HADRON COLLIDER BEHAVIOR IN THE NONLINEAR NUMERICAL MODEL EVOL

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EVOL is a tracking program including sextupoles, multiple beam–beam collisions, external tune modulation, and other effects. Its construction emphasizes operational speed and the nested scanning of two or three configuration variables, such as betatron tune, amplitude and chromaticity, at the expense of simplifications in the physical model. After describing EVOL, simulation results are compared as quantitatively as possible with SPS operating experience, and with the Chirikov–Courant theory of chaotic resonance overlap in tune-modulated beam–beam collisions.

The tune-modulated sextupole dynamic aperture is found responsible for increased SPS proton backgrounds near the 2/3 resonance. Simulations agree excellently with the theoretical chaotic beam-beam limits for single collisions and for “coherent” multiple collisions, but are more stable than expected for “incoherent” multiple collisions.

Sextupolar amplitude-dependent tune shifts are shown to aggravate the SPS chaotic behavior, but can have a stabilizing influence. While constant collision offsets and oscillating dispersive offsets are found to produce comparable odd resonances, SPS odd resonances are ascribed to proton–antiproton orbit separations.

Independent one-dimensional simulations in both transverse planes are shown to closely reproduce resonances found in an SPS scan. Moderate nonresonant oscillations in a second dimension are shown to have little influence on chaotic behavior in the first dimension, but interpretive problems in the bi-resonant case are illustrated.

INTRODUCTION

Tracking programs have established their usefulness most clearly in the evaluation of sextupole dynamic apertures in existent or proposed accelerators. More recently, tracking studies of the response of colliders to beam–beam collisions, either for “flat” (electron, radiatively damped) or “round” (proton, non-radiative) beams, have appeared. EVOL is a FORTRAN program approximately 1000 lines long which offers the choice of studying sextupoles and beam–beam collisions separately, or both together. EVOL implicitly assumes that the presence of betatron tune modulation, from whatever source, is an essential ingredient in a useful and realistic model of nonlinear effects. Tune modulation has already been found to have a fundamental role in real and simulated experiments at the SPS with the “nonlinear lens”. Tune modulation is also one of the foundations of a theoretical description of round beam–beam collisions that is in good quantitative agreement with observed beam–beam limits at the SPS.

EVOL was originally written at CERN to simulate nonlinearities in the SPS
FIGURE 1  A schematic overview of EVOL model options.
collider. It is now being used and developed further, in round and flat beam versions, at the Cornell Electron Storage Ring, CESR. Single particles are tracked for many turns, for example $10^5$, around a nonlinear lattice in the presence of a set of physical effects chosen by the user from a "library". These effects interact with each other strongly or weakly, in ways that are theoretically understood to a greater or lesser degree. The particular library available in the SPS version of EVOL is shown in Fig. 1, together with enhancements that are either available or potentially available in the CESR versions.

Figure 2 illustrates how the behavior of an array of particles (trajectories) is studied by making a two or three dimensional scan of dynamical variables. The example shows superficially how the dynamic aperture might be found over a range of horizontal tunes, $Q_{x0}$, for four different tune-modulation amplitudes, $q$. The inner DO loop decreases the initial horizontal amplitude, $a_{xi}$, until 5 particles have been found that do not hit an aperture limit during 200 synchrotron periods. Note that initial-condition variables (e.g., $a_{xi}$) and lattice variables (e.g., $Q_{x0}$) are treated on the same footing. All DO loop indices, etc., are read in from a command file which is edited between runs. Thus, for example, the third-dimensional scan of tune-modulation amplitude could be turned off by changing the DO loop increment from 0.001 to 0.01.

Speed of operation is heavily emphasized in the EVOL code, since productivity increases at least in proportion to speed if truly interactive computers are available. A typical working session then initially has several brief runs with intensive user interaction, followed by a few long runs with minimal user interaction, perhaps overnight. The early short runs are essential in choosing interesting long-run parameters, so that relevant and useful results are produced.

Some of the speed and flexibility of EVOL is achieved, quite deliberately, at the cost of making simplifying assumptions in its modeling of real colliders.

![Flowchart](https://example.com/flowchart.png)

**FIGURE 2** A typical three-dimensional scan for dynamic aperture.
Hence, for example, all sextupoles are thin, and $\beta$ does not depend on energy. In what follows below, various details and simplifications of the mathematical models used in EVOL are described. Then the results of various EVOL simulations of the SPS are presented. Where possible, these results are compared with experimental data and with the analytical models currently in use at the SPS.

THE EVOL MODEL

In the SPS version of EVOL, one trajectory at a time is followed in 2 (or 1) transverse dimensions through a lattice which has 108 (or 0) sextupoles and 6 (or 0) round beam–beam collisions, with no radiation excitation or damping. Betatron tune modulation is introduced as a polynomial of an external quantity $\delta$, which oscillates sinusoidally at a frequency $Q_s$.

$$\delta = \delta_0 \cos (2\pi Q_s t)$$  \hspace{1cm} (1)

Usually $\delta$ is thought of as $\delta E/E$, the off-energy parameter of a particle, so that if $K_z$ is the (linear) vertical chromaticity, then the vertical betatron tune on turn $t$ is

$$Q_z = Q_{z0} + K_z \delta_0 \cos (2\pi Q_s t)$$

$$= Q_{z0} + \hat{q}_z \cos (2\pi Q_s t),$$  \hspace{1cm} (2)

where $\hat{q}_z$ is the tune-modulation depth. However, $Q_s$ is the frequency of ANY parametric tune modulation source under study. It will be seen later that smaller frequencies are more dangerous,\(^{10}\) so that power-supply ripple at frequencies around a few Hertz, for example, is well worth studying. Nonetheless, in all the simulation experiments presented below, $Q_s$ has the nominal value of $1/194 = 5.15 \cdot 10^{-3}$, close to the synchrotron frequency of the SPS. This is appropriate for tune modulation through chromaticity or, for round beams, through longitudinal collision-point oscillations.\(^8\)

The betatron motion between consecutive thin nonlinearities, (sextupoles or beam–beam collisions), is assumed to be completely linear, though chromatic. Consequently, while the motion could be described in terms of the physical displacements, $X$, $X'$, $Z$ and $Z'$, it is far more useful to work in normalized coordinates $X_n$, $X'_n$, $Z_n$, and $Z'_n$,

$$\begin{pmatrix} x_m \\ x'_m \end{pmatrix} = \begin{pmatrix} \beta_x^{-1/2} & 0 \\ \alpha_x^{-1/2} \beta_x^{-1/2} \end{pmatrix} \begin{pmatrix} X \\ X' \end{pmatrix},$$  \hspace{1cm} (3)

since then the linear tracking between nonlinear elements $i-1$ and $i$ is performed by a rotation matrix.

$$\begin{pmatrix} x_n \\ x'_n \end{pmatrix}_i = \begin{pmatrix} \cos (\Delta \phi_x) & \sin (\Delta \phi_x) \\ -\sin (\Delta \phi_x) & \cos (\Delta \phi_x) \end{pmatrix} \begin{pmatrix} x_n \\ x'_n \end{pmatrix}_{i-1}$$

$$\Delta \phi_x = \phi_{x,i} - \phi_{x,i-1}.$$  \hspace{1cm} (4)

Normalized displacements, amplitudes and angles all have units of (millimeters)$^{1/2}$ so that, for example, a typical normalized beam size in the SPS is $\sigma = \ldots$
0.005 (mm)\(^{1/2}\). This corresponds to a physical size at a typical \(\beta = 40.0\) m of 1.0 mm, or a size of 0.16 mm at a collision point with \(\beta^*\) around 1.0 m since

\[
X = \beta_x^{1/2} x_n, \quad (5)
\]

In normalized coordinate space, the transverse motion of an on-energy particle through a thin sextupole of strength \(S\) is given by the transformation

\[
\Delta x'_n = (S\beta_x^{1/2} \beta_z) z_n^2 - (S\beta_x^{3/2}) x_n^2 \\
\Delta z'_n = (2S\beta_x^{1/2} \beta_z) x_n z_n. \quad (6)
\]

However, a particle that is off-energy by \(\delta\) performs betatron oscillations about an orbit displaced horizontally by \(\eta \cdot \delta\), where \(\eta\) is the dispersion function. Assuming that there is no vertical dispersion, it is then natural to make a second coordinate transformation,

\[
x = x_n - (\eta/\beta_x^{1/2}) \cdot \delta \\
z = z_n, \quad (7)
\]

so that displacements, angles and amplitudes are always measured about the displaced central orbit. Motion through a thin sextupole is now given by

\[
\Delta x' = (S\beta_x^{1/2} \beta_z) xz^2 - (S\beta_x^{3/2}) x^2 - (2S\beta_x \eta \delta) x \\
\Delta z' = (2S\beta_x^{1/2} \beta_z) xz + (2S\beta_z \eta \delta) z \quad (8)
\]

with the same sextupole terms as (6), plus additional quadrupole terms with strengths proportional to \(\delta\), which add betatron phase at the sextupoles.

Sextupoles are included in accelerator lattices so that these “chromatic quadrupoles” can be used to make the net chromaticity zero—or slightly positive, compensating for the weakening of quadrupoles with energy which makes the natural chromaticity negative. However, the variation of phase with energy is not the whole story, since the Twiss parameters also vary. While the dependency of \(\alpha\) on energy is irrelevant, since it does not enter (8) or the equivalent beam–beam equations, the variation of \(\beta\) with \(\delta\) causes the nonlinear element strengths to be modulated. EVOL assumes that such strength modulations are small and comparatively unimportant. It takes \(\beta\) to be constant, so that the coordinate transformations (3) and (7) are also, apparently, constant.

The EVOL code describes the net linear motion from the \((i-1)\)th to the \(i\)th sextupole by an energy-dependent rotation matrix (4), which includes half of the linear part of each sextupole kick in (8) by adding the phase advance so caused to the natural phase advance. Horizontally, for example, the net rotation angle used is

\[
\Delta \phi_x = \Delta \phi_{x0} + [(S\beta_x \eta)_i + (S\beta_x \eta)_{i-1}] \cdot \delta - \chi_{xi} \cdot \delta \quad (9)
\]

where the natural chromatic contribution is proportional to \(\chi_{xi}\), which is given by a sum over quadrupoles of strength \(k\) between \(i-1\) and \(i\),

\[
\chi_{xi} = \sum \int k \beta_x \, ds \quad (10)
\]
and where \( k \) is positive for horizontally focusing quadrupoles. Since the linear parts of (8) have been included in the net rotation, EVOL describes motion through a thin sextupole by (6), but with the coordinates \( x_n, x'_n, z_n, z'_n \) replaced by \( x, x', z, z' \).

When one of the scanning variables is the betatron tune, as in Fig. 2, the total machine tune is trimmed by adding small constant phases between every pair of nonlinear elements, in proportion to the nominal phase advance. This procedure is reasonable for \( Q \) variations of order 0.1\%, which are common.

Particle tracking is speeded by precalculating the \((108 \times 3 \text{ or so})\) normalized strengths in (6). Since there are \( 1/Q_s = 194 \) turns per synchrotron period, a number deliberately chosen integer, of order \( 194 \times 108 \times 4 \) different sines and cosines of phase advances are also precalculated and stored. Inspection of (4) and (6) then shows that a particle is tracked around a lattice in two dimensions using only 14 multiplications per sextupole, or about \( 108 \times 14 \) multiplications per turn, excluding beam–beam interaction calculations.

Two files are input by EVOL, a command file and a lattice file defining every nonlinearity by a strength and seven parameters, \( \beta_x, \beta_z, \eta, \Delta \phi_x, \Delta \phi_z, \chi_{xt}, \chi_{z1t} \). The single output file usually summarizes the life history of each trajectory in eight numbers, four for each transverse plane. Horizontally these are \( Q_{xt} \), the measured betatron tune, \( \hat{x} \), the maximum displacement measured every turn, and \( \hat{a}_x, \hat{a}_z \), the maximum and minimum amplitudes, measured every synchrotron period. As is discussed below, the synchrotron period, an integer number of turns, is the true natural period of the motion.

**THE DYNAMIC APERTURE OF THE SPS**

The collision lattice used in the SPS for most of the late summer of 1983 is identified by its experimental collision betas, which were 1.3 (horizontal) \( \times 0.65 \) (vertical) m. It is this “nominal” lattice that was used for almost all the EVOL simulation results presented below. For example, Fig. 3 shows the results of a coarse dynamic aperture search in two transverse dimensions over a comparatively large tune range. They were obtained, as already illustrated in Fig. 2, by progressively reducing the initial amplitude \( a_i \) until a trajectory was found that was stable for a given number of turns. Contours in Fig. 3 join the amplitudes of the first 5 stable trajectories at each tune, clearly showing the incremental amplitude step size. Although resonances are clearly visible, they are unphysically weak, reaching down only to about \( 4\sigma \) in amplitude for the extreme cases of the 2/3 and 3/4 resonances.

At least two qualifications must be made when reporting the dynamic apertures measured in this way. First, enough turns must be tracked to be reasonably sure of the asymptotic behavior of trajectories. Experience showed that \( 2 \times 10^4 \) turns, or about half a second real time, were sufficient. Second, it must be recognized that stable regions of four dimensional phase space may be surrounded by unstable regions, since phase space is severely contorted near the dynamic aperture. This is illustrated in Figures 3 and 4 by the occasional nonuniform
FIGURE 3  A coarse search for the nominal SPS sextupole dynamic aperture without tune modulation.

FIGURE 4  The sextupole dynamic aperture measured in a finer scale search near $Q = 26 + 2/3$, without tune modulation.
separation of the stable contours, caused by one or more trajectories that were unstable despite the stability of a larger-amplitude trajectory of the same tune.

Figure 4 shows the results of a finer tune scan than in Fig. 3, over a range of tunes close to the SPS operating point. While the 2/3 resonance remains weak, it is now also possible to notice the slight effect of the 7/10 resonance. In both simulation runs, the initial amplitudes and the basic fractional tunes were the same horizontally and vertically. There were no beam–beam collisions, and the tune-modulation depth was zero, as was the amplitude of energy oscillations. (Note that with zero chromaticity, the phase advance between sextupoles may still oscillate.)

When tune modulation was introduced, Fig. 5, the measured dynamic aperture was dramatically reduced, to realistic values. A plausible but large value of $\bar{q} = 0.004$ was used in this case, while trajectories were followed for a shorter time, 30 synchrotron periods of 194 turns, with a realistic fractional tune split of $q_x - q_z = 0.1$. In real life, proton backgrounds in the SPS experiments are unacceptable when $q_x$ is less than about 0.674. Recalling that $\sigma$ is typically between 0.005 and 0.01 (mm)$^{1/2}$, Fig. 5 confirms that protons in the tail of a bunch, with amplitudes of 3 or more $\sigma$, are indeed lost at this boundary tune.

When beam-beam collisions are included (not shown), the tune of a given amplitude particle is increased, and the measured dynamic aperture at a “linear magnetic tune” just above 2/3 also increases. This agrees with the observation in the SPS that backgrounds and lifetimes are better for antiprotons than for protons, close to and above the 2/3 resonance. (Protons in the SPS see almost no

\[ a_{xi} = a_{zi} \] (mm)$^{1/2}$

\[ q_x = 0.01 \]

\[ q_z = 0.674 \]

**FIGURE 5** The measured dynamic aperture near $Q = 2 + 2/3$, dramatically reduced by the introduction of tune modulation.
beam–beam effects, since there are about ten times more protons than antiprotons in each bunch.)

A final example is instructive, though of largely academic interest. Figure 6 shows the dynamic aperture in arbitrary units of purely horizontal motion in a lattice of fractional tune $Q$ with only one sextupole. Even in this simplest of all possible cases, with no tune modulation, an analytic solution does not exist for the aperture, although this result has been reproduced$^{16,17}$ using the symbolic manipulation program REDUCE.

THE ANALYTIC THEORY OF TUNE-MODULATED BEAM–BEAM RESONANCES

Here I review the theoretical framework with which the beam–beam simulation results from EVOL will be compared, and develop the notation and concepts which will be used. What follows has already been described in more complete detail by several different authors,$^{10,12,18,19}$ so for the most part, established results will be simply quoted, with minimal justification. The discussion, like almost all of the literature except Ref. 20, is restricted to one transverse dimension, with an externally imposed tune modulation.

First consider a one-dimensional collider, of unmodulated basic tune $Q_0$, with a single nonlinearity, one beam–beam collision per turn. The betatron tune of a test particle is a function of its amplitude, $a$, so that

$$Q(a/\sigma) = Q_0 + \xi \cdot D(a/\sigma),$$

(11)
where $\xi$ is the tune-shift parameter, and $\sigma$ is the size of the other beam, assumed round and Gaussian. The “detuning function” $D(a/\sigma)$ has the exact analytic form\textsuperscript{12,14,15}

$$D(\alpha) = 4\alpha^{-2}[1 - \exp(-\alpha^2/4) \cdot I_0(\alpha^2/4)],$$

with asymptotic limits

$$D(0) = 1$$

and

$$D(\alpha) \rightarrow 4\alpha^{-2}[1 - (2/\alpha)/(2\pi)^{1/2}] \quad \text{as} \quad \alpha \rightarrow \infty,$$

as shown in Fig. 7. Here $\alpha$ is the normalized amplitude $a/\sigma$, and $I_0$ is a modified Bessel function. (Note that this definition of $\alpha$ is different from that found in some of the literature.)

A beam–beam resonance of even order $N$ is present if the tune is equal to a rational fraction $m/N$ at some amplitude $\alpha_N$, as in Fig. 7. Resonant behavior near this amplitude is sketched in the flow diagram of Fig. 8, where the phase-space positions of several different trajectories are plotted in polar coordinates after every $N$th turn. Most particles trace out a contour that is apparently continuous, after a sufficiently large number of points have been plotted. Nonresonant

![Figure 7](image)

FIGURE 7 The round-beam detuning function $D$ versus the normalized amplitude $\alpha$. 
particles, with amplitudes smaller (larger) than \( \alpha_N \), have a phase advance slightly larger (smaller) than \( m2\pi \) between records, and appear to drift to the right (left). Resonant particles "lock on" to the tune \( m/N \), and circulate one of \( N \) islands in phase space.

The full normalized-amplitude island width for a single collision is

\[
\Delta \alpha_w = 4[V_N(\alpha)/(\alpha D'(\alpha))]^{1/2},
\]

independent of the tune-shift parameter, where the prime denotes differentiation by \( \alpha \). For round beams the "resonant width function" \( V_N(\alpha) \) is given by\(^{12,14,15}\)

\[
V_N(\alpha) = \int_0^\alpha (8/\alpha) \exp (-\alpha^2/4) \cdot I_{N/2}(\alpha^2/4) \, d\alpha.
\]

Although the number line is dense in rational fractions, so that there are many neighboring sets of resonant islands, the island size decreases so rapidly with resonant order that the single-resonance approximation is usually valid. However, the region very close to the stable island boundaries is highly sensitive to the presence of higher-order resonances, leading to the phenomenon of chaos, or stochasticity. This causes the distinctive appearance of trajectories that appear to discontinuously fill a two-dimensional area, rather than lying on a one-dimensional line. The amplitude of a particle can vary very rapidly within the boundaries of a chaotic region.

This picture of beam–beam effects naively predicts the strong stability of hadron beams, since large-amplitude fluctuations can only occur if the resonant islands and the chaotic regions are large, at tune-shift parameters near 0.1,\(^{21}\) a level rarely approached even in electron storage rings. Once again, tune modulation must be introduced if the model is to be realistic. This enhancement and the
ultimate generalization of the theory to include multiple collisions help considerably to give agreement with the SPS experience that tune-shift parameters of around 0.004 per collision are potentially dangerous.

Suppose that external tune modulation is introduced ad hoc into the model, so that the instantaneous betatron tune on turn \( t \) is

\[
Q(\alpha, t) = Q(\alpha) + \dot{\alpha} \cos(2\pi Q_s t) \tag{15}
\]

A family of synchrobetatron sideband resonances now appears, at time-averaged tunes of

\[
Q(\alpha) = m/N + p \cdot Q_s/N. \tag{16}
\]

This is illustrated in Fig. 9, where the phase-space points are now plotted every synchrotron period. Note that the size of the sideband islands drops very rapidly outside a band of tunes,

\[
m/N - \dot{\alpha} < Q(\alpha) < m/N + \dot{\alpha}, \tag{17}
\]

since particles beyond this band do not cross the resonance at any time during their tune-modulation period, and so do not feel its effect very strongly.

Equation (13) is generalized to give the full width of the \( p \)th sideband as

\[
\Delta \alpha_s = 4[ V_N(\alpha) J_p(N\dot{\alpha}/Q_s)/(\alpha D'(\alpha))]^{1/2}. \tag{18}
\]

Here \( J_p \) is the \( p \)th integer order Bessel function. When the condition (17) applies, \( J_p \) is of order

\[
J_p(N\dot{\alpha}/Q_s) \approx (Q_s/\pi N\dot{\alpha})^{1/2}, \tag{19}
\]

but is very small when (17) is violated. The sidebands are separated in amplitude

![Figure 9](image)
from each other by
\[
\Delta \alpha_s = (Q_s/N)/Q'(\alpha) = Q_s/N \xi D' (\alpha). \tag{20}
\]

As the tune-shift parameter is increased, the sidebands remain constant in size, while their separations decrease, until the Chirikov–Courant overlap criterion is satisfied
\[
\Delta \alpha_s < \Delta \alpha_w. \tag{21}
\]

Combining (18), (19) and (20), there is overlap if
\[
\xi > \frac{1}{4} (\pi \hat{g})^{1/4} (Q_s/N)^{3/4} (\alpha/V_N(\alpha)D'(\alpha))^{1/2}, \tag{22}
\]
which is the chaotic beam–beam limit for a single collision per turn. Figure 10 sketches how the stable synchrobetatron islands shrink as they are surrounded and submerged in the chaotic sea, as the overlap condition becomes more strongly satisfied.

The generalization of the overlap criterion to include multiple collisions per turn is aided by the introduction of the beam–beam ‘resonance vector’
\[
\vec{\xi}_{Ni} = \xi_i \exp (jN \phi_i), \tag{23}
\]
where \(\xi_i\) and \(\phi_i\) are the tune-shift parameter and the betatron phase of the \(i\)th collision. While the separation of the \(N\)th order sidebands now involves the sum of the resonant vector lengths,
\[
\Delta \alpha_s = (Q_s/N)/Q'(\alpha) = Q_s/(N \cdot \sum |\vec{\xi}_{Ni}| \cdot D'(\alpha)), \tag{24}
\]

FIGURE 10 A sketch of synchrobetatron islands partially submerged in the chaotic ‘sea’ caused by sideband overlap.
the width of the \( p \)th sideband island also involves the length of the resonant vector sum.

\[
\Delta \alpha_w = 4\left[ \sum |\xi_{Ni}| \cdot V_N(\alpha) J_p(N\hat{q}/Q_s)/(\sum |\xi_{Ni}| \alpha D'(\alpha)) \right]^{1/2}. \tag{25}
\]

The overlap criterion now has the fundamental general form

\[
\left( \sum |\xi_{Ni}| \cdot \sum |\xi_{Ni}| \right)^{1/2} > \frac{1}{4}(\pi \hat{q})^{1/4}(Q_s/N)^{3/4}(\alpha/V_N(\alpha)D'(\alpha))^{1/2}. \tag{26}
\]

Note that the beam–beam limit decreases almost linearly with the modulation frequency. This means that low-frequency magnet power-supply ripple is potentially dangerous, and that large synchrotron frequencies are very useful in ridding hadron colliders of the chaotic effects of beam–beam resonances.

It is instructive to rewrite the overlap criterion (26), emphasizing the role of resonance order \( N \), and amplitude \( \alpha \)

\[
N^{3/2}V_N(\alpha)/\alpha > Q_s^{3/2}(\pi \hat{q})^{1/2}/(16D'(\alpha)(k\xi)^2), \tag{27}
\]

where the \( k \) collisions per turn are assumed to act 'coherently,' with collinear resonant vectors. In the range of parameters of interest, the left-hand side increases rapidly with decreasing \( N \) or increasing \( \alpha \). This equation has the sense that trajectories with amplitude \( \alpha \) (or less) are stable on \( N \)th (or higher) order resonances, if \( Q_s \) is larger than a specific value, when \( \hat{q}, k \) and \( \xi \) are fixed.

Equivalently, for a particular \( N \) and \( Q_s \), \( \hat{q} \), \( k \) and \( \xi \), there is an amplitude \( \alpha_{Nc} \) below which chaos may never occur. This is sensible, since in the limit of small amplitudes, particles see only a stable quadrupole field in the beam–beam collision. If \( \alpha_{Nc} \) is larger than (say) 5.0, the resonance is weak and unimportant. Except for this condition, and as shown in Fig. 11, chaotic particles roam, apparently at will, over an amplitude band of approximate width

\[
\Delta \alpha = 2\hat{q}/Q'(\alpha) = 2\hat{q}/(\sum \xi_i \cdot D'(\alpha)). \tag{28}
\]

FIGURE 11 The broadening of the chaotic amplitude band at higher tunes, due to smaller tune versus amplitude slopes.
This width corresponds to a tune range of \( \pm \dot{q} \) about the resonant tune, and increases as the base tune is raised, since the tune slope decreases with increasing resonant amplitude.

LOOKING FOR CHAOS

Three classes of trajectories can usually be clearly distinguished in EVOL—"stable nonresonant", "stable resonant", and "chaotic". Stable nonresonant trajectories have maximum and minimum amplitudes that under realistic conditions differ from their initial amplitudes by less than or about 0.1\( \sigma \), even quite close to a resonance. In other words, with the exception of resonant islands and stochastic bands, there is only weak structure in one-dimensional phase space. Stable resonant trajectories are distinguished by their measured tunes, which are exactly locked on to synchrobetatron sidebands, as given in (16). Their amplitudes oscillate periodically as they rotate around synchrobetron islands.

Chaotic trajectories, in contrast, suffer dramatic and irregular fluctuations in amplitude which are often much larger than 1.0\( \sigma \). These trajectories are interesting, or dangerous, in their own right, so it is really rather a moot point whether or not this behaviour can formally be always called chaotic. Infrequent spot checks always showed neighboring "chaotic" trajectories diverging exponentially with time, a necessary but not sufficient test of formal chaos. Neighboring "stable" trajectories were always found to diverge linearly.

Figures 12a and 12b show all three kinds of behavior, in plots of the maximum amplitude ever attained by a particle in a given time versus the initial amplitude of the particle. There is a very broad chaotic band in Fig. 12a between lower and upper amplitude boundaries at about 2.9\( \sigma \) and 15\( \sigma \). The simulation was made with one collision of \( \xi = 0.024 \), near the 7/10 resonance at \( Q_0 = 0.695 \), and with a nominal value of \( \dot{q} = 0.0023 \). The data are a subset of those shown below in Fig. 24, as part of a discussion of how sextupole detuning can enhance the width of chaotic bands. It takes a typical time of about \( 4 \times 10^4 \) turns, one second in the SPS, for a trajectory to explore most of the chaotic amplitude range. Observed antiproton lifetimes of many hours can therefore only be "explained" in this simple model by invoking an additional slow diffusive process, which pushes particles over the lower boundary, where they make their way rapidly to large amplitudes, and are lost to a physical or a dynamical aperture.

Figure 12b shows several trajectories trapped on stable islands in a stochastic sea with small amplitude oscillations. The same single-collision strength was used as before, with a smaller value of \( \dot{q} = 6 \cdot 10^{-4} \), near a 13th order resonance excited by displacing the mean collision point transversely by 0.3\( \sigma \). Because the modulation depth is comparatively small, the chaotic band only stretches between amplitudes of about 3.8 and 5.2\( \sigma \).

Figure 13 shows how the upper and lower chaotic bounds vary as the magnetic tune is scanned through the \( Q_x = 26 + 5/7 \) resonance, in the nominal 1.3\( \times \)0.65 m lattice. Six collisions of strength \( \xi = 4.5 \times 10^{-3} \) were used, with the nominal \( \dot{q} \), and
a total tracking time of $2 \cdot 10^4$ turns. The odd resonance was driven by an orbit separation between the two beams which was different at different collision points, with a root-mean-square value of 0.15σ. The two dashed lines correspond to the expected chaotic boundaries at amplitudes found by solving the following equation for $\alpha$

$$Q(\alpha) = \frac{5}{7} \pm \hat{\alpha}$$

(29)
Chaos was not observed in the simulation below a critical amplitude marked by the solid horizontal line at $2.3\sigma$.

COHERENT TENTH-ORDER RESONANCES

The tenth-order resonance is an ideal subject for simulation because at the SPS it lies on the border between potency and insignificance, both theoretically and experimentally. Theoretically (see below) 10th order resonances in the SPS should be able to produce chaos down to a critical amplitude of about $3.0\sigma$.\textsuperscript{15} Resonances with critical amplitudes much larger than this are not dangerous, since the loss of a very small fraction of particles is unimportant. This is unlike the situation with electron colliders, where phase-space diffusion makes the stability of (say) $6\sigma$ trajectories important. Experimentally, the $7/10$ resonance near the SPS operating point $(Q_x, Q_z) = (26.685, 27.675)$ is observed as a significant drop in the antiproton lifetime, and an increase in the antiproton background in UA1 and UA2. Note that the distribution of antiproton tunes straddles the horizontal $7/10$ resonance when the total beam-beam tune shift is $6 \times 0.004$, a typical value.

Single-collision simulations show that the sideband spectrum (16) is indeed present for many different resonance orders, and that the amplitude spacing of the islands agrees well with Eq. (20). In a more demanding test of the theory, Fig. 14 shows the measured amplitude widths of the zeroth and first sidebands of a tenth-order resonance as a function of the tune-modulation amplitude. A width recorded there is the maximum of the maximum-minus-minimum amplitude differences for the locked-on trajectories found during a scan of initial amplitudes.
with a constant initial phase. Such a measurement is not sensitive to the choice of the initial phase.

The theoretical solid lines in Fig. 14 show that the Bessel function behavior predicted by (18) agrees quite closely with observation, after a vertical scale factor has been removed. This scale factor is a measure of the resonant function $V_N$ at the center of the island. Two such measurements of $V_{10}$, at $\alpha = 3.0$ and 4.5, are shown in Fig. 15 with the theoretical function for even orders 4 through 12 as predicted by (14). Here again the agreement between simulation and theory is good.

Now that presence, size, and separation of the synchrobetatron sidebands have been shown to be accurately described by the basic one-dimensional model, it is time to test the quantitative accuracy of the overlap criterion (26) in predicting chaos. Figure 16 plots the minimum total tune shift found to be necessary to cause chaos as a function of initial amplitude in three different cases. The agreement between theory and simulation is surprisingly good, considering, for example, the approximation of Bessel functions (19) that has been used.

The theoretical solid curve in Fig. 16 is the same in all three cases because there is either only one resonance vector or all six of them are parallel. Not shown on the figure because of its extreme stability is a fourth case that was tried, also six-fold symmetric, but with a total machine tune of 26.7. This was very stable because the arc ‘tune’ was 4.45, causing complete 10th order resonant-vector cancellation, leaving only a 20th order resonance. The horizontal $267/10$ resonance is only present in the SPS because the design lattice is not six-fold symmetric, but rather totally asymmetric, due to the mina-beta optics.
Figure 15 shows how the range of chaotic amplitudes depends on tune close to the 26.7 resonance, in a six-collision coherent simulation with resonance vectors of nominal length 0.004. The critical amplitude is about 3.0σ, with particles at tunes above 0.696 reaching amplitudes greater than 10.0σ, near the sextupole dynamic aperture, from initial amplitudes as low as about 3.3σ.

The critical order $N_c$ below which all coherent beam–beam resonances must be avoided at the SPS is found by solving (27) with appropriate 'fixed' parameters

$$N_c^{3/2} V_{Nc}(3) = 6.3 \cdot 10^{-2} (Q/Q_s) 5 \cdot 10^{-3} 3/2$$

(30)

It is desirable that 3.0σ or smaller trajectories must not be capable of chaos.
FIGURE 16 A comparison of the theoretical and measured critical tune shifts for one round collision near 10th order resonances.

FIGURE 17 The 10th order chaotic amplitude band for nominal SPS parameters with full coherence.
Figure 18 shows the graphical solution for the one-dimensional fractional tune line above the half integer. Note that the two-dimensional tune plane degenerates into such a tune line for equal horizontal and vertical tunes, on the tune plane diagonal. Even resonances, the only ones allowed by the theory so far, are labeled on the left vertical axis and parameterized in strength by the synchrotron tune on the right vertical axis. The only safe parts of the tune line are those where a tune spread of 0.024 can be placed without being pierced by a resonance line, at a height given by $Q_s$.

SPS operating conditions during summer 1983 are intersected by the 7/10 resonance, as shown, and marginally fail this test. While the line segment between the 7/10 and the 3/4 resonances appears attractive for the SPS antiprotons, it was not possible in practice to raise the tune there, because of low lifetime caused by the odd 5/7 resonance. The strong presence of “forbidden” odd resonances like this is discussed in more detail below.

It has been shown that the one-dimensional theoretical chaotic prediction of Eq. 26 compared very well with simulation results when the resonant vectors all lined up. This equation also predicts exactly how the threshold for chaos should be raised when the vectors move away from total coherence, the case of real practical interest to the SPS. The 7/10 resonance is not as dangerous as calculated in the ‘worst case’ configuration above because the collisions are not coherent, even ignoring accidental lattice perturbations.

Figure 19 shows simulation measurements of how the threshold tune-shift parameter $\xi_c$ for chaotic behavior of a 3.5$\sigma$ particle varied with the incoherency of a two-collision machine. At all times, the total tune was 27.6, on a 10th order resonance, but the “tune” difference $\delta Q$ between the two arcs was varied to

![Figure 18](image_url)  
**FIGURE 18** Critical even-order resonances parameterized by $Q_s$, with otherwise nominal SPS conditions and full coherency.
rotate the resonant vectors between parallel and antiparallel. According to (26) the threshold in this geometry is expected at

$$\xi_c = \xi_0/\left(\cos(10\pi\delta Q)\right)^{1/2}, \tag{31a}$$

where, from Fig. 16

$$\xi_0 = 0.009 \tag{31b}$$

In the figure a value $\xi_0 = 0.012$ is used as a best fit to the data at low $\delta Q$. While stability against the beam–beam effect increases with the incoherency as expected, there are signs that the simulated machine is more stable than theory predicts at large dephasing angles.

A slightly more complicated geometry in a three collision machine (see Fig. 20) shows strong divergence between theory and simulation. In this case two collisions had the same strength $\xi_1$, with back to back resonant vectors which were perpendicular to the third resonant vector of length $\xi$. The threshold strength of the third collision for the chaotic behaviour of a 3.5$m$ particle, at a given value of $\xi_1$, is expected to be

$$\xi_c = (2\xi_0^2 + \xi_1^2)^{1/2} - \xi_1 \tag{32}$$

where again $\xi_0$ is treated as a free parameter in the figure. Once again the simulated dephased machine is more stable than expected, but now the difference is much more pronounced.

FIGURE 19 The critical 10th order tune-shift parameter measured as a function of the dephasing tune for two collisions.
In the light of the two preceding examples, extreme caution should be used when applying the analytic theory to incoherent situations, especially those with many collisions and no lattice symmetry, as in the SPS. The only consolation is that the theory seems to be overly pessimistic. Even in one dimension, the study of dephased collisions is a task that, like the sextupole dynamic aperture problem, is best left to numerical simulation, at least until a better theoretical model comes along. This, therefore, is the point at which the description of EVOL simulation results turns from quantitative comparisons with theory to semi-quantitative and qualitative comparisons with operational experience at the SPS.

Over a period of several months in 1983, the configuration of the SPS changed gradually from four collisions per turn (two proton bunches, one antiproton bunch) with collision-point betas of $3.5 \times 7.0 \text{ m}$, to six collisions per turn with $0.65 \times 1.3 \text{ m}$ betas. While conclusive experimental proof was difficult to obtain, theoretically the $267/10$ resonance became weaker during this progression, due to the advance towards almost complete incoherency. Table I records the coherency in the series of lattices as the ratio of the length of the resonant vector sum to the maximum possible length, assuming an ideal machine.

Figure 21 compares the chaotic amplitude bands that EVOL finds as a function of tune near the resonance in the first and last lattices. Despite using a strength $\xi = 0.005$ for four or six collisions, both configurations are less dangerous than the case of six weaker coherent collisions already shown in Fig. 17. As expected, four collisions in the first lattice are significantly more unstable than six collisions in the last, with chaos down to amplitudes of about $3.2 \sigma$ versus $4.2 \sigma$, respectively. On
TABLE I

<table>
<thead>
<tr>
<th>( \beta_0^v ) (metres)</th>
<th>( \beta_0^h )</th>
<th>( \frac{\sum \xi_{10k}^v}{\sum \xi_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>no mini-beta</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3.5</td>
<td>7.0</td>
<td>0.578</td>
</tr>
<tr>
<td>1.0</td>
<td>2.0</td>
<td>0.187</td>
</tr>
<tr>
<td>0.75</td>
<td>1.5</td>
<td>0.016</td>
</tr>
<tr>
<td>0.65</td>
<td>1.3</td>
<td>0.035</td>
</tr>
</tbody>
</table>

the average, the lower amplitude boundary is about 2.0\( \sigma \) larger in the low beta lattice than in the high-beta lattice.

The true situation is not so straightforward, however, since there are betatron phase errors in a real machine causing errors in the direction that the real resonant vectors point. Because the angular errors are multiplied by the resonance order \( N \), the true resonant vector sum is in effect statistically randomized for resonances above some order.

FIGURE 21 Chaotic band measurements showing comparatively weak 10th order chaos in the idealized nominal SPS lattice.
For example, suppose that there is a beta error wave

$$\beta_R \approx \beta_D (1 + \delta \sin (2(\phi_D - \phi_0)))$$  \hspace{1cm} (33)

where $\delta \ll 1$, where $\delta$ and $\phi_0$ are constant over a segment of the machine, and where the subscripts $D$ and $R$ stand for design and real. There is an associated phase-error wave given by

$$\delta \phi = \phi_R - \phi_D \approx \int \left( [1 + \delta \sin (2(\phi_D - \phi_0))]^{-1} - 1 \right) (ds/\beta_D).$$  \hspace{1cm} (34)

After changing the integration variable to $\phi_D$, this becomes, to first order, simply

$$\delta \phi = \delta/2 \cos (2(\phi_D - \phi_0)).$$  \hspace{1cm} (35)

Now, to obtain the correct phase advance between $N$th order resonant vectors, it is necessary that

$$N\delta \phi \ll 1,$$  \hspace{1cm} (36a)

or that

$$|\beta_R - \beta_D|_{MAX}/\beta_D \ll 1/N.$$  \hspace{1cm} (36b)

So, to believe that the data in Table 1 correctly represent the true SPS, the betas need to be known to be correct to better than 10% accuracy. This is not currently the case.

**SEXTUPOlar DETUNING**

Particle motion through $M$ sextupoles can be represented by four polynomials of order $2^M$, showing that to second order in sextupole strength, the betatron tune is a quadratic function of amplitude, even for a single sextupole lattice. One effect of adding sextupoles to beam–beam collisions in EVOL is therefore to modify the curve of tune versus amplitude, mainly by the addition of an octupolar term. (For the sake of simplicity, the discussion is still restricted to one dimension, now specifically identified as the horizontal.)

EVOL measures the horizontal sextupolar detuning to be

$$Q_s(a_s) = Q_x + 3.6 \cdot 10^{-5} (a_s/0.01)^2 + \cdots$$  \hspace{1cm} (37)

close to the normal SPS working point in the 0.65 x 1.30 m beta lattice. Combining this with the beam–beam collisions of antiprotons against a beam with $\sigma = 0.01$ (mm)$^{1/2}$, the tune versus amplitude curves of Fig. 22 are found for three different total beam–beam tune shifts. Each curve now has a minimum at a finite amplitude, which has important chaotic consequences, since now the chaotic bands can become very wide.

In particular, with the nominal total beam–beam tune shift there is a tune minimum at an amplitude of about $a = 6.8\sigma$, as shown in Fig. 23. This has the destabilizing effect of causing a discontinuous jump in the upper chaotic amplitude boundary as a function of tune, as seen in the solid curve in Fig. 24.
dashed line there shows that sextupolar detuning with the opposite sign would lead to a more stable situation. In either case, sextupoles modify the beam–beam behavior of the 0.65 × 1.30 m beta lattice significantly through this effect.

The data for Fig. 24 were found by using a single octupole in EVOL to cause the detuning of (37), instead of tracking through the 108 SPS sextupoles, and
slowing the simulation down considerably. This is legitimate at small amplitudes far enough away from 4th order resonances, but it artificially raises the sextupole dynamic aperture.

In conclusion, we note that various aspects of the tenth-order resonance are summarised by comparing Figures 17, 21 and 24.

ODD RESONANCES

So far it has been implicitly assumed that the one-dimensional beam–beam kick is an odd function of only the displacement at the collision point, leading to resonances of only even orders. Odd resonances are introduced, for example, if the bunches do not collide head on, if they collide at an angle, or if there is non-zero dispersion at the interaction points. Of these three sources only the last one is intentionally present in the SPS, although the design horizontal dispersions at the interaction points are so small that the odd resonances should still be strongly suppressed. However, despite the expected irrelevance of these resonances, a real experiment on the SPS,15,23 summarized in Fig. 25, shows that the 7th and 11th order resonances are stronger than the 10th.

In this experiment, a single weak antiproton bunch made six collisions per turn with three bunches of protons, with a tune-shift parameter of about 0.003 per crossing. Data were taken in the $1.3 \times 0.65$ metre lattice, at the seven locations between the 2/3 and 3/4 shown in Fig. 25a, in a tune scan roughly parallel to the
FIGURE 25a  The seven locations in the tune plane visited during a beam-beam experiment at the SPS.

FIGURE 25b  Measured proton and antiproton currents as a function of time after the beginning of the experiment.
diagonal. At each location, the antiprotons had a tune spread which was actually shaped like a “necktie” in the tune plane, but which is represented by a straight line in the figure. The more stable lower amplitude particles are at the higher tunes. Figure 25b shows the current that was measured for both kinds of particles as a function of time after the beginning of the experiment. Beam lifetimes were inferred from the slopes of these curves, after the addition of large current offsets.

The experiment began at position 1, the nominal operating point, with an antiproton lifetime of 300 hours. Lower values at the other locations are ascribed to the beam–beam effect because the proton bunches, with tune-shift parameters ten or more times smaller, had lifetimes that remained consistently large. On the 10th order resonance at positions 2 and 3, the lifetime was about 25 hours, dropping to 3 hours on the 7th/11th at position 4, and rising again to 11 hours on the 11th at position 5. Even when only small-amplitude particles crossed the 4th order resonance at 7, the lifetime was less than 6 hours.

Proton–Antiproton Orbit Differences

In another SPS experiment, six collisions occurred with a total tune shift of about 0.018, and orbits electrostatically separated by a variable amount.\textsuperscript{24} At the nominal operating tunes, the antiproton tune spread straddled the 13th order resonance, which caused the antiproton lifetime to drop from 59 hours with no intentional separation to 32 hours with $0.3\sigma$ peak separation, and 20 hours with $1.1\sigma$ peak separation. The effect of separation on the 13th order resonance was clear, but comparatively weak.

It has been pointed out that the difference in proton and antiproton closed orbits, even without electrostatic elements, due to the tune splitting that the beam–beam effect introduces, is an odd resonance mechanism in the SPS.\textsuperscript{25} The expected size of this effect is enough to explain semi-quantitatively the SPS odd resonance experience.

To show this, suppose that there is a distribution of dipole errors $g(s)$ around a lattice, shifting the closed orbit at a reference point with phase $\phi(s) = 0$ to

$$y = (2 \sin (\pi Q))^{-1} \int_0^{2\pi R} g(s) \beta^{1/2} (s) \exp [j(\pi Q - \phi(s))] \, ds. \quad (38)$$

Here $y$ is the normalized displacement in either transverse plane, conveniently represented in complex notation, and $s$ is the azimuthal coordinate. The orbit difference between protons and antiprotons is now formally written as

$$\Delta y = (dy/dQ) \cdot \Delta Q = (dy/dQ) \cdot \xi_T, \quad (39)$$

where $\xi_T$ is the total antiproton beam–beam tune shift, and where protons are assumed to have no tune shift. While the differentiation in (39) is awkward to perform explicitly, it will be sufficient for our present purposes to make some semi-quantitative observations about the size of the terms expected after differentiation.
First, Fourier decompose \( g(s) \) so that
\[
g(s) = \sum_{n} g_n \exp \left( jnS/R \right),
\]
where \( g_{-n} \) is the complex conjugate of \( g_n \), and rewrite (38) in a more useful form as
\[
y = \sum_{n} g_n (2 \sin \left( \pi Q \right))^{-1} \int_{0}^{2\pi R} \beta^{1/2} \exp \left[ j(nS/R + \pi Q - \phi) \right] ds.
\]
This expression depends explicitly or implicitly on \( Q \) in three places, in the denominator, in \( \beta \), and in the exponent. Differentiation leads to three terms of comparable size, so that the separation is expected to be
\[
\Delta y \approx y \cot (\pi Q) \cdot \pi \xi_T
\]
within a factor of two or three. Despite this uncertainty, and the statistical nature of the problem, it is clear from (41) that each differential term is exactly halved if all the \( g_n \) are halved. Since closed-orbit correction is precisely the process of making the \( g_n \) small, it follows that good orbit correction is vital to the elimination of odd resonances, if they are in fact due to orbit separation errors.

Typical horizontal proton–antiproton orbit separations in the SPS are estimated from (42) to be
\[
\Delta X_{co}(mm) \approx 0.24(\xi/0.004)(k/6)(\hat{X}_{co}(mm)/5.0) \cdot (\beta/\hat{\beta})^{1/2},
\]
using a fractional tune of 0.69, \( k \) collisions per turn, and peak closed-orbit errors of \( \hat{x}_{co} \) at a beta of \( \hat{\beta} \). It is often more useful to express the separation in units of the beam size, so that
\[
\Delta X_{co}/\sigma \approx 0.17(\xi/0.004)(k/6)(\hat{x}_{co}(mm)/5.0)(10^5/\hat{\beta}(mm))^{1/2}(2 \cdot 10^{-5}/\varepsilon (mm))^{1/2},
\]
which once again is parameterized with nominal SPS values. These two results are really estimated separation amplitudes, which need to be multiplied by an unknown phase factor to find the separation at a given point. They are in reasonable agreement with data taken by a wire scanner at a single location in the SPS,\(^{25}\) which measured separations of between 0.003 and 0.09 mm. Most of these data were taken with a tune-shift parameter of about 0.003 at 4 collision points.

How strong are the odd resonances caused by a set of orbit separations \( d_i \cdot \sigma \) expected to be? The sideband island width due to an odd resonance of order \( N \) is
\[
\Delta \alpha_w = 4[\sum d_i |\xi_N| \cdot U_N(\alpha)J_P(N\hat{\eta}/Q_0)/(\sum |\xi_N| \alpha D'(\alpha))]^{1/2}
\]
in the limit of small separations. This is the same expression as (25), except that for odd resonances the dimensionless separation \( d_i \) multiplies the resonant vector, and the odd resonance function \( U_N(\alpha) \) replaces the even function \( V_N(\alpha) \). Figure 26 shows the results of simulation measurements of \( U_N \) that were made in the same way as for the \( V_N \) in Figure 15, except that small orbit offsets were
FIGURE 26 Theoretical even-order and measured odd-order resonance strengths, versus the normalized amplitude.

introduced into the single collision. Inspection of the figure shows that

$$U_N(\alpha) = V_{N-1}(\alpha)$$  \hspace{1cm} (46)

In particular, sidebands of the 7th order resonance caused by a single collision of separation $d = 0.03$ are roughly as wide as sidebands of the 10th order resonance. Since real SPS separations are estimated in (44) to be larger than this, the 5/7 resonance is expected to be more dangerous than the 7/10 resonance, in agreement with observation.

Knowing the $U_N$, it is now possible to quantify the smallness of SPS orbit
separation necessary to keep the critical chaotic amplitude due to odd resonances above 3.0σ, at a given point on the tune diagonal, and under nominal conditions of $q$, $k$, and $\xi$. Figure 27 is the odd resonance version of Fig. 18, graphically solving the equation

$$n_{c}^{3/2}u_{n_{c}}(3) = 6.3 \cdot 10^{-2}(Q_{s}/5.10^{-3})^{3/2}/d_{1},$$

(47a)

where the "single-collision equivalent separation" $d_{1}$ is given by

$$d_{1} = \left| \sum d_{i}\xi_{Ni} \right|/\left| \sum |\xi_{Ni}| \right|.$$

(47b)

As $d_{1}$ gets smaller, the odd resonances get proportionally weaker, making the condition on the synchrotron tune $Q_{s}$ less stringent. The SPS conditions shown in Fig. 27 assume a root-mean-square orbit separation of 0.1σ, and coherent resonant vectors. According to (44) and Fig. 27, to be free from the worst case coherent 5/7 horizontal resonance, the SPS orbit must be flattened to a root-mean-square deviation of about 1 mm.

Finally, the mechanism of orbit separation due to tune differences may be speculated to lie behind the extreme importance of vertical orbit flattening in electron storage rings. It may also be implicated in the importance of injecting equal currents of electrons and positrons. If two beams in collision have the same size and current, they have the same beam–beam tune shift, and lie on identical orbits. Balanced beams like these see no odd resonances. However, if one beam has less current or is larger than the other, not only does it see additional odd resonances, but all the resonances it sees are stronger than those seen by the other beam. In consequence, the weaker beam may become (even) larger, and more
vulnerable. This enhances the bistable behavior often observed in electron storage rings, in which one beam is blown up, while the other is collapsed.

Suppose that in order to suppress odd resonances the vertical orbit differences must be kept below one third of a beam sigma. Using (42) the condition on the vertical orbit flatness becomes

$$ \dot{Z}_{co} < 0.3 (\varepsilon_z \beta_z)^{1/2} / (\pi Q \cot (\pi Q_z)),$$

or, with typical CESR parameters,

$$ \dot{Z}_{co} < 0.3 \cdot (2.10^{-6} \cdot 3.10^4)^{1/2} / (\pi 2 \cdot 0.025 \cot (\pi 9.36)) = 1.0 \text{ mm}. $$

This is about the level of sensitivity of colliding-beam conditions in CESR to closed orbit errors.

**Simulations with Orbit Separation**

Proton–antiproton orbit offsets are input into EVOL as a single number, the root-mean-square displacement $d_s$ measured in beam sigmas. Then displacements $d_i$ at the six collision points are calculated in a pseudo-random fashion for a particular odd resonance $N$ such that

$$ \left| \sum d_i \bar{\xi}_{Ni} \right| = d_s \sum |\xi_{Ni}| / 6^{1/2}, $$

so that the coherency of the resonance is fixed at its statistically expected value. The $d_i$ are calculated by first finding the unit vector $\hat{b}$ that maximizes the expression

$$ b_T = \left| \sum (\hat{b} \cdot \bar{\xi}_{Ni}) \bar{\xi}_{Ni} \right|$$

and then setting

$$ d_i = (d_s \sum |\xi_{Ni}|)^{1/2} \cdot (\hat{b} \cdot \bar{\xi}_{Ni}) / b_T. $$

This process favors a smaller number of larger displacements with resonance vectors pointing more or less in the same direction, and implicitly assumes that all separation phase information is lost between collision points.

Figure 28 shows how the simulated critical amplitude for chaos varies in the nominal SPS lattice as a function of $d_s$ on the 7th and the 11th order resonances. The 7th order resonance is dangerous for separations as low as 0.060, corresponding to physical closed-orbit errors as small as about two millimeters. The 11th order resonance is comparatively innocuous, as expected.

Results from a tune-scan simulation analogous to the real experiment presented in Fig. 25 are given in Fig. 29, showing semi-quantitative agreement with the observed resonant behavior. The tune plane was scanned with $Q_{z0} = Q_{x0} + 1 - 0.009$, under nominal conditions with a root mean square orbit separation of 0.150. Particles were tracked in one dimension at a time, but the horizontal and vertical resonances that were found are plotted on the one figure. Dispersive effects were not included. The vertical 7/10 resonance is stronger than the horizontal 7/10, because the latter is so well dephased, as already discussed.
FIGURE 28 The nominal 7th and 11th order critical amplitudes versus the normalized root-mean-square orbit separation.

FIGURE 29 Chaotic band width results, in both dimensions separately, from an EVOL simulation of the SPS beam-beam experiment.

By contrast, all the odd resonances in Fig. 29 have roughly the same shape and size in both planes, a consequence of the fixed coherence levels assured by the method of selection of collision offsets. Trajectories with amplitudes as low as 2.3σ see chaotic effects off the 5/7 resonances, while the highest order resonances present, the thirteenth, are negligibly weak.
Simulations with Dispersion

Odd resonances are also driven by dispersion at the collision points, since a test particle sees the opposing beam offset by an amount that slowly oscillates at the synchrotron frequency. In the sense that the strength of an odd resonance depends on both the offset amplitude and frequency, \((\eta \cdot \delta E/E)/\sigma\) and \(Q_s\), the separated-orbit model discussed above is just the steady-state limit of this more general time-varying model. Figure 30 compares the simulated response of the 7th order critical chaotic amplitude to constant and dispersive offsets of the same amplitude, as a function of that amplitude. According to this criterion, the nominal SPS synchrotron frequency is low enough for oscillating collision offsets it may cause to be treated as constant, at their maximum values.

The design horizontal dispersion at the collision points in the nominal SPS lattice is zero at the two experiments, and is small but non-zero at the other four collisions, where \(\eta^* = 0.215 \text{ m}\). A particle with a very large relative energy oscillation of amplitude \(a_e = 10^{-3}\) therefore has an oscillating offset at these four with a normalized amplitude of

\[
d_i = \eta^* a_e (\epsilon \beta^*)^{1/2} = 0.22
\]

Figure 31 shows the 7th order chaotic amplitude band of such a particle, and compares it with the effect of \(0.15\sigma\) root-mean-square orbit separations. While the lower chaotic bounds are still almost identical, the dispersive upper chaotic bound is somewhat smaller than for constant offsets. In conclusion, even if the closed orbits in the SPS were perfectly flat, there would still be important odd...
FIGURE 31 A comparison of the 7th order chaotic band width for constant and dispersive collision offsets of the same amplitude.

horizontal resonances which could only be removed by reducing the design dispersion at the four parasitic collision points.

However, lattice imperfections cause not only closed orbit errors but also spurious dispersion waves, which should also be flattened for odd-resonance-free operation. The expected dispersion errors are of order

$$\frac{\Delta \eta^*}{\beta^{1/2}} \approx (\Delta \beta/\beta)(\eta_{\text{typical}}/\beta_{\text{typical}}^{1/2}) \approx 0.3(\Delta \beta/\beta),$$  \hspace{1cm} (52a)

giving

$$\Delta \eta^* \approx 2.0(\Delta \beta/\beta) \text{ m}$$  \hspace{1cm} (52b)

at the parasitic collision points in the nominal SPS lattice. Horizontal beta error waves of relative amplitude 0.1 therefore cause the appearance of important horizontal odd resonances, even with flat closed orbits and zero design dispersion at all collision points.

TWO-DIMENSIONAL CHAOS

Chaotic motion has been studied above in only one transverse dimension, so that analytic formulae could be written, and physical insight gained. On the one hand, it is often reasonable to ignore the coupling between the two transverse planes introduced by sextupoles and skew quadrupoles, etcetera. For example, the discussion may be limited to amplitudes much smaller than the dynamic aperture
in a lattice that is well decoupled. On the other hand, the coupling caused by the beam–beam interaction may not be ignored in a complete model. For a round Gaussian beam, the interaction in plane 1 is independent of the amplitude in plane 2 only if amplitude 2 is much smaller than \( \sigma \), the beam size. This is a property enjoyed by only a small fraction of the particles.

Nonetheless, when modest nonresonant oscillations are added in a second plane, the chaotic behavior in the first plane is only slightly modified. This is the main conclusion when the simulation data of Fig. 32 are compared with those in Fig. 17. Figure 32 shows the chaotic horizontal amplitude band near the horizontal 7/10 resonance with six nominal coherent collisions and two different non-zero values of the vertical amplitude. As the vertical amplitude is increased, the width of the band decreases only slightly. Unshown here is the absence of chaos in the vertical motion, with or without vertical collisions.

Complications arise in the interpretation and presentation of two dimensional simulation data when the transverse amplitudes are comparable, and when both planes are resonant. Figure 33 reexamines a segment of the tune scan made in Fig. 29, but this time with initial amplitude ratios \( a_x : a_z \) of 2:1, 1:1 and 1:2, instead of the values 1:0 and 0:1 previously used. Where chaos is shown to exist in both planes, the same particle may be chaotic in both planes simultaneously, or different particles may be chaotic in only one plane. No attempt is made here to distinguish between these two cases. It may be said that Fig. 33 adds little or nothing to the understanding of the resonant behaviour already gained from examination of the one-dimensional motion in Fig. 29.

The beam–beam betatron tune shift of a given amplitude particle in a plane of primary interest is modified by an additional displacement, in either plane. Thus a
FIGURE 33 Chaotic bands from fully two-dimensional motion, in a second EVOL simulation of the SPS beam-beam experiment.
secondary betatron or synchrotron oscillation in either transverse plane, causes modulation of the primary betatron tune at the secondary frequency (and higher harmonics). In this sense, tune modulation can never be avoided when two-dimensional motion is considered, even in the absence of all "exterior" mechanisms such as chromaticity, longitudinal collision-point oscillations, or power supply ripple.

Figure 34 shows how the widths of the 7/10 resonance sidebands vary as a function of the amplitude of a secondary synchrotron oscillation in the same plane as the betatron motion, with no exterior tune modulation. There is no chaos because the sideband islands are always well separated, even at unrealistically large dispersion values, and even though all the nominal total tune shift is gathered into a single collision. When the external tune modulation is turned on, there is no significant reduction of the beam–beam tune shift chaotic limit below the data in Fig. 16, even for large dispersions. Similarly stable motion is observed when the synchrotron oscillations are in the other transverse plane.

When the modulation frequency is larger, due to betatron motion in the other plane, the sideband islands are no wider and are so much further separated that only the zeroth order is observable, except at tunes very near a half-integer. So, while "internal tune modulation" is observable by simulation, it is neither a strong nor a dangerous effect.
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