(q)] e^{-S(\phi)} = (\text{const.}) \int_{0 < q < 1/2} D[\phi(q)] e^{-\frac{1}{2} \int_0^{1/2} d^d q \phi(q) \phi(-q) g(q)} \quad (19)$$

As we want to repeat the transformation, the labelling of the variables should be the same as before. Introduce therefore

$$\begin{aligned} q'_\mu &= 2 q_\mu \\ \phi'(2q) &= \zeta \phi(q) \end{aligned} \quad (20)$$

where ζ is a constant, to be fixed later. It is analogous to the constant b in Eq. (2). After this change of variables, the effect of the RG transformation can be described as

$$S(\phi) \xrightarrow{RG} S'(\phi') = 2^{-d} \int \frac{1}{2} \int_0^1 d\bar{q}' \phi'(\bar{q}') \phi'(-\bar{q}') \varrho(\bar{q}'/2), \quad (21)$$

or

$$\varrho(q) \xrightarrow{RG} 2^{-d} \int \varrho(q/2) = 2^{-d} \int \left(r_0 + \frac{1}{4} \sum_\mu q_\mu q_\mu + \frac{1}{16} \beta \left(\sum_\mu q_\mu q_\mu \right)^2 + \dots \right) \quad (22)$$

By choosing $\zeta = 2^{-(d+2/2)}$ the transformed ρ will have the standard form of Eq. (18) again and we obtain the following change of the parameters under the RG transformation¹⁵:

$$\begin{aligned} r_0 &\xrightarrow{RG} 4 r_0 \xrightarrow{RG} 4^2 r_0 \rightarrow \dots \\ \beta &\xrightarrow{RG} \frac{1}{4} \beta \xrightarrow{RG} \left(\frac{1}{4}\right)^2 \beta \rightarrow \dots \\ \gamma &\xrightarrow{RG} \frac{1}{4} \gamma \xrightarrow{RG} \left(\frac{1}{4}\right)^2 \gamma \rightarrow \dots \end{aligned} \quad (23)$$

We have a fixed point at $r_0 = 0$, $\beta = 0$, $\gamma = 0, \dots$. The fixed point action has the form

$$S^*(\phi) = \frac{1}{2} \int d\bar{q} \phi(\bar{q}) \phi(-\bar{q}) \bar{q}^2, \quad (24)$$

which is the usual, continuum action of a massless scalar field. As Eq. (23) shows, for this fixed point there is one relevant operator (ϕ^2) having an eigenvalue $\lambda = 4$. All the other operators are irrelevant (Fig. 5).

Let us collect a few important points.

- i) The basic steps of a RG transformation are:
 - thinning out the variables by some procedure;
 - relabelling the new variables in such a way that the new indices run over the same values as the original ones, therefore the RG transformation can be automatically repeated;
 - in case of linear transformations (when the new correlation functions are just some linear combination of the old ones, as in the examples we considered until now) a rescaling of the new fields is necessary in order to get a useful fixed point.

In the previous example, the presence and the role of the factor ζ might seem particular. Consider therefore the following more general argument¹⁶. At the critical point, there are massless

excitations, the two-point function is expected to decay with some power of n :

$$\langle \phi(n) \phi(0) \rangle \underset{n \rightarrow \infty}{\sim} \frac{1}{n^{d-2+\eta}} \quad , \quad (25)$$

where η is a critical index ($\eta = 0$ for a free theory). Consider now the block transformation defined by Eq. (2). We obtain

$$\begin{aligned} \langle \phi'(n') \phi'(0) \rangle &= b^2 \left\langle \sum_{n \in \text{block } n'} \phi(n) \sum_{m \in \text{block } 0} \phi(m) \right\rangle \\ &\underset{n \rightarrow \infty}{\approx} b^2 \cdot 2^d \cdot 2^d \cdot \langle \phi(2n') \phi(0) \rangle \quad , \quad (26) \end{aligned}$$

where we used the fact that in all the 2^{2d} correlation functions essentially the same distance enters if n is large. Equations (25) and (26) give

$$\frac{1}{n^{d-2+\eta}} = b^2 \cdot 2^{2d} \frac{1}{(2n')^{d-2+\eta}} \quad ,$$

which requires

$$b = b^* \equiv 2^{\frac{-(d-2)+\eta}{2}} \quad (27)$$

Therefore it is expected that the fixed point can be found only for some specific value of the constant b in Eq. (2) and this specific value determines the critical index η via Eq. (27).

In general, we have no similar problems for non-linear RG transformations¹⁶. In models defined by constrained variables (like in the Ising model, σ model and gauge theories) the RG transformations are usually non-linear.

ii) The fixed point action defines a massless theory. If we want to describe a continuum, massive (free) theory, r_0 should be appropriately tuned towards zero as the cut-off is sent to infinity. In our simple case $r_0 \sim (m/\Lambda^{\text{cut}})^2$, where m is the physical mass.

iii) In the simple case discussed in the example, the eigenvalues are determined by the engineering dimension of the operators. Even if we introduce small, non-Gaussian couplings (like $u_0 \phi^4 + v_0 \phi^6 + \dots$), in the linearized approximation around the Gaussian fixed point (where all the couplings are zero) it remains true^{3,5} that for every operator θ with an engineering dimension d_θ there corresponds¹⁷ an eigenoperator of the RG transformation with eigenvalue

$$\lambda_\theta = 2^{d-d_\theta} \quad (28)$$

A few examples are given in Table 1.

θ	d_θ	λ_θ
ϕ^2	$d-2$	4
ϕ^4	$2d-4$	2^{4-d}
$(\nabla^2\phi)^2$	$d-6$	$1/4$
ϕ^6	$3(d-2)$	2^{6-2d}

- Table 1 -

iv) The leading term which violates rotational symmetry (but complies with cubic symmetry) is irrelevant ($\lambda = 1/4$). Rotational symmetry is obtained back automatically (without tuning) in the continuum limit.

v) Consider $d = 4$. According to Eq. (28), all the operators having engineering dimension >4 are irrelevant, while ϕ^4 is marginal. In order to learn what happens with the ϕ^4 coupling one should go beyond the linear approximation. In our case, the technical steps are very similar to that of doing higher order perturbative expansion^{3,5}. Beyond the linear approximation, the ϕ^4 coupling begins to move. As is well known, this model is not asymptotically free. The bare coupling [which is equivalent to the running coupling constant at momenta $\sim 0(\text{cut-off})$] is decreasing as the cut-off is decreased. This motion is slow, it has the typical form $1/u'_0 = 1/u_0 + c \ln 2$ ($c > 0$), where u_0 and u'_0 are the couplings (of the original marginal operator) before and after the RG transformation, respectively (Fig. 6). Therefore, beyond the linear approximation, u_0 behaves like an irrelevant coupling, and Fig. 6 is similar to what we would have obtained in $d > 4$ in the linear approximation (apart from the flow-speeds). If we attempt to define a scalar, continuum theory on the Gaussian fixed point (i.e., the starting value of the bare parameters are in the neighbourhood of this fixed point and the relevant coupling r_0 is tuned towards the fixed point) it will be a free field theory. Of course, it is an exciting problem, whether there are other fixed points in the parameter space where a non-trivial theory can be defined. Increasing evidence shows that the answer is no¹⁸.

How does perturbation theory define a non-trivial perturbative expansion? Perturbation theory tunes the bare coupling towards infinity as the cut-off is increased. Perturbation theory is looking for a fixed point at infinity. Although u_4 is running towards zero under repeated RG steps, if we start far enough from the origin (and we start farther and farther away as the cut-off is increased) it can be arranged that the coupling remains non-zero even after a large number of RG steps arriving at, say, the scale

of 1 GeV. It is clear, however, that this procedure cannot be controlled in perturbation theory. Contrary to what perturbation theory would suggest, increasing evidence shows that actually there is no fixed point (other than the Gaussian) neither at finite nor at infinite distance from the origin¹⁸.

vi) Finally, an important remark. In every point of the critical surface $\xi = \infty$, the system is critical. The exact position of the fixed point in this surface depends on the specific form of the RG transformation. For the Gaussian model, we considered a RG transformation in momentum space and we got $r_0^* = 0$, $\beta^* = 0$, $\gamma^* = 0$, $\gamma^* = 0, \dots$ For other possible RG transformations, however, the fixed point might be (and in general will be) different. The form of the eigenoperators depends also on the RG transformation. However, predictions like the number of relevant and marginal operators, the eigenvalues (critical indices), i.e., properties which are related to long-distance properties ($\sim$ physics, if we use the reducing glasses of the field theorist) are independent of the specific RG transformation.

As an example, consider the Gaussian model on a hypercubic lattice. The RG transformation defined in momentum space leads to the fixed point action in Eq. (29) corresponding to $\rho^*(\vec{q}) = \vec{q}^2$. On the other hand, if the RG transformation is done in configuration space, using the block transformation defined by Eqs. (2) and (27) (with $\eta = 0$), one obtains¹⁹

$$\rho^*(\vec{q}) = \left\{ \sum_{\vec{\ell}=0}^{\infty} \frac{\prod_{i=1}^d (\sin q_i/2)^2}{(\vec{q} + 2\pi\vec{\ell})^2} \frac{1}{(\vec{q}/2 + \pi\vec{\ell})^2} \right\}^{-1} \quad (29)$$

Although the function ρ^* in Eq. (29) behaves as $\vec{q}^2$ for small $\vec{q}$ just as before, the two fixed points are clearly different. The eigenoperators are also more complicated in this case. One can check, however, that the eigenvalues are the same.

1.3. "GOOD" AND "BAD" REGULARIZATIONS; GAUGE THEORY ON THE LATTICE

As we discussed, it is rather easy to study the linearized RG equations around the Gaussian fixed point. The results are equivalent to what a trivial dimensional analysis would suggest. In $d = 4$, for a non-asymptotically free scalar theory, the Gaussian fixed point is not very interesting. Starting from the neighbourhood of this point, after a few RG steps (which integrate out the irrelevant, short-distance properties of the model and leave the physical properties intact) the ϕ^4 (and even more so the $\phi^6, \dots$) coupling becomes ~ 0 , the theory is free.

The Gaussian fixed point is much more interesting for asymptotically free theories, like the $d = 2$ non-linear σ model, or $d = 4$

Yang-Mills theory. The reason is that in this case the marginal operator behaves like a relevant variable beyond the linear approximation; the coupling flows outwards making it possible to define a non-trivial continuum theory on the Gaussian fixed point. No relevant operators are expected to occur in the linear approximation. Candidates like $\sum_{\mu, a} A_{\mu}^a A_{\mu}^a$ are excluded by gauge invariance. Therefore the leading operator (having the lowest engineering dimension) which is allowed by gauge, P, C and cubic symmetry is $F_{\mu\nu}^a F_{\mu\nu}^a$. The corresponding coupling is unchanged in the linear approximation (it is marginal), but moves away from the Gaussian fixed point if higher order corrections are taken into account. Therefore, we expect that in a Yang-Mills gauge model, a continuum field theory can be defined around the Gaussian fixed point by tuning one parameter only.

Of course, this conclusion is true only if the regularization we use (in this case the lattice) preserves gauge invariance. The problem of "good" and "bad" regularizations is very often discussed nowadays, especially in relation with supersymmetry. What is the criterion? First we should note that, in general, every regularization breaks certain symmetries: the Pauli-Villars regularization breaks unitarity, gauge symmetry, ..., the lattice breaks rotational symmetry, ..., dimensional regularization breaks supersymmetry, and so on. Therefore the regularized theory belongs to a larger class of theories than the original one. If the extra operators of this enlarged class are all irrelevant for the given fixed point, the regularization is "good". Otherwise it is "bad". A "bad" regularization can be used also, but the price is a very inconvenient tuning procedure.

The correct, gauge invariant lattice regularization of Yang-Mills models is expected to be "good". Really, on the basis of our previous discussion, the extra operators which violate Euclidean rotational symmetry are all expected to be irrelevant for the Gaussian fixed point.

Consider a hypercubic lattice. The convenient variables, in terms of which gauge invariance is easily kept, are not the vector potentials themselves, but their exponentiated forms. An $SU(N)$ matrix $U_{n\mu}$ is associated to the link with end-points n and $n+\hat{\mu}$ on the lattice as an independent variable:

$$\begin{array}{c} \text{---} \longrightarrow \text{---} \\ n \qquad \qquad n+\hat{\mu} \end{array} \quad U_{n\mu} \quad \left(= e^{iga A_{n\mu}^a T^a} \right) \quad (30)$$

Under a gauge transformation $U_{n\mu}$ transforms as

$$U_{n\mu} \longrightarrow V_n U_{n\mu} V_{n+\hat{\mu}}^+ \quad (31)$$

where $\{V_n\}$ are arbitrary $SU(N)$ matrices. Therefore the trace of the

product of U matrices along a closed loop is gauge invariant. A gauge invariant action can depend only on these traced loops. Locality tells us to use small loops. The simplest possibility is

$$S_W = -\frac{1}{g^2} \sum_{\text{plaquettes}} (\text{Tr } U_P + \text{cc}) ,$$

$$U_P = U_{n\mu} U_{n+\hat{\mu},\nu} U_{n+\hat{\mu},\nu}^+ U_{n\nu}^+ , \quad \begin{array}{c} n+\hat{\mu} \\ \uparrow \\ \square \\ \downarrow \\ n \end{array} \quad \begin{array}{c} n+\hat{\mu}+\hat{\nu} \\ \uparrow \\ \square \\ \downarrow \\ n+\hat{\mu} \end{array} , \quad (32)$$

which is called the Wilson action²⁰. In the classical continuum limit, when $U_{\mu\nu}$ is expanded in terms of the vector potential for small "a", we get back $\int d^4x \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$. This requirement fixes the constant in front of the sum to be $1/g^2$ ²¹.

1.4. THE GAUSSIAN FIXED POINT IN A YANG-MILLS GAUGE THEORY

As we discussed, the first step in a RG transformation is the thinning out of the variables by some procedure. The transformation should be gauge invariant otherwise we would lose gauge symmetry after the first RG step already. This would be unwise as we discussed before. Actually it is not quite trivial to define a gauge invariant transformation. The simple Fourier space transformation we used for the scalar model before is not gauge invariant. Later I will give gauge invariant examples. They are all block transformations in configuration space.

We know that even starting with the simple Wilson action, all kinds of other couplings will enter the successive actions later. Let us start therefore with a general action

$$\frac{1}{g^2} \left\{ \theta_1 + c_2 \theta_2 + c_3 \theta_3 + \dots \right\} , \quad (33)$$

where $\theta_1 = \sum_P (\text{Tr } U_P + \text{cc})$, while $\theta_2, \theta_3, \dots$ are combinations of different kinds of interactions containing larger loops and more complicated functions of the trace. We require that $\theta_1, \theta_2, \dots \rightarrow 0$ in the classical continuum limit in order to recover the classical action.

In the multiparameter space $(g^2, c_2, c_3, \dots)$ the hyperplane $(g^2 = 0, c_2, c_3, \dots)$ is a critical surface. Really, taking a fixed, gauge invariant expectation value (say, a Wilson loop of fixed size on the lattice), leading order perturbation theory will correctly

give the result for $g^2 \rightarrow 0^*$). On the other hand, leading order perturbation theory works with massless free vector particles. Therefore, the dimensionless correlation length ξ is going to infinity when $g^2 \rightarrow 0$.

There is a fixed point somewhere in this plane: ($g^2 = 0, c_1^*, c_2^*, \dots$) (Fig. 7). The position of this fixed point depends on the specific form of the block transformation. The fixed point coordinates $c_2^*, c_3^*, \dots$ can be found in tree level perturbation theory. If

$$S^* = \lim_{g^2 \rightarrow 0} \frac{1}{g^2} \left\{ \theta_1 + c_2^* \theta_2 + c_3^* \theta_3 + \dots \right\} \xrightarrow{RG} S^*, \quad (34)$$

then the identical type of gauge invariant expectation values (for instance the plaquette expectation value before and after the transformation) should remain unchanged. Before and after the transformation, the leading order result has the form $1 - g^2 f(c_2^*, c_3^*, \dots) |_{g^2 \rightarrow 0}$ and $1 - g^2 \tilde{f}(c_2^*, c_3^*, \dots) |_{g^2 \rightarrow 0}$, respectively. ($\tilde{f}$ is obtained by expressing the block gauge matrices in terms of the original variables everywhere, and performing perturbation theory in terms of the original variables.) The fixed point condition is $f(c_2^*, c_3^*, \dots) = \tilde{f}(c_2^*, c_3^*, \dots)$, which determines $c_2^*, c_3^*, \dots$ if the procedure is repeated for the different gauge invariant expectation values.

Having found the fixed point, one can discuss the linearized RG equations around this fixed point. We expect to find one marginal operator ($\equiv "F_{\mu\nu}^a F_{\mu\nu}^a"$, the precise form of it - as expressed by $\theta_1, \theta_2, \dots$ - depends on the block transformation), while all the other eigenoperators are irrelevant (Fig. 7).

Beyond the linear approximation ($\sim$ tree level perturbation theory) the coefficient of the marginal operator begins to move; it moves outwards from the critical plane (asymptotic freedom). It behaves like a relevant parameter. We can define the RT the usual way and the flow lines look like as given in Fig. 7.

*) Of course, this fact does not imply that the theory becomes perturbative in the $g^2 \rightarrow 0$ limit. Those Wilson loops which describe non-perturbative, long-distance continuum phenomena are increasingly large loops as g^2 gets smaller and smaller. For a fixed size loop, however, the correction to the leading result is $\sim g^2 (\ln n + c)$, where n is the fixed size, and this becomes arbitrarily small when $g^2 \rightarrow 0$.

1.5. RENORMALIZABILITY

Figure 7 is the basic starting point in expecting renormalizability and universality (i.e., the independence of the physical predictions of the specific lattice action chosen).

The reader might be bothered by the following question. In studying perturbative renormalizability in a Yang-Mills gauge theory, one shows that it is possible to change the cut-off and the bare coupling constant g^2 in such a way that the physical predictions remain unchanged. On the other hand, a RG transformation is just a procedure for changing the cut-off and it tells us that we should enter a multiparameter coupling constant space (?).

There is no contradiction. Let us illustrate how renormalizability (in the usual sense) of the Wilson action follows from the flow-line structure of Fig. 7.

Consider the standard Wilson action, Eq. (32). It is defined along the line $c_2 = 0, c_3 = 0, \dots, g^2 \in (0, \infty)$ in the multiparameter space. Perform a RG transformation starting from a small g^2 . After this transformation, we arrive to the point 1 in the multidimensional parameter space (in Fig. 8, only one extra co-ordinate, c_2 is kept). After a few RG steps we land on the RT and continue to move along it. Let us repeat the same procedure starting from the point $g'^2 > g^2$ ($c_2 = c_3 = \dots = 0$). By tuning g'^2 it can be arranged that those points of this second sequence which lie on the RT coincide with the corresponding points of the first sequence, but one step behind (Fig. 8). Therefore, the Wilson action with couplings g^2 and g'^2 describes identical long-distance (continuum) physics, the only difference is in their scale:

$$\zeta(g^2) = 2 \zeta(g'^2),$$

or

$$a(g^2) = \frac{1}{2} a(g'^2).$$

Really, take the last (almost) coinciding point in Fig. 8. At this point, we have some cut-off Λ^{cut} and we can calculate different predictions. These predictions will be the same as at the starting points g^2 and g'^2 since the RG transformation does not change the predictions for the unintegrated variables. On the other hand, the cut-off at g^2 is $2^7 \Lambda^{\text{cut}}$, while at g'^2 it is $2^6 \Lambda^{\text{cut}}$. Therefore, it is possible to change the cut-off and the coupling appropriately without changing the physical content of the theory.

Notice that we are not claiming that all the predictions of the Wilson action at g^2 and g'^2 are identical apart from a scale change. We only say that the predictions are identical concerning

the long-distance properties (remember: long-distance on the lattice $\leftrightarrow$ any finite distance in the continuum). On the other hand, a RG transformation exactly reproduces all the predictions for the unintegrated variables, including even the short-distance predictions. To achieve that, one should enter a multiparameter coupling constant space.

I.6. THE β FUNCTION AND $\Delta\beta(\beta)$

Consider the Wilson action, i.e., the one-parameter line (g^2 , $c_2 = 0$, $c_3 = 0, \dots$) in the parameter space. As we discussed, if we start in the neighbourhood of the critical surface (g^2 is small), it is possible to change the cut-off ($\sim 1/a$) and the coupling g^2 in such a way that the physical predictions remain unchanged. One can remove the regularization this way ($a \rightarrow 0$), this is the continuum limit, the general problem of defining a field theory in a limiting process.

Therefore, the existence and properties of the function $g = g(a)$, or of the β function^{*})

$$B(g) = -a \frac{d}{da} g(a) \quad (35)$$

are basically important. The β function describes the way the bare coupling should be tuned in order to keep all the physical predictions independent of the cut-off in the continuum limit.

The continuum limit is the $g^2 \rightarrow 0$ limit. If g^2 is finite but small, there will be small cut-off dependent errors in the predictions. If g^2 is small, these corrections are very small also. Given a set of quantities, for any prescribed, required precision, there exists a region $g^2 \in (0, g_{\max}^2)$ in which the Wilson action gives correct continuum predictions. This region is called the continuum, or scaling region.

In the scaling region, any dimensionful physical quantity behaves in a definite way as the function of g^2 . This scaling behaviour is fixed by the β function. Really, in this region

$$a \frac{d}{da} m_{\text{phys}} = 0, \quad (36)$$

while on dimensional ground we have

$$m_{\text{phys}} = \frac{1}{a} f(g), \quad (37)$$

which gives

^{*}) The β function will be denoted by $B(g)$ in order to distinguish it from the coupling $\beta = (2N/g^2)$ as defined in Eq. (43).

$$-f(g) + f'(g) \left(a \frac{d}{da} g(a) \right) = 0. \quad (38)$$

The solution of the first-order differential equation has the form

$$f(g) = c e^{-\int^g \frac{1}{\beta(g')} dg'} \quad (39)$$

giving

$$m_{\text{phys}} = c \frac{1}{a} e^{-\int^g \frac{1}{\beta(g')} dg'} \equiv c \Lambda^{\text{latt}} \quad (40)$$

Here Λ^{latt} is the scale parameter of the theory which can be related to more conventional Λ parameters like Λ^{mom} or $\Lambda^{\overline{\text{MS}}}$. Its precise normalization [i.e., the prescription for the integration constant in Eq. (40)] will be fixed later [see Eq. (42)].

What do we know about the β function? Perturbation theory can be used to calculate its behaviour for small g^2 :

$$\beta(g) \underset{g \rightarrow 0}{=} -b_0 g^3 - b_1 g^5 \dots, \quad \begin{aligned} b_0 &= \frac{11N}{48\pi^2}, \\ b_1 &= \frac{34}{3} \left(\frac{N}{16\pi^2} \right)^2 \end{aligned} \quad (41)$$

The negative sign of the first term implies that $g(a)$ is increasing when "a" is increasing (the cut-off is decreasing) - a property we used repeatedly in drawing the flow lines before.

If g is very small (the cut-off is very large) in Eq. (40), the β function is well represented by the first two terms of Eq. (41) and we get:

$$m_{\text{phys}} = c \frac{1}{a} e^{-\frac{1}{2b_0 g^2} - \frac{b_1}{2b_0^2} (b_0 g^2)} \left[1 + \mathcal{O}(g^2) \right], \quad (42)$$

which is often called asymptotic scaling.

There are, however, higher order perturbative terms in the β function and completely unknown non-perturbative contributions also, which are relevant at intermediate values of g in Eq. (40). This is the region where the cut-off is large enough already to make the theory continuum-like, but not so large that the $g^7, \dots$ corrections can be neglected.

It is basically important to reveal and understand the quantitative properties of the full β function. It is important to show

that it exists (i.e., there exists a renormalizable, continuum field theory as the lattice regularization is removed) and that deep in the continuum limit it goes over to the perturbative form in Eq. (41).

Renormalization group techniques, especially the MCRG methods were just invented for these kinds of problems.

In the MCRG approach, not the β function itself, but a related quantity $\Delta\beta(\beta)$ is determined. Here β is the usual notation for the inverse coupling:

$$\beta = \frac{2N}{g^2}, \quad \left(\frac{6}{g^2} \text{ for } SU(3) \right), \quad \beta \rightarrow \infty \text{ in the cont. limit} \quad (43)$$

while the function $\Delta\beta(\beta)$ gives the change of the coupling β : $\beta \rightarrow \beta - \Delta\beta(\beta)$ when the (dimensionless) correlation length (or the cut-off) is decreased by a factor of two:

$$\begin{array}{ccc} \beta & a & \xi \\ \downarrow & \downarrow & \downarrow \\ \beta' \equiv \beta - \Delta\beta(\beta) & 2a & \xi/2 \end{array} \quad (44)$$

Actually, in Fig. 8, the couplings g^2 and g'^2 are just related by $\Delta\beta$: $\beta = 6/g^2$, $\beta' = 6/g'^2$ and $\Delta\beta(\beta) = \beta - \beta'$.

Using the definition of the β function [Eq. (35)], it is easy to see that $\Delta\beta(\beta)$ is directly related to the integral of the inverse of the β function and it carries the same information:

$$\int_{\beta - \Delta\beta}^{\beta} \frac{dx}{x^{3/2} B(\sqrt{6/x})} = - \frac{2 \ln 2}{\sqrt{6}}, \quad (SU(3)) \quad (45)$$

If $\Delta\beta(\beta)$ is known, the β function can be immediately obtained from Eq. (45) and the other way around.

As we discussed before, a unique β function [therefore $\Delta\beta(\beta)$] is defined only in the continuum limit in the scaling region. If we are far from the critical surface, there is no way to change a and g^2 together in such a way that all the physical predictions stay unchanged. We might define a β function for all the values of the coupling by keeping the string tension fixed but this β function will be different from that obtained by keeping a glueball mass fixed. A unique β function [or $\Delta\beta(\beta)$] exists only in the scaling region. Nevertheless, it is instructive to see how $\Delta\beta(\beta)$ behaves over the full β range if it is defined, say, via the string tension.

For small β strong coupling expansion gives²¹

$$\beta = \frac{1}{a^2} \ln \frac{18}{\beta} + O(\beta^4) \quad (46)$$

The requirement $\sigma^{\frac{1}{2}} = \text{const.}$ gives $g = g(a)$, from which one obtains:

$$\Delta\beta(\beta)_{\text{tension}} = \beta - \frac{\beta^4}{18^3} + \dots \quad (47)$$

The higher order terms of the strong coupling expansion are quite ineffective up to $\beta \lesssim 5$, therefore Eq. (47) represents $\Delta\beta(\beta)$ well up to this point. Around $\beta \sim 5$, the higher order terms begin to bend this curve down (Fig. 9).

For large β , the β function, therefore $\Delta\beta(\beta)$, is known. Equations (41) and (45) give

$$\Delta\beta(\beta)_{\beta \rightarrow \infty} = 0.579 + \frac{0.204}{\beta} + O(1/\beta^2), \quad (\text{sub}) \quad (48)$$

This asymptotic prediction is plotted also in Fig. 9. It is clear from this Figure that the model should do some real gymnastics to get $\Delta\beta$ down to ~ 0.6 from $\Delta\beta \sim 5$. However, what caused some concern a year ago was that the model seemed to do this job too well: according to the string tension and T_c determinations, $\Delta\beta$ began to drop like a stone and instead of smoothing to ~ 0.6 from above, it continued to decrease^{22,23}. Around $\beta \sim 6.0$ its value seemed to be only ~ 0.3 . The scenario, where $\Delta\beta(\beta)$ has a zero, would be quite unpleasant: the associated phase transition would presumably imply that either confinement or asymptotic freedom is lost in the continuum limit.

We do not believe our favourite toy dying so unwarranted. Renormalization group methods are clearly called for.

II.1. MONTE CARLO RENORMALIZATION GROUP METHODS; OPTIMIZATION

The Monte Carlo renormalization group method (MCRG) is a combination of the beautiful RG ideas with the brute-force power of MC simulations. There are different MCRG methods. First, I will discuss the classic blocking MCRG method and its optimization. Next, another method, the so-called ratio method, will be discussed, which is simpler, perhaps less powerful, but easier to implement.

MCRG with blocking²⁴⁻²⁷

Consider again Fig. 8. Given g^2 , we have to determine the corresponding g'^2 - this relation gives $\Delta\beta(\beta)$ (or the β function) of the Wilson action (the procedure is identical with other actions, of course).

How can we find the relation $g^2 \rightarrow g'^2$? A possibility would be to generate configurations by MC at g^2 , to construct the block configuration at every time and calculable block correlation functions (block Wilson loop expectation values). Then, by trial and error, or by some clever, systematic procedure²⁸, we search for an action in the multiparameter space which gives the same correlation functions (without blocking). This way, the sequence of points in the multiparameter space (Fig 8) is explicitly found. The flow lines - at least in a subspace of all the couplings - are explicitly followed.

There is another, less ambitious method, however. We observe that it is not the complicated multiparameter action which is important for us, but the configurations created by this action. However, simulating the new action, a certain configuration would occur with the same probability as the probability of having this configuration occurring via the blocking of the configurations generated by the standard Wilson action at g^2 . Of course, these configurations will be obtained on a lattice which is half as large as the original one, and in the next step, we lose another factor of 2, and so on. If we accept this disadvantage, the simplified procedure is the following.

Configurations are generated on an N^4 lattice by MC by using the standard action at coupling g^2 . On every configuration, the block variables are constructed [they are defined on an $(N/2)^4$ lattice] and different block correlation functions are calculated. Next, block variables are constructed from the first block variables [they will be defined on an $(N/4)^4$ lattice], and so on. The whole procedure is repeated at g'^2 starting on an $(N/2)^4$ lattice. The block correlation functions obtained in the first sequence after the K^{th} step are compared with those obtained in the second sequence after the $(K-1)^{\text{th}}$ step. The coupling g'^2 is tuned until matching is found after a few RG steps. By comparing numbers obtained on lattices of the same size, the matching condition will not be distorted by finite size effects.

At every blocking step, the linear size of the lattice is reduced by a factor of 2. If the RT is far from the action we work with (the Wilson action in our case) and many blocking steps are required in order to match the two sequences of points in the multiparameter space, the procedure would require a prohibitively large starting lattice. Or, in other words, if we can perform only

an insufficient number of RG steps, then no consistent matching will be possible (the matching of the different block expectation values would give different $\Delta\beta$ values). Later, in discussing the results I will illustrate this problem with an example.

A possible solution is to use an improved action which lies closer to the RT. For instance, one might determine (at least approximately) the position of the fixed point in perturbation theory and then use an improved action as illustrated in Fig. 10²⁹. This solution has some disadvantages. First, we wanted to determine the β function of the Wilson action and looking for a tool, we abandon the original problem (namely we consider another action). Second, the new action will contain a lot of different terms ($c_2 \neq 0$), $c_3 \neq 0, \dots$) and will be difficult to simulate.

There is another possibility, however: search for an improved block transformation whose fixed point and RT lie close to the original (Wilson) action^{30,31}. This is possible in principle since the position of the fixed point and the RT depends on the specific form of the block transformation.

One might start with a block transformation which contains one or more free parameters. Then these parameters are fixed by requiring the RT running as close to the Wilson action as possible in the region of g^2 where we study the β function. For small g^2 , one might use perturbation theory to determine the fixed point and then optimize the free parameter(s) to push this fixed point towards the point ($g^2 = 0$, $c_2 = 0$, $c_3 = 0, \dots$)³¹. For intermediate g^2 values, non-perturbative optimization can be also used (I will show examples in discussing the results)^{30,32}.

Let me give an example of a gauge invariant block transformation containing adjustable free parameters ³³ (Fig. 11). The crosses in Fig. 11 denote the points of the lattice after the RG transformation. The new gauge field variable V_{AB} associated with the link A-B is chosen with the probability

$$\text{prob.}(V_{AB}) \sim e^{\mathcal{P}(\text{Tr } V_{AB} X^{\dagger} + c c)} \quad (49)$$

where X is the sum of the product of $SU(N)$ matrices along different paths connecting the point A with B:

$$X = \begin{array}{c} \times \\ \text{A} \end{array} \begin{array}{c} \times \\ \text{B} \end{array} + \alpha \sum_{2d \text{ paths}} \begin{array}{c} \times \quad \times \\ \text{---} \quad \text{---} \\ \times \quad \times \end{array} + \beta \sum_{4d \text{ paths}} \begin{array}{c} \times \quad \times \quad \times \\ \text{---} \quad \text{---} \quad \text{---} \\ \times \quad \times \quad \times \end{array} + \dots \quad (50)$$

where $\alpha, \beta, \dots$ [and $\mathcal{P}$ in Eq. (49)] are free, adjustable parameters. It is easy to check that this block transformation preserves the original symmetries: gauge symmetry, cubic symmetry, and so on.

The ratio method^{31,34,35}

This method is less sophisticated than the blocking procedure discussed before. Its disadvantage is that it is very demanding for what concerns the statistical quality of the MC data. It is difficult to see also how to perform the optimization of the ratio method beyond perturbation theory. However, the method has also some definite advantages. First of all, it is rather easy to implement, second it can be trivially generalized to the case when fermions are present (Grassmann variables in the path integral)³⁶.

The basic idea is as follows. The β function entered our discussion via the equation

$$a \frac{d}{da} \langle \text{physics} \rangle_{a \rightarrow 0} = 0 \quad (51)$$

"Physics" in Eq. (51) might be a measurable mass. Unfortunately, a mass reflects a long-distance behaviour even in the continuum (i.e., even through the reducing glasses of a field theorist in Fig. 1), therefore it is long-distance on the lattice in a double way. There are, however, physical quantities which reflect short-distance continuum properties. A first guess would be a correlation function, like^{*)} $\langle \phi(x)\phi(0) \rangle$, where x is small (compared to the correlation length; on the other hand much larger than the inverse cut-off). This function, however, does not correspond directly to physics. It picks up wave function renormalizations and only the correlation of the renormalized fields satisfy Eq. (51). As the wave function renormalization factors are not known, this fact is of no use. On the other hand, the wave function renormalization factors are independent of the argument of the fields, which suggests to use the ratio

$$f(x,y) = \frac{\langle \phi(x)\phi(0) \rangle}{\langle \phi(y)\phi(0) \rangle} \quad (52)$$

Equation (51) implies immediately the following relation on the lattice:

$$f(2n_1, 2n_2; \beta, 2L) = f(n_1, n_2; \beta', L), \quad (53)$$

where

*) Scalar fields are temporarily used for illustration. The argument x is dimensionful. The reader might think of the $d = 2$ σ model here.

$$f(n_1, n_2; \beta', L) = \frac{\langle \phi(n_1) \phi(0) \rangle}{\langle \phi(n_2) \phi(0) \rangle} \Big|_{\substack{L^d \text{ lattice,} \\ \text{coupling } \beta'}} \quad (54)$$

while $\beta - \beta'$ is just the function we are looking for: $\Delta\beta(\beta)$. Since x and y in Eq. (52) should be much larger than the inverse cut-off, $n_1, n_2 \gg 1$ is necessary in Eq. (53).

In a gauge theory, one proceeds in a completely analogous way. There we consider Wilson loop expectation values:

$$W(i_1, i_2) \equiv \left\langle \text{Tr} \begin{array}{c} \square \\ i_1 \quad i_2 \end{array} \right\rangle \quad (55)$$

This quantity picks up renormalization factors also: a self-energy correction, which is proportional to the length of the loop and factors due to "corner" singularities. In perturbation theory, these renormalizations factorize out and we are led to define:

$$f(i_1, i_2; i_3, i_4) = \frac{W(i_1, i_2)}{W(i_3, i_4)}, \quad i_1 + i_2 = i_3 + i_4 \quad (56)$$

or

$$g(i_1, i_2, i_3, i_4; i'_1, i'_2, i'_3, i'_4) = \frac{W(i_1, i_2) W(i_3, i_4)}{W(i'_1, i'_2) W(i'_3, i'_4)}, \quad i_1 + i_2 + i_3 + i_4 = i'_1 + i'_2 + i'_3 + i'_4 \quad (57)$$

and so on. The renormalization group equation now takes the form:

$$f(2i_1, 2i_2; 2i_3, 2i_4; \beta, 2L) = f(i_1, i_2; i_3, i_4; \beta', L) \quad (58)$$

Similar equations hold for the function g and for other possible ratios. By measuring Wilson loop expectation values on and L^d lattice and twice as large loops on and $(2L)^d$ lattice at β' and β , respectively, the function $\Delta\beta(\beta) \equiv \beta - \beta'(\beta)$ can be determined by searching for that value of β' where Eq. (58) is satisfied.

As we discussed before, n_1 and n_2 in Eq. (53), and $i_1, i_2, \dots$ in Eqs. (56), (57) and (58) should be large in order to avoid lattice artifacts which violate Eqs. (53) and (58). In a practical MC study, the size of these integers is always limited and the renormalization group equation is contaminated by lattice artifacts. One might attempt to minimize these corrections by some optimization procedure just as in the blocking case. A possibility is the following^{31,35}.

We observe that any linear combination of the functions f defined in Eq. (56) also satisfy Eq. (58) (the same is true for the

functions $g, \dots$). In the improved ratio method, the mixing coefficients are determined by the requirement of cancelling the lattice artifact corrections to Eq. (58) systematically order by order in perturbation theory. At the tree level, the linear combination of two basic ratios is formed and the mixing coefficient is determined by requiring Eq. (58) to be correct on the tree level (i.e., by requiring $\Delta\beta = 0$; there is no scale generated, there is no shift in β at the tree level). At the one-loop level the linear combination of three basic ratios is formed, and the two mixing coefficients are determined by requiring $\Delta\beta = 132(\ln^2/16\pi^2) \approx 0.579$ in Eq. (58), which is the exact one-loop result [see Eq. (48)], and so on. This procedure systematically eliminates the lattice artifacts in perturbation theory, which are relevant at large correlation lengths. Whether non-perturbative lattice effects become important at moderate couplings can be judged only through the consistency of the final results.

A large number of systematically improved, mixed ratios, can be obtained this way. These mixed ratios are then used in the actual MC analysis to determine $\Delta\beta(\beta)$ via an equation analogous to Eq. (58) written for the mixed ratios.

II.2. RESULTS

In this section, some of the results obtained by the MCRG techniques discussed will be briefly summarized³⁷. First, we shall check the methods on an exactly soluble, $d = 2$, asymptotically free model, on the $O(N)_{N \rightarrow \infty}$ σ model. Next, the $d = 2$, $O(3)$ σ model will be discussed while the results on $d = 4$, $SU(3)$ Yang-Mills theory will be summarized at the end.

$O(N)_{N \rightarrow \infty}$ σ model

The standard action of the model is defined as

$$\text{action} = \lim_{N \rightarrow \infty} \left\{ -N\beta \sum_n \sum_{\mu=1}^2 \vec{S}_n \cdot \vec{S}_{n+\hat{\mu}} \right\}, \quad (59)$$

where $\vec{S}_n$ is a spin of unit length $\vec{S}_n^2 = 1$ with N components. The model can be solved exactly in the limit $N \rightarrow \infty$ ³⁸. For small β (high temperature) the spins are disordered. The model stays in this phase over the full coupling constant space $\beta \in (0, \infty)$, there is no phase transition into a phase with spontaneous magnetization for any finite β . When $\beta \rightarrow \infty$, the correlation length $\xi \rightarrow \infty$. The $\beta = \infty$ point is a critical point. A continuum theory can be defined at this point. This continuum theory is asymptotically free³⁹ and

shows interesting analogies with $d = 4$ Yang-Mills theories in many respects.

The β function, defined in the continuum limit, can be exactly determined. For the standard action in Eq. (59), the β function contains one term only; the perturbative one-loop result is exact, there are no higher order corrections. The function $\Delta\beta(\beta)$ [as defined in Eq. (44)] has the simple form:

$$\Delta\beta(\beta)_{\text{exact}} = \frac{\ln 2}{2\pi} \quad (60)$$

Let us check how (MC)RG performs in this model⁴⁰. Consider first the simple blocking

$$\vec{S}'_{n'} = \frac{\vec{S}_{n_1} + \vec{S}_{n_2} + \vec{S}_{n_3} + \vec{S}_{n_4}}{\|\vec{S}_{n_1} + \vec{S}_{n_2} + \vec{S}_{n_3} + \vec{S}_{n_4}\|}, \quad n_1, n_2, n_3, n_4 \in \text{block } n' \quad (61)$$

which is a natural possibility. No optimization is involved here. The predicted $\Delta\beta(\beta)$ as obtained by using this block transformation is given in Fig. 12. The matching predictions of the nearest-neighbour (NN) correlations are given there for starting lattices of different sizes and at different blocking stages.

At moderate correlation lengths ξ , the results are not too bad, but the matching procedure breaks down completely at large values of ξ . Actually, it can be shown that any generic (i.e., not optimized) block transformation leads to matching results where the error is increasing linearly with β for large β ³¹. This error is related to lattice artifacts showing up in tree level perturbation theory already and can be cured by perturbative, tree-level optimization. Consider the following, one-parameter blocking:

$$\text{probability of } \vec{S}'_{n'} \sim e^{P \vec{S}'_{n'} (\vec{S}_{n_1} + \vec{S}_{n_2} + \vec{S}_{n_3} + \vec{S}_{n_4})} \quad (62)$$

where P is an adjustable parameter. For $P \rightarrow \infty$, this block transformation is identical to that given in Eq. (61). By demanding the fixed point lying as close to the standard action as possible (for $\beta \rightarrow \infty$), one obtains the optimal value of P ³¹:

$$P_{(\text{tree level pert. th.})}^{\text{opt}} \approx 4.6 \beta \quad (63)$$

The results obtained by using this optimized block transformation are given in Fig. 13. The $3/2$ (= three steps on 32^2 versus two steps on 16^2) and $4/3$ results are almost identical and give precise predictions up to correlation lengths $O(10^5)$ and beyond. Below $\beta \approx 0.5$ ($\xi \lesssim$ few lattice units) the predicted $\Delta\beta(\beta)$ is different for

the NN and DI ("diagonal": $\langle \vec{s}_{(0,0)} \vec{s}_{(1,1)} \rangle$) matchings indicating that this region is already outside the continuum limit.

The optimized ratio method works similarly well in this model³¹. By using tree-level perturbation theory, the basic ratios can be mixed in such a way as to cancel the leading lattice artifact corrections to the renormalization group equation, Eq. (53). A subset of the matching predictions is given in Fig. 14 at arbitrarily selected β values. It is worth mentioning again that at, say, $\beta = 2.049$ the correlation length is very large [$0(10^5)$] while the matching predictions were obtained by comparing appropriately mixed ratios on 32^2 and 16^2 lattices, respectively. In this model, it is easy to perform the optimization in the one-loop level perturbation theory also. The matching values obtained in this case are practically indistinguishable from the exact number in Eq. (60).

In Figs. 12, 13 and 14, the predictions are free of statistical errors, of course. In the $N \rightarrow \infty$ limit, the RG procedure can be followed analytically (with a little help from the computer).

$O(3)$ σ model

Although no exact solution is available, this model has been studied by many different methods. The first two perturbative terms of the β function of the standard action

$$\text{action} = -\beta \sum_n \sum_{\mu=1,2} \vec{S}_n \cdot \vec{S}_{n+\hat{\mu}}, \quad \vec{S}_n = (S_n^1, S_n^2, S_n^3) \quad (64)$$

is known³⁹ (the model is asymptotically free), but no analytic information is available on the behaviour of the full β function.

The results on $\Delta\beta(\beta)$ as obtained by different MCRG methods^{27,31,41} are summarized in Fig. 15. The (perturbatively) optimized blocking and ratio methods give consistent results. The predicted β function smooths over the asymptotic two-loop form at large β values while showing a non-trivial, non-perturbative structure at intermediate β values.

Until now, we emphasized the importance of the β function in relation with controlling the continuum limit. Let us illustrate now how the quantitative knowledge of the β function helps in determining the mass gap (mass of the lowest-lying excitation) precisely⁴¹. The mass gap is proportional to the Λ parameter of the lattice Λ^{latt} , which is defined in complete analogy to that in a $d = 4$, Yang-Mills gauge theory [Eq. (40)]:

$$m = c \Lambda^{\text{latt}}, \quad \Lambda^{\text{latt}} = \frac{1}{a} e^{\int^f \frac{1}{B(f')} df'}, \quad f \equiv 1/\beta. \quad (65)$$

We have to determine the connecting constant c . In a MC simulation, $m.a$ is measured at different couplings f ($= 1/\beta$). In order to extract the constant c , we need to know the β function. When β is large, the β function is known from perturbation theory. However, for large β , the correlation length $\xi = 1/m.a$ is large and measuring $m.a$ there requires large lattices and very long simulations⁴². If, however, the β function is known from MCRG studies for intermediate coupling values, then the constant c can be determined by measuring $m.a$ at moderate correlation lengths. For instance, by combining the MCRG results in Fig. 15 with the value of $m.a$ at $\beta = 1.2$ (where the correlation length is only a few lattice units and the MC measurement of $m.a$ is relatively easy) one obtains⁴¹:

$$m = (113 \pm 10) \Lambda^{\text{latt}} \quad (66)$$

This result is in agreement with the direct MC mass gap predictions (obtained at very large lattices)⁴². Having the constant c and the β function, one can predict $m.a$ as the function of β . This prediction is compared with the actual MC numbers in Fig. 16⁴¹. I think that Fig. 16 is a nice illustration of the power of the MCRG method.

SU(3) gauge theory ($d = 4$)

The available results on $\Delta\beta(\beta)$ obtained by optimized block MCRG^{32,43} and by the improved ratio method³⁵ are summarized in Fig. 17. These results are consistent with the numbers of Ref. 44, where a special block transformation with a scale factor of $\sqrt{3}$ was applied^{45,*}). In Fig. 17, the two-loop perturbative result is also given. There exists an estimate on the three-loop contribution - its effect seems to be small⁴⁶. For instance at $\beta = 6.0$, we get $\Delta\beta^{(1)} = 0.579$, $\Delta\beta^{(1+2)} = 0.618$ and $\Delta\beta^{(1+2+3)} \approx 0.607$ at the one-, two- and three-loop level, respectively. Let me remind the reader that $\Delta\beta(\beta)$ (as defined by the string tension - it is unique only in the continuum limit) starts from zero at $\beta = 0$, grows $\sim$ linearly as β up to $\beta \lesssim 5$ and then begins to drop.

In the case of the optimized ratio test, the points represent the average prediction from a large number (several thousands,

*) Since $\Delta\beta(\beta)$ depends on the scale factor of the RG transformation [see Eq. (44)] a quantitative comparison is possible only when these results are expressed in terms of the β function. For that we need $\Delta\beta(\beta)$ in several neighbouring β points. This is expected to be available in the near future.

typically) of improved ratios. The smaller error bars represent the average fluctuation of these large number of matching predictions while the larger error bars give the statistical errors.

At $\beta = 6.0$, there are results from two different optimized block transformations. In addition to the points at $\beta = 6.0$ and 6.3 results are expected very soon at $\beta = 6.6$ and 7.2 also⁴³.

According to Fig. 17, $\Delta\beta(\beta)$ has a rather strange structure. Unlike in the $d = 2, 0(3)$ σ model, where it goes smoothly to the asymptotic form from above, here $\Delta\beta$ drops significantly below the asymptotic value in the neighbourhood of $\beta \sim 6$, and then slowly recovers.

An important remark on the notations: as Eq. (45) shows, $\Delta\beta(\beta)$ is related to the integral of the inverse of the β function over the region $(\beta - \Delta\beta, \beta)$. Since $\Delta\beta(\beta = 6.6)$ is close to the asymptotic form, it implies that the β function is close to the asymptotic form for $\beta \gtrsim 6.0$.

It is clear that further points are needed before one can attempt to draw a smooth curve through the points giving $\Delta\beta(\beta)$ or the β function. It can already be seen, however, that there is no asymptotic scaling before $\beta \lesssim 6.0-6.1$. This observation explains and conforms with the earlier strange results on the string tension $\sigma^{22,47}$ and critical temperature $T_c^{23,48,49}$. When these data were analyzed by assuming asymptotic scaling, the dimensionful, predicted σ and T_c decreased significantly as β was increased (instead of staying constant as any physical prediction should) (Table 2). On the other hand, a crude analysis based on Fig. 17 would predict consistently $\sigma^2 \sim 100 \Lambda^{\text{latt}}$ and $T_c \sim 57 \Lambda^{\text{latt}}$ independently of β for $\beta \gtrsim 5.6$.

Let me mention at the end that there are preliminary MCRG results for SU(2) gauge theory also^{44,50}.

CLOSING

With the help of different MCRG techniques the SU(3) Yang-Mills gauge theory seems to be increasingly under control. Reliable results can be expected on the β function and, as a consequence, on the string tension and critical temperature in the near future. There are results on $\Delta\beta(\beta)$ already in the presence of light quarks in SU(2)³⁶. MCRG is and will be a powerful tool in the study of QCD.

β	$\sigma^{\frac{1}{2}}$	β	T_c
5.6	$(144 \pm 3)_{\Delta}^{\text{latt}}$	5.55	$(86 \pm 1)_{\Delta}^{\text{latt}}$
5.7	$(135 \pm 4)_{\Delta}^{\text{latt}}$	5.70	$(76 \pm 1)_{\Delta}^{\text{latt}}$
5.8	$(113 \pm 1)_{\Delta}^{\text{latt}}$	5.79-5.82	$(69 \pm 1)_{\Delta}^{\text{latt}}$
6.0	$(105 \pm 1)_{\Delta}^{\text{latt}}$	$\left(\begin{array}{c} 5.92-5.94 \\ 5.88 \end{array} \right) ?$	$(66 \pm 1)_{\Delta}^{\text{latt}}$
$\left(\begin{array}{c} 6.2 \\ \vdots \\ \vdots \\ \vdots \end{array} \right)$	$\left(\begin{array}{c} (101 \pm 1)_{\Delta}^{\text{latt}} \\ ? \\ \vdots \\ \vdots \end{array} \right)$	$\left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right) ?$	

TABLE 2

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FIGURE CAPTIONS

- Fig. 1 A magnet as seen by the field theorist through his reducing glasses. As $T \rightarrow T_c$, the reducing factor increases.
- Fig. 2 An example of the blocking procedure in configuration space.
- Fig. 3 A typical behaviour in the neighbourhood of a fixed point is illustrated in a two-coupling constant space. Directions 1 and 2 are a relevant and an irrelevant eigendirection, respectively.
- Fig. 4 A fixed point with one relevant operator. In this example ($K_1, K_2 = K_3 = \dots = 0$) represents the standard Ising model, the point where the critical surface is crossed by the first axes is the critical point of the Ising model.
- Fig. 5 Flow lines in the neighbourhood of the Gaussian fixed point. β and γ are couplings of quadratic operators with higher derivatives.
- Fig. 6 Flow lines of small, non-Gaussian couplings in the neighbourhood of the Gaussian fixed point in $d = 4$. The couplings u_0^4 and u_0^6 belong to ϕ^4 and ϕ^6 , respectively. The Figure is schematic only, see Ref. 17).
- Fig. 7 The fixed point and flow lines in the neighbourhood of the Gaussian fixed point in $d = 4$ Yang-Mills theory.
- Fig. 8 Illustration for renormalizability arguments in a Yang-Mills theory.
- Fig. 9 Schematic behaviour of $\Delta\beta$ in $d = 4$, SU(3) gauge model at small and large β values, as derived by fixing the string tension.

- Fig. 10 A RG improved action might be defined as $(g^2(0, \infty), c_2^*, c_3^*, \dots)$, where $(g^2 = 0, c_2^*, c_3^*, \dots)$ is the fixed point of the given RG transformation.
- Fig. 11 Blocking in configuration space in a gauge theory. The $SU(N)$ matrix V_{AB} is the new block variable associated with the block link A-B.
- Fig. 12 The matching predictions of this Figure were obtained for the standard $O(N)$, $N \rightarrow \infty$, action using the simple block transformation in Eq. (61). $\Delta\beta(\beta)$ is the change of β resulting in reducing the correlation length by a factor of two. The exact result in the continuum limit is $\Delta\beta = (2\pi)^{-1} \ln 2$ (without corrections), $\xi(\beta)$ is the correlation length. The curve "128² vs 64², 4/3 blocking, NN" refers to a matching where the nearest-neighbour block correlation after 4 RG steps on a 128x128 lattice was compared to that after 3 RG steps on a 64x64 lattice, and so on.
- Fig. 13 The matching predictions for the standard $O(N)$, $N \rightarrow \infty$, model using optimized block transformations. The disagreement between the NN and DI matchings below $\beta \approx 0.5$ indicates that this region is outside the scaling region and no unique β function can be defined.
- Fig. 14 Matching predictions for the $O(N)_{N \rightarrow \infty}$ model obtained by using the improved ratio method.
- Fig. 15 Matching predictions for the $O(3)$ σ model as obtained by different MCRG techniques. The solid line is the two-loop perturbative prediction.
- Fig. 16 By using the MC data (crosses) for m.a at $\beta = 1.2$, the behaviour of m.a (β) can be predicted for the other β values (solid line) with the help of the β function obtained in the MCRG study.
- Fig. 17 The shift $\Delta\beta$ as a function of β obtained from one-loop improved ratios and from two different blocking procedures (A and B). Let us emphasize again: $\Delta\beta(\beta)$ reflects the behaviour of the β function between β and $\beta - \Delta\beta(\beta)$!

magnet

magnet under reducing glasses

↓ T approaches T_c

solid state physicist

field theorist

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16

Fig. 17