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DYNAMICAL IMPLICATIONS OF ANOMALIES IN
NON-LINEAR SIGMA MODELS

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ABSTRACT

We compute the anomalies of non-linear σ -models both for compact and non-compact manifolds. Integrated anomalies vanish for non-compact irreducible coset Kähler σ -models. We find that the anomalies are identical to the usual gauge anomalies of the auxiliary gauge fields, whenever these are expected to be generated dynamically. The σ -model Wess-Zumino action is also analyzed, and it is shown to vanish for irreducible coset Kähler manifolds (compact or non-compact). Implications for the low-energy realization of supersymmetric theories are discussed.

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1. - INTRODUCTION

Non-linear σ -models play an important role in the understanding of low-energy phenomena of gauge theories, since they have confinement and chiral symmetry breaking already built-in. These theories are of interest also for their own sake, mainly because gauge bosons are conjectured to be spontaneously generated in some of them. σ -models also appear naturally in supergravity theories.

Recently it was realized¹⁾ that when fermions are coupled to non-linear σ -models, one may obtain an anomalous theory. Since supersymmetric σ models necessarily contain fermions, the anomaly is an important tool in selecting physically meaningful theories. The problem of anomalies is well-known in gauge theories coupled to fermions, and it is therefore of interest to check whether the two kinds of anomalies are in any way related. This question is even more interesting in light of the conjectured spontaneous generation of dynamical gauge bosons. In the present paper, we study this question with the following conclusions (derived in Sections 2, 3 and analyzed in Section 4):

1) For the compact manifolds CP^N and $G_{p,p+q}$ the σ -model anomalies are identical to the gauge anomalies of the auxiliary fields which are introduced in order to implement the constraint, whenever these fields are expected to be generated dynamically. In particular, the conditions for anomaly cancellations are the same whether or not an auxiliary field formalism is used.

2) For irreducible non-compact coset manifolds of relevance for physics, i.e., those which are ghost-free, there is no anomaly (the Kähler irreducible coset manifolds which are non-compact all fall into this category).

3) We also present an example of a reducible non-compact Kähler manifold where no auxiliary fields are needed to impose the constraints and show that it is anomaly-free.

The term "anomaly" has been used in the above description quite loosely. The precise quantity under study is

$$K = \int_{S^d} A(x) dx \quad (1)$$

where our space-time is d -dimensional and $A(x)$ is the anomaly density. This integrated anomaly may vanish even for an anomalous theory, the simplest example

being QED with Weyl fermions in four dimensions. There the anomaly density is proportional to $\text{Tr } \tilde{F}F$ which vanishes when integrated over space-time due to the absence of $U(1)$ instantons. We will also specifically discuss $A(x)$ in some parts of the paper.

Anomalies which represent the obstruction for a global definition of $\det \not{D}$ on the manifold, and therefore for a meaningful theory will be referred to as "intrinsic anomalies". One may think of gauging some global symmetries of the theory and ending up with an anomalous theory, even if it makes perfect sense when these symmetries remain global. These global anomalies when integrated can be used for the construction of the Wess-Zumino term²⁾, and for what we shall call the " σ Wess-Zumino action" which represents the effect of global anomalies on the low-energy σ -model of the theory. We will show, extending the work of Nemeschansky and Rohm³⁾, that even though global anomalies exist in supersymmetric theories, their effect on the σ -model cannot be represented by a σ -WZ term in many interesting cases, both compact and non-compact. The difference between the supersymmetric case and the standard QCD derivation is discussed in detail.

The paper is organized as follows: in Section 2 we discuss the topology of some compact Kähler σ -models: CP^N and Grassmannian $G_{p,p+q}$. We compute their intrinsic anomalies, proving statement 1 above. Section 3 is devoted to anomalies on non-compact manifolds, irreducible and reducible. Sections 2 and 3 are rather technical, and the non-mathematically oriented reader can skip to Section 4, where the results are discussed and classified and their meaning concerning the question of dynamical generation of physical gauge vectors is elaborated on. In Section 5 we discuss the global anomalies and the Wess-Zumino term, stressing the new features that supersymmetry brings along.

2. - ANOMALIES IN COMPACT KÄHLER MANIFOLDS

We will now derive the explicit expressions for the integrated intrinsic anomalies of compact σ -models coupled to fermions (supersymmetrically or not). We demonstrate their equivalence to the anomalies of the auxiliary gauge connections which are used to implement the constraints. Our examples will be CP^N (pointing out the peculiarity of CP^1) and Grassmannian $G_{p,p+q}$ manifolds.

2.1 CP^N models⁴⁾⁻⁶⁾

The CP^N model is a theory of N complex scalar fields mapping a d -dimensional space-time into the N -dimensional complex projective space

$$\phi_i : S^d \rightarrow CP^N \quad (2)$$

CP^N is geometrically defined as the set of lines in C^{N+1} passing through the origin

$$CP^N = (C^{N+1} - \{0\}) / C \quad (3)$$

Co-ordinates cannot be globally defined on CP^N . We therefore define a covering of CP^N by open neighbourhoods U_k , the set of lines with $z_k \neq 0$. The co-ordinates U_k are:

$$\rho_i^{(k)}(p) = \left(\frac{z_0}{z_k}, \dots, \frac{z_N}{z_k} \right) \quad (4)$$

where $(z_0, \dots, z_N)$ is an arbitrary point of the line p . The transition functions on the overlaps are $\Phi_{jk} = z_k / z_j$. Let us now define the natural line fibre-bundle L on CP^N :

$$L_I = ((p, z); p \in CP^N, z \in p) \quad (5)$$

The transition functions of this fibre bundle are valued on C ; $L_I : C^{N+1} - \{0\} \rightarrow CP^N$. An alternative representation of CP^N is through the $U(1)$ bundle $L_{II} : S^{2N+1} \rightarrow CP^N$. This corresponds to the usual representation of CP^N in the physics literature in terms of $(N+1)$ complex fields satisfying $|\bar{z}z| = 1$.

In both cases the corresponding co-ordinates can be interpreted as sections of these line fibre bundles. Specifically $\phi_i(x)$ are sections of the pullback on S^d of the line fibre bundle on CP^N defined before.

Fermions $\psi_i(x)$ are coupled minimally to the spin connection of the manifold. They are defined on S^d and take values in a Clifford algebra $S^\pm$ ($\pm$ depending on chirality). The coupling of $\psi_i(x)$ with the fields $\phi_i(x)$ is obtained by the introduction of internal indices in such a way that $\psi_i(x)$ are sections of the tensorial product fibre bundle $S^\pm \otimes \phi^*(T(CP^N))$, where $\phi^*(T(CP^N))$ is the pullback of the tangent $T(CP^N)$ on S^d .

We will now study the anomaly using the family index theorem⁷⁾. Consider a two-parameter family $\phi_y(x)$ of the ϕ fields; $y \in S^2$. The sphere is parametrized by two hemispheres y^+ , y^- . Their overlapping region is the equator, on which we define a transition function

$$\begin{aligned} \phi_\theta^+(x) &= f(\theta, x) \phi_\theta^-(x) \\ f(\theta, x) : S^1 \times S^d &\rightarrow U(1) \end{aligned} \quad (6)$$

On the sphere we define the line fibre bundle $\text{det } \mathcal{D}(\phi)$. The effective action given by $\text{det } \mathcal{D}(\phi) = e^{\Gamma_{\text{eff}}(\phi)}$ is a section of this fibre bundle. On the equator this section will satisfy

$$\exp[\Gamma_{\text{eff}}(\phi_\theta^+)] = e^{i\theta K} \exp[\Gamma_{\text{eff}}(\phi_\theta^-)] \quad (7)$$

K is the integrated anomaly of Eq. (1), and is given here by the first Chern class of $\text{Det } \mathcal{D}(\phi)$. The family index theorem yields:

$$K = \int_{S^2 \times S^d} \text{ch } \hat{\phi}^*(\theta, \phi, x) [T(\mathbb{C}P^N)] \quad (8)$$

where $\hat{\phi}^*$ is the pullback of $T(\mathbb{C}P^N)$ on $S^2 \times S^d$. The set of functions $\phi_i(\theta, \phi, x)$ defines a section of a $U(1)$ fibre bundle on $S^2 \times S^d$. In order to get $K \neq 0$, $\phi_i(\theta, \phi, x)$ needs to be a section of a non-trivial $U(1)$ bundle on $S^2 \times S^d$. These non-trivial bundles are classified by $H^2(S^2 \times S^d; \mathbb{Z})$. Using the Kunneth theorem, we get

$$H^2(S^2 \times S^d; \mathbb{Z}) = [H^2(S^2; \mathbb{Z}) \otimes H^0(S^d; \mathbb{Z})] + [H^0(S^2; \mathbb{Z}) \otimes H^2(S^d; \mathbb{Z})] \quad (9)$$

For $d = 2$ we find

$$H^2(S^2 \times S^2; \mathbb{Z}) = H^2(S^2; \mathbb{Z}) \oplus H^2(S^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

The contribution to the Chern character on $S^2 \times S^2$ is given by

$$\int_{S^2} f d\theta, \wedge d\phi, \quad \int_{S^2} g dx_1 \wedge dx_2 \quad (10)$$

f and g are the pullback of the generators of the respective H^2 by the

application $\phi(\theta, \phi, x)$. For $\hat{\phi}^*(T(\mathbb{CP}^N))$ we take the Whitney sum yielding

$$K = (N+1) \int_{S^2} \{ d\theta_1 \wedge d\phi_1 \} \int_{S^2} g dx_1 \wedge dx_2 \quad (11)$$

If $N > 1$, we find for the Hopf family of instanton configurations used by Moore and Nelson¹⁾ $K = N+1$. The $N = 1$ case will be discussed in the following subsection.

For $d = 4$ we have

$$H^2(S^2 \times S^4; \mathbb{Z}) = H^2(S^2; \mathbb{Z}) \otimes H^0(S^4; \mathbb{Z}) = \mathbb{Z}$$

which implies that the first Chern class of a non-trivial $U(1)$ fibre bundle will be given by a two-form $f d\theta_1 \wedge d\phi_1$. The Chern character on $S^2 \times S^4$ corresponding to the anomaly will be given by products of three C_1 's, which vanishes by the antisymmetry of the exterior product. We conclude $K = 0$. This will be the case for any $d > 2$.

2.2 The $\mathbb{CP}^1$ model

The $\mathbb{CP}^1$ model [which is isomorphic to the $O(3)$ σ -model] is peculiar in two dimensions. Since the dimension of $T(\mathbb{CP}^1)$ is 2, the only characteristic class of $T(\mathbb{CP}^1)$ which can be pulled back on $S^2 \times S^2$ is C_1 . However, Eq. (10) implies that the non-trivial $U(1)$ bundle on $S^2 \times S^2$ is characterized by the integral of C_1^2 . The antisymmetry of the differential form leads to a vanishing Chern character for the pullback of $T(\mathbb{CP}^1)$ on $S^2 \times S^2$. The model has therefore no intrinsic anomaly.

2.3 Grassmannian $G_{p,p+q}$ models

$G_{p,p+q}$ is geometrically defined as the set of p planes in $\mathbb{C}^{p+q}$ passing through the origin. If we define the operation of an element of $U(p) \times U(q)$ on a point of $G_{p,p+q}$ by $(\phi, \gamma)(\alpha) = \phi(\alpha)$ with $\alpha \in G_{p,p+q}$, $\phi \in U(p)$, $\gamma \in U(q)$, we can identify $G_{p,p+q}$ with the coset space $U(p+q)/U(p) \times U(q)$. One can alternatively define $G_{p,p+q}$ by the co-ordinates of the orthogonal q planes. From this it follows that $G_{p,p+q}$ and $G_{q,p+q}$ are isomorphic.

The co-ordinates $\phi^{(p)}$ of p -planes in $\mathbb{C}^{p+q}$ are given by a set of p linearly

independent vectors in $\mathbb{C}^{p+q}$. $\phi^{(q)}$ will be the co-ordinates of the orthogonal plane. They are sections of the natural fibre bundle ξ^p, ξ^q . The transition functions of ξ^p, ξ^q are elements of $U(p), U(q)$ respectively.

Fermions are defined as sections of the fibre bundle $S^\pm \otimes \phi^{(p)*}(\xi^p) \otimes \phi^{(q)*}(\xi^q)$ which has dimension pq . The transition functions of the fibre are elements of $U(p) \times U(q)$. We now study the anomaly following the lines of the CP^N analysis. Define a two-parameter family of $\phi^{(p)}$ as $\phi^{(p)}(x, \theta, \phi)$ and look for non-trivial $U(p), U(q)$ fibre bundles on $S^2 \times S^d$, using the property $ch(\alpha \otimes \beta) = ch\alpha \cdot ch\beta$ for the Chern character of the tensorial product of the two fibre bundles. Notice that the fact that $\Pi_5(U(k)) \neq 0$ for $k \geq 3$ will allow us to have a topologically non-trivial $f(\theta, x)$ as defined in Eq. (7). The Kunneth theorem for $d = 4$ reads:

$$\begin{aligned} H^6(S^2 \times S^4; \mathbb{Z}) &= H^2(S^2; \mathbb{Z}) \otimes H^4(S^4; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \\ H^2(S^2 \times S^4; \mathbb{Z}) &= H^2(S^2; \mathbb{Z}) \otimes H^0(S^4; \mathbb{Z}) = \mathbb{Z} \end{aligned} \quad (12)$$

In the latter case the characteristic classes depend only on the co-ordinates of the two-sphere and do not contribute to the Chern character on $S^2 \times S^4$. The forms in H^6 are

$$C_3 = \int d\theta \wedge d\phi \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \quad (13)$$

and we find for the anomaly

$$\begin{aligned} K &= \int_{S^d} A(x) dx = \int_{S^2 \times S^d} ch \left[\hat{\phi}^{(p)*}(\xi^p) \otimes \hat{\phi}^{(q)*}(\xi^q) \right] = \\ &= \int_{S^2 \times S^d} ch \left[\hat{\phi}^{(p)*}(\xi^p) \right] ch \left[\hat{\phi}^{(q)*}(\xi^q) \right] = [p + \int_{S^2} C_3(\xi^p)] [q + \int_{S^4} C_3(\xi^q)] \end{aligned} \quad (14)$$

for local $U(p)$ and global $U(q)$ we get for example $q \int_{S^2 \times S^4} C_3(\xi^p)$. When integrated over the (θ, ϕ) variables of S^2 , this gives the usual non-Abelian integrated anomaly for a $U(p)$ gauge field, which is non-vanishing when $\Pi_5(U(p)) \neq 0$.

3. - ANOMALIES IN NON-COMPACT KÄHLER MANIFOLDS

3.1 The irreducible case

Non-compact manifolds are locally indistinguishable from their compact analogues. Their characteristic classes will therefore be locally the same in both cases. However, the integrals of these characteristic classes with respect to cycles of the group manifold may be different due to the different global topology. In fact we will show that they vanish for the case at hand. We demonstrate the topology of non-compact manifolds by the example $G = U(p, q)$. Topology then tells us⁸⁾ that non-vanishing integrals of characteristic classes are always given by the cohomology group of the maximal compact subgroup, which in our case is $U(p) \times U(q)$. For irreducible coset non-compact Kähler manifolds, G/H , H is the maximal compact subgroup of G (see discussion in Section 5). In the example we are considering, the resulting manifold is the non-compact analogue of $G_{p, p+q}$.

Specifically, the integrated anomaly is given by an integral on $S^2 \times S^d$ of the pullback of some characteristic class of our Kähler manifold. Using the properties of the pullback we get

$$\int_{S^2 \times S^d} \hat{\phi}^* (c_i) = \int_{\phi(S^2 \times S^d)} c_i \quad (15)$$

where $c_i \in H^{2i}(M)$ and $M = G/H$ is the Kähler manifold. ϕ is the field configuration defined from space-time into the manifold M . The integral necessarily vanishes if M has no compact submanifold. We conclude that there is no integrated anomaly.

3.2 The reducible case

As a generic example consider

$$\frac{SL(N_f, C)_L \times SL(N_f, C)_R}{SL(N_f, C)_V} \sim SL(N_f, C) \quad (16)$$

This manifold is of interest for supersymmetric QCD and will be discussed further in Section 5. This manifold contains compact submanifolds, the maximal being $SU(N_f)$. The spin connection for a σ -model is given by the one-form $w_\mu = g^{-1} \partial_\mu g$,

$g(x)$ is an application from space-time into the group $SL(N_f, C)$. w_μ is a pure gauge configuration and therefore the curvature vanishes identically. The manifold is therefore free of intrinsic anomalies.

4. - PHYSICAL INTERPRETATION OF THE ANOMALIES

In the preceding sections we have defined the scalars as living directly on the Kähler manifold and the fermions as minimally coupled to the spin connection of the manifold. The formulation which is more prevailing in the physics literature employs unconstrained scalar fields and auxiliary gauge fields which couple as Lagrange multipliers to impose the constraints. It is believed that at least in some cases these fields become dynamical degrees of freedom, namely their propagators develop poles at $k^2 = 0$. In two dimensions the gauge fields are not dynamical, however they lead to the confinement of charge⁶⁾. In four dimensions the situation is more complex, mainly because non-linear σ -models are non-renormalizable there. Unless the theory happens to be finite, one needs to introduce a cut-off on which physical quantities will depend. The problem of intrinsic anomalies is believed to be more severe, however, since it involves breakdown of unitarity.

Explicit computations support the conjecture that the auxiliary fields become dynamical in compact σ -models, but infra-red divergences prevent this from happening in non-compact models⁹⁾, unless a mass gap is introduced by some variation of the model¹⁰⁾.

We would now like to see what anomalies can tell us about this question. In the CP^N case ($N > 1$) discussed in Section 2, we have found intrinsic anomalies for $d = 2$. The integrated anomaly $K = N+1$ is precisely the instanton induced phase on the determinant of $N+1$ Weyl fermions coupled to the scalars (supersymmetric σ -models in two dimensions have real fermionic representations and are hence non-anomalous). The anomaly is therefore identical to that of the auxiliary $U(1)$ field which implements the constraint in the usual formulation. The peculiarity of the CP^1 model is however quite meaningful. Instantons exist also in CP^1 ⁶⁾ and therefore if one employs the auxiliary field formalism naively one would conclude that this model is anomalous too. However, $CP^1 \sim O(3)$ σ -model where the fields live on S^2 with no local $U(1)$ symmetry. One therefore does not expect spontaneous generation of a gauge field in any number of dimensions. It is encouraging to see that the model indeed exhibits no anomalies.

In four space-time dimensions we have found no integrated anomaly for the CP^N model. This is again the same result we would get from the auxiliary field formalism, since $\int \text{Tr } \tilde{F}F = 0$ for $U(1)$ gauge fields. Unfortunately, our formalism does not allow us to detect the anomaly density which, if non-vanishing, is enough to spoil gauge invariance. It is amusing to note that the auxiliary field formulation of this problem¹⁾ does give a vanishing anomaly density for CP^1 since CP^1 satisfies^{*)}

$$0 = \epsilon_{\mu\nu\alpha\beta} (\partial_\mu Z^\dagger \partial_\nu Z) (\partial_\alpha Z^\dagger \partial_\beta Z)$$

For the $G_{p,p+q}$ models, the anomaly we found is identical to that of the auxiliary fields, which are presumably generated dynamically. This is the non-Abelian anomaly, recently discussed in the physics literature from a topological point of view⁷⁾. Assuming that this property is true for all the irreducible Kähler manifolds, we can classify the $d = 4$ anomalies we expect to get on each one of these manifolds, as classified by Calabi and Vesentini¹¹⁾:

$$\text{Type I}_{m,m'} = U(m+m')/(U(m) \times U(m')) = G_{m,m+m'}$$

non-Abelian anomaly for $m(m') > 3$, only Abelian otherwise.

$$\text{Type II}_m = SO(2m)/U(m)$$

non-Abelian anomaly $m > 3$, Abelian otherwise.

$$\text{Type III}_m = Sp(m)/U(m); \text{ same as Type II}_m.$$

Type IV_m = $SO(m+2)/(SO(m) \times SO(2))$ only Abelian anomalies. Even though for $m = 6$, $\Pi_5(SO(6)) = \mathbb{Z}$, the fermionic representation is real under the non-Abelian symmetry.

$$\text{Type V} = E_6/(Spin(10) \times SO(2)) \text{ only Abelian anomaly } (\Pi_5(spin(10)) = 0).$$

$$\text{Type VI} = E_7/(E_6 \times SO(2)) \text{ only Abelian anomaly } (\Pi_5(E_6) = 0).$$

The non-compact case is more puzzling. We have found that the integrated anomaly vanishes on the non-compact irreducible Kähler manifolds. We do not know

*) This was first pointed out to us by J. Ellis.

what this implies for the anomaly density. Notice that even in the auxiliary field formalism the computation of the anomaly is non-trivial due to the appearance of negative norm modes for bosons and in the supersymmetric case, also for fermions (see discussion in the next section). A manifestly ghost-free formulation is gotten only after the constraints are imposed. In any case, the vanishing integrated anomaly may be telling us that the theory is well defined but no physical gauge bosons are generated dynamically. This was the case for the CP^1 model, and this is also the case for our reducible non-compact example $SL(N_f, C)$. It is also possible that an anomaly-free subgroup of H is generated, or perhaps an integrated anomaly-free subgroup ($U(1)$). Obviously dynamical computations are needed in order to settle this question.

The relation of the anomaly, which is a topological number, to a kinetic term for a gauge field should come as no surprise. The fact that a surface integration is non-vanishing shows that the theory has long-range correlations. We demonstrate this property with two examples:

Consider first the Dirac magnetic monopole. The magnetic charge is the integral of the first Chern class $\int_{S^2} F$. Stoke's theorem then relates the divergence of $B_\mu = \epsilon_{\mu\nu\rho} F^{\nu\rho}$ to the source of the magnetic charge. Starting from $\int_{S^2} F \neq 0$ we therefore realize that the system is governed by an underlying Maxwell equation, which in turn necessitates a kinetic term for the gauge field.

As a second example, consider the implication of $\int_{S^4} \text{Tr } F\tilde{F} \neq 0$. As is well known from instanton theory¹²⁾, this implies a pole in the propagator of the Chern-Simons form K^μ . In this case, however, we have no direct information on the field A_μ . K^μ is a composite operator in the theory.

The object of interest for the $d = 4$ anomaly is the Chern-Simons density leading to $\int_{S^2 \times S^4} \text{Tr } (F \wedge F \wedge F) \neq 0$. If such an intrinsic anomaly exists, the theory has first to be made sensefull by the addition of fermions to cancel the anomaly. Once this is done, the theory probably possesses propagating gauge bosons. If no integrated intrinsic anomaly exists we can say nothing about such generation. In fact, we are perfectly consistent if it does not occur.

5. - THE WESS-ZUMINO TERM

Once the flavour anomalies of a theory are known they can be integrated (if they satisfy appropriate consistency conditions) to give the so-called Wess-Zumino term $W(g,A)^{2)}$, which represents the phase by which the generating functional of the theory changes under an element g of the global symmetry group G in the presence of external fields coupled to its generators. W depends non-trivially on any g related with a non-vanishing anomaly. As the external fields are taken to zero, $W(g) \equiv W(g,0)$ may vanish even for a theory with flavour anomalies. If it does not vanish one says that the Wess-Zumino term "splits".

In the standard derivation, the underlying theory is QCD with N_f flavours. $W(g)$ depends on the axial generators ξ only, since the vector flavour symmetries are non-anomalous. In fact²⁾:

$$W(\xi, V_\mu, A_\mu) = \int_0^1 dt \exp \left[-t \int \xi(x) U(x) dx \right] \int \xi(x) A(x) dx \quad (17)$$

where x is integrated over space-time, $A(x)$ is Bardeen's anomaly density, and $U(x)$ is the part of the variation of the vacuum functional under an infinitesimal gauge transformation, involving only the external vectorial (V_μ) and axial (A_μ) fields.

One now makes the additional assumption that QCD is realized at low energies as an $SU(N_f)_L \times SU(N_f)_R / SU(N_f)_V$ σ -model. This is the theory of Goldstone bosons predicted if only the vectorial $U(N_f)_V$ is realized linearly in the spectrum. We stress that this is an assumption about the dynamics of the theory, supported by increasing theoretical evidence^{13),14)}, and obviously by the observed hadronic spectrum. The axial generators ξ now couple to the σ -model fields, and $W(\xi)$ can be represented in the σ -model as an additional term in the action. This action was elegantly derived by Witten¹⁵⁾ as an integral of a five-form. The integral is over a five-dimensional manifold of which space-time is the boundary. Note, however, that one may guess a different realization of the global symmetry G , possibly a partial realization of chiral symmetry or perhaps a breakdown of a subgroup of $SU(N_f)_V$. In each case one can attempt to write a term in the corresponding σ -model representing the anomalous current algebra of the underlying theory. We name this term "the σ -Wess-Zumino (σ -WZ) action" and it is this term that we now seek for supersymmetric σ -models.

$W(g,A)$ for supersymmetric underlying theories may in principle be derived from the explicit form of the flavour anomalies¹⁶⁾, but this seems a hard task at

present due to the complexity of the expressions. The question of whether or not a σ -WZ action can be written down turns out to be much easier, once the σ -model manifold is known. The main problem for supersymmetric gauge theories, however, is that their dynamics are not yet well understood and therefore it is not in general known what the appropriate low-energy theory is. As an example for this difficulty, we discuss supersymmetric QCD (SQCD), which is the supersymmetric extension of ordinary QCD, with N_f flavours and N_c colours.

We shall not write out the Lagrangian explicitly, since we are here only interested in its symmetry properties¹⁷⁾. The non-anomalous global symmetry of the theory is given by

$$G_1 = SU(N_f)_L \times SU(N_f)_R \times U_V(1) \times U_R(1) \quad (18)$$

where $U_R(1)$ is the non-anomalous R symmetry.

Supersymmetric gauge theories, however, have a remarkable property. The scalar potential of the theory, at least at tree level, vanishes over a large manifold of fields. Let us denote by G_2 the group composed of all the global transformations g_2 with the following property: if $V_{\text{tree}}(\phi) = 0$ then $V_{\text{tree}}(g_2\phi) = 0$. It turns out that G_2 is the complexification of G_1 ¹⁸⁾, together with over-all scale transformations for left and right handed fields

$$G_2 = GL(N_f)_L \times GL(N_f)_R \times \bar{U}_R(1) \quad (19)$$

where $\bar{U}(1)$ is the group of transformations of the form e^α , α being any complex number. Note that G_2 is not a symmetry of $\mathcal{L}$; in particular it is broken by the kinetic terms. The non-renormalization theorem ensures that g_2 can be defined in terms of the potential to all orders of perturbation theory.

Let us now assume that in the vacuum a function of the fields transforming non-trivially under G_2 condenses, leaving a subset H_2 unbroken [in the sense that for $h_2 \in H_2$, $V_{\text{tree}}(h_2\phi) = 0$ when $V_{\text{tree}}(\phi) = 0$]. We expect that the zero mass scalar sector of the theory will contain the generators of the broken symmetries, namely they will live on the manifold G_2/H_2 . Notice that G_2 is non-compact. This manifold contains the Goldstone bosons of the broken symmetries of G_1 and their scalar partners sharing the same superfield.

One can imagine that some flat directions of the scalar potential are lifted by non-perturbative effects¹⁹⁾. In this case a generator of G_2 transforming the

fields along these directions will not correspond to a massless scalar (unless the second derivative of the potential vanishes at the minimum). Replacing V_{tree} by V_{eff} which includes the non-perturbative effects, we therefore define G'_2 and H'_2 as before. The massless scalars will now live on G'_2/H'_2 . Notice that if all the flat directions are lifted, G'_2 will be compact.

We have so far not imposed supersymmetry. If supersymmetry is not broken dynamically, the resulting σ -model has to be a Kähler manifold. If no additional massless scalars exist, G'_2/H'_2 must be Kähler. If such scalars are tolerated (they are not required by any symmetry), one should look for a Kähler manifold M containing G'_2/H'_2 .

Another important constraint on the possible realization of global symmetries comes from unitarity. When the non-compact symmetry of the flat potential is realized in a σ -model, it translates into negative kinetic terms for the σ -fields, which is in general inconsistent with unitarity. However, some non-linear non-compact σ -models are unitary. For example, if G'_2 is semi-simple it is known that ghosts are avoided if one divides G'_2 by its maximal compact subgroup²⁰⁾. It is amusing to note that all irreducible non-compact coset Kähler manifolds are automatically ghost-free. For reducible Kähler manifolds unitarity imposes an independent constraint.

The manifold of the σ -model (on which the σ -WZ action will be defined) is therefore G'_2/H'_2 (or larger). What is the manifold on which the Wess-Zumino term $W(g,A)$ is defined ?

When the global symmetry (except for the R symmetry which cannot be gauged when supersymmetry is non-local^{*)}) is gauged, the complexified symmetry $G'_1 = SL(N_f)_L \times SL(N_f)_R \times \bar{U}_V(1)$ becomes classically a symmetry of the full Lagrangian of the theory. A part of this classical symmetry may be broken by anomalies, leaving a non-anomalous subgroup to be denoted by H_{na} . $W(g,A)$ will depend non-trivially on transformations belonging to G'_1/H_{na} . Notice that this manifold is in general different from that of the σ -model term. Non-supersymmetric QCD is very special for the identity of the two manifolds. We assume, however, that some group of the G'_1/H_{na} elements (namely global symmetries which become anomalous when gauged) belongs also to the manifold M so that we can have on the Kähler manifold a representation of the flavour anomalies of the underlying

^{*)}We thank S. Ferrara for teaching us that.

theory. The set of these transformations will be denoted by G_a . Let us pick one such element g_a . The σ -WZ term will have to represent for us the non-trivial phase that the functional integral picks when g_a is varied on a two-parameter family of G_a configurations. With space-time compactified to S^4 these are two parameter families of $\Omega_3(G_a) = \{S^3 \rightarrow G_a\}^{21}$. Intuitively S^3 here is R^4 space-time infinity, where the gauge configuration becomes pure gauge. This allows us to represent the family as a five-dimensional surface in G_a . A non-trivial σ -WZ action will therefore exist on the G_a manifold if $H_5(G_a, R) \neq 0$. In our case G_a is embedded in a larger manifold. It is therefore possible that non-contractible five-dimensional surfaces in G_a become contractible in M . In that case no σ -WZ action can be defined. This is analogous to the magnetic monopole which loses its Dirac string when $U(1)_{em}$ is embedded in a non-Abelian group. We therefore require $H_5(M) \neq 0$ where M is our Kähler manifold.

We now prove, extending the work of Nemeschansky and Rohm³⁾, that the σ -WZ term must vanish in many cases of interest. As noted in Ref. 3), the σ -WZ action must be an integral of a symmetric five-form over the five-dimensional manifold, and will therefore vanish if the fifth De-Rahm cohomology group H^5 of the manifold is trivial. This is the case for all the coset irreducible compact Kähler manifolds (in general the odd cohomology groups are trivial on these manifolds).

What happens in the non-compact case? The condition of unitarity (or equivalently Kähler) implies that for irreducible spaces G/H , H must be the maximal compact subgroup of G . The non-trivial topology of G is completely represented by the cohomology groups of H (see Section 3). Since we are considering equivalence classes with respect to H , we are left with a topologically trivial G/H . In particular H^5 is trivial and there is no σ -WZ action. We stress again that this does not mean that there are no global anomalies in the underlying theory. In order to understand this better, let us look closer at ordinary (non-supersymmetric) QCD. The anomalies we have been considering so far were the $SU(N_f)_L \times SU(N_f)_R$ anomalies. However, the full global symmetry group includes the baryon number $U_V(1)$. This introduces new anomalies. The triangle graph with two chiral $SU(N_f)$ currents and one $U(1)$ current is non-vanishing. Since the $U_V(1)$ factor is not in the σ -model manifold (baryon number is assumed to be realized linearly), the σ -WZ term cannot represent these anomalies. The simplest example for this is $N_f = 2$ where there is no σ -WZ action but a flavour anomaly involving the $U_V(1)$ current does exist.

Anomalies of global symmetries and therefore a Wess-Zumino term (but no σ -WZ action) will exist in general also directly in the overlying theory, namely the supersymmetric σ -model. This is due to the inevitable presence of fermions coupled to the manifold. Even for a theory free of intrinsic anomalies, the gauging of G with external fields will cause the fermion determinant to transform non-trivially under the global symmetries.

In order to shed some more light on the non-existence of σ -WZ terms on irreducible Kähler manifolds, let us try to derive such a term along the lines of the D'Hoker and Farhi construction²²⁾. There one begins with an anomaly-free theory. A subset of the fermionic degrees of freedom then decouples using large Yukawa couplings to scalars of the low-lying spectrum. This generates σ -WZ and Goldstone-Wilczek interactions in terms of the scalars and gauge bosons. The procedure can be implemented even when the remaining set of Fermi fields is by itself anomalous. These terms become local as the decoupled fields become infinitely massive. For supersymmetric σ -models we can think of the following analogue: we start with a supersymmetric G/H model which may or may not be anomalous and couple new superfields to it such that the resulting theory is anomaly-free. We then wish to decouple the added superfields by increasing their couplings to low-energy scalar fields which develop vacuum expectation values. However, the scalars transform always non-trivially under the group H which we wish to keep unbroken, and consequently cannot develop v.e.v.'s. The additional superfields will therefore remain massless and integrating over them will not result in a local action term. We conclude that a construction of a σ -WZ action along the d'Hoker and Farhi lines must fail.

The situation for reducible non-compact manifolds is quite different. Take for example the Kähler manifold

$$\frac{SL(N_f, C)_L \times SL(N_f, C)_R}{SL(N_f, C)_V} \sim SL(N_f, C) .$$

As we have seen in Section 3.2, that manifold is non-anomalous and there is no spontaneous generation of gauge fields. However, the manifold does contain compact sub-manifolds [the maximal being $SU(N_f)$]. This generates a non-trivial H^5 and a possible σ -WZ action³⁾. $SL(N_f, C)$, however, does not correspond to a unitarity σ -model.

6. - CONCLUSIONS

We have seen that the anomalies from which non-linear σ -models with fermions suffer, can be used as a tool in the understanding of dynamical generation of gauge bosons in such models. The anomalies can be studied even in a formulation without auxiliary fields, and they do not vanish only in theories where gauge bosons are expected to be generated dynamically. Our result is similar in spirit to t'Hooft's anomaly consistency conditions¹³⁾, in the sense that the anomalies teach us about the dynamical realization of the theory. We can formulate the following rule which we have not proven, but which is consistent with all our results:

A set of auxiliary gauge fields in the formulation of a non-linear σ -model can become dynamical only if it leads to the same intrinsic anomalies deduced from the σ -model itself when fermions are coupled to the theory.

If the fermion representation is chosen to be anomaly-free (which usually requires an extension of the minimal model), the resulting theory will be perfectly consistent. From the point of view of gauge boson generation, it is rather discouraging that the intrinsic integrated anomaly vanishes in non-compact Kähler manifolds. It is still possible that an integrated anomaly free subgroup of gauge fields is generated [for example, Abelian or SU(2) factors]. The common topological features of gauge theories and σ -models certainly deserve further study.

Requiring that a σ -model be an irreducible symmetric Kähler manifold (compact or non-compact) immediately prevents the construction of a σ -WZ action, which will describe the effects of global anomalies in any underlying theory leading to the σ -model. Fortunately, the global symmetry group of typical supersymmetric underlying theories (like SQCD) is a product of semi-simple factors and therefore a σ -WZ action may exist. The flat directions of the scalar potential in these theories will generally lead to a non-compact σ -model and one has to make sure that the σ -model is indeed ghost-free. The requirement that the effective low-energy model be a unitary Kähler manifold (if supersymmetry is unbroken) is quite stringent and may give us useful information in analyzing its spectrum, and in determining the non-perturbative fate of the flat directions.

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REFERENCES

- 1) G. Moore and P. Nelson, Phys. Rev. Lett. 53 (1984) 1519.
- 2) J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95.
- 3) D. Nemeschansky and R. Rohm, Princeton Univ. preprint PRE-27754 (1984).
- 4) T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys. Rep. 66 (1980) 213;
S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. II
(John Wiley & Sons, 1969).
- 5) E. Witten, Phys. Rev. D16 (1977) 2991;
E. Cremmer and J. Scherk, Phys. Lett. 74B (1978) 341;
A. D'Adda, P. di Vecchia and M. Lüscher, Nucl. Phys. B152 (1979) 125.
- 6) A. D'Adda, P. di Vecchia and M. Lüscher, Nucl. Phys. 146 (1978) 63.
- 7) M.F. Atiyah and I.M. Singer, Proc. Natl. Acad. Sci. USA81 (1984) 2597;
C. Gomez, Salamanca Univ. preprint DFTUS 06183 (1983);
L. Alvarez-Gaumé and P. Ginsparg, Harvard preprint HUTP-83/A081 (1983);
O. Alvarez, I.M. Singer and B. Zumino, Berkeley Univ. preprint UCB-PTH-84/9
(1984);
R. Stora, Cargèse Lectures, LAPP-TH.94 (1983).
- 8) B. Dubrovine, S. Norikov and A. Fomenko, Géométrie Contemporaine, Vol. II
(1982).
- 9) A.C. Davis, A.J. Macfarlane and J.W. van Holten, Nucl. Phys. B232 (1984)
473.
- 10) J.W. van Holten, Phys. Lett. 135B (1984) 427.
- 11) E. Calabi and E. Vesentini, Annals of Math. 71 (1960) 472.
- 12) E. Witten, Nucl. Phys. B156 (1979) 269;
G. Veneziano, Nucl. Phys. B159 (1979) 213.
- 13) G. 't Hooft in Recent Developments in Gauge Theories, eds. G. 't Hooft et
al. (Plenum Press, New York, 1980).
- 14) C. Vata and E. Witten, Nucl. Phys. B234 (1984) 173;
E. Cohen and C. Gomez, Harvard preprint HUTP-83/A026 (1983).
- 15) E. Witten, Nucl. Phys. B223 (1983) 422.
- 16) N.K. Nielsen, Nucl. Phys. B244 (1984) 499.
- 17) T.R. Taylor, G. Veneziano and S. Yankielowicz, Nucl. Phys. B218 (1983) 493.
- 18) B. Ovrut and J. Wess, Phys. Rev. D25 (1982) 409.
- 19) I. Affleck, M. Dine and N. Seiberg, IAS preprint (Nov. 1983); Phys. Rev.
Lett. 52 (1984) 1677;
G.C. Rossi and G. Veneziano, Phys. Lett. 138B (1984) 195;
Y. Meurice and G. Veneziano, Phys. Lett. 141B (1984) 69.

- 20) E. Cremmer and B. Julia, Nucl. Phys. B159 (1979) 141.
- 21) M.F. Atiyah and J.D.S. Jones, Comm. Math. Phys. 61 (1978) 97.
- 22) E. D'Hoker and E. Farhi, MIT-CTP 1165;1166 (1984).
- 23) P. di Vecchia, S. Ferrara and L. Girardello, CERN preprint TH-4024/84 (1984),
to appear.