ANOMALIES OF HIDDEN LOCAL CHIRAL SYMMETRIES IN
 σ -MODELS AND EXTENDED SUPERGRAVITIES

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ABSTRACT

Non-linear σ -models with hidden gauge symmetries are anomalous, at the quantum level, when coupled to chiral fermions in not anomaly free representations of the hidden chiral symmetry. These considerations generally apply to supersymmetric Kählerian σ -models on coset spaces with hidden chiral symmetries as well as to extended supergravities in four-dimension with local $SU(N)$ symmetry. The presence of the anomaly implies that the scenario of dynamical generation of gauge vector bosons has to be reconsidered in these theories.

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Recently much attention has been devoted to the problem of anomalies and their topological origin in Yang-Mills and gravitational theories defined on a D-dimensional space-time [1]. There are at least two reasons for not restricting the dimension of space-time to the case of physical interest, $D = 4$. One reason is mathematical, i.e. topological properties and invariants in different space-time dimensions are related. Another is physical: it is hoped that Kaluza-Klein theories will in some way describe a unified theory for the fundamental interactions, will explain their symmetry properties, will give the mass spectrum, and will answer the question why we live in $D = 4$ space-time dimensions through the phenomenon of spontaneous compactification.

However, it has become clear that since these theories are gauge theories we have to worry about anomalies. The very reason for requiring the absence of anomalies in a gauge theory is not so much because of renormalizability but rather because of unitarity. In fact if gauge invariance is broken by anomalies the theory will unavoidably violate unitarity and therefore will not be suitable for describing the physical world.

Very recently, it has been observed that some non-linear σ -models in which the scalar manifold is a coset space G/H may show up anomalies when they are coupled to fermions which are not in anomaly-free representations of the subgroup H of G [2]. Indeed, these σ -models may in general be formulated as models having an artificial hidden symmetry $G_{\text{global}} \times H_{\text{local}}$, the hidden local symmetry H_{local} being realized through connection fields which are merely auxiliary fields in the classical Lagrangian. In a fixed gauge of H_{local} the linear symmetry reduces to H_{global} , with G_{global} being non-linearly realized. However, the equivalence of the models formulated with symmetries G_{global} and H_{global} , respectively, is only true if the fundamental fermions are in an anomaly-free representation of H_{local} . To be more specific, let us consider a model with scalars $\phi \in G/H$ and fermions ψ transforming according to certain representations of H . The σ -model can be formulated as follows:

$$+ \text{Tr} (D_\mu g D_\mu g^{-1}) - i \bar{\psi} \not{D} \psi, \quad (1)$$

where $g(x) \in G$ and D_μ is the covariant derivative with respect to H :

$$\begin{aligned} D_\mu g &= \partial_\mu g - g A_\mu \\ D_\mu \psi &= \partial_\mu \psi - A_\mu \psi \end{aligned} \quad (2)$$

A_μ is a Lie-algebra-valued gauge field.

The equation of motion for A_μ gives

$$A_\mu = g^{-1} \partial_\mu g|_{\parallel} + i/2 \bar{\psi} \gamma_\mu T \psi = A_\mu(g) + i/2 \bar{\psi} \gamma_\mu T \psi \quad (3)$$

where

$$A_\mu(g) = g^{-1} \partial_\mu g|_{\parallel}.$$

T is the H -representation of the fermion field ψ . Since

$$g^{-1} D_\mu g = g^{-1} \partial_\mu g|_{\perp} - i/2 \bar{\psi} \gamma_\mu T \psi, \quad (4)$$

the Lagrangian becomes

$$- (g^{-1} \partial_\mu g|_{\perp})^2 - i \bar{\psi} \not{D} (A_\mu = g^{-1} \partial_\mu g|_{\parallel}) \psi - 1/4 (\bar{\psi} \gamma \cdot T \psi)^2. \quad (5)$$

If we now parametrize an element of $g(x)$ as follows,

$$g(x) = g[h(x), \phi(x)], \quad \text{with } \phi \in G/H,$$

then the Lagrangian becomes

$$\mathcal{L} = g_{ij}(\phi) \partial_\mu \phi^i \partial_\mu \phi^j - i \bar{\psi} \gamma^\mu \{ \partial_\mu + [h^{-1} \partial_\mu h + h^{-1} A_\mu(\phi) h] \} \psi, \quad (6)$$

where $A_\mu(\phi)$ is a vector field which depends only on the ϕ variables. At this point we observe that we could set $h(x) = 1$ by choosing an H gauge so that the Lagrangian depends only on ϕ .

However, this is merely a formal manipulation which is only true at the classical level. This is a consequence of the fact that, because of the anomaly, the fermion determinant is not invariant under local chiral transformations.

Under an infinitesimal H -transformation

$$h(x) = 1 + i \omega^a T^a, \quad (7)$$

$$\delta_\omega \text{Tr} \log \mathcal{D}[A_\mu(g)] = \int d^4x \omega^a G^a[A_\mu(g)], \quad (8)$$

where the non-Abelian anomaly satisfies the Wess-Zumino consistency condition

$$\int d^4x [\omega'^a \delta_\omega G^a - \omega^a \delta_{\omega'} G^a - [\omega, \omega']^a G^a] = 0. \quad (9)$$

The integrated version of Eq. (8) is the Wess-Zumino term [3],

$$\log \det \mathcal{D}(h^{-1} \partial_\mu h + h^{-1} A_\mu h) = \log \det \mathcal{D}(A_\mu) + \text{WZ}[h(x), A_\mu], \quad (10)$$

Equation (10) reflects the fact that in Eq. (6) we cannot eliminate the $h(x)$ dependence by means of a local chiral rotation on the fermions. If we confine our discussion to a non-supersymmetric theory (or a supersymmetric theory in the Wess-Zumino gauge) the anomaly, associated with an H -local gauge transformation, is given by the following Wess-Zumino term,

$$\text{WZ}[h(x), A_\mu(g)], \quad (11)$$

which may vanish on some anomaly-free subgroup of H . The absence of the anomaly requires the fermions to be in an anomaly-free representation of H , or the WZ term to vanish identically for $A_\mu = A_\mu(g)$. This is, for instance, the case for a CP^1 [2].

However, we note that for non-Abelian groups the infinitesimal anomaly cannot vanish identically. In fact, if we compute Eq. (8) for $A_\mu(g = h)$ it does not vanish. Its integrated version corresponds to the Wess-Zumino-Witten term

$$\text{WZ}[h(x), A_\mu = 0] \propto \int d^5x \epsilon_{ijklm} \text{Tr} (h^{-1} \partial_i h h^{-1} \partial_j h h^{-1} \partial_k h h^{-1} \partial_l h h^{-1} \partial_m h) \quad (12)$$

which does not depend on the ϕ fields of the coset space G/H .

These general considerations apply to any σ -models with fermions in representations of H that are not anomaly free. They include, for instance, CP^{N-1} and Grassmannian four-dimensional supersymmetric models as well as all N -extended four-dimensional supergravities ($N > 4$) with hidden local chiral symmetries [4].

Exceptions are $N = 2$ supersymmetric σ -models on hyper-Kähler [5] manifolds as well as $N = 2$ σ -models on some quaternionic manifolds [6] (hypermultiplets coupled to $N = 2$ supergravity). Also, two-dimensional supersymmetric CP^{N-1} models as well as extended supergravities in dimension $D \neq 4$ do not suffer from this kind of disease.

The situation is particularly relevant in extended supergravities where hidden local symmetries have been used for phenomenological purposes [7] and for a possible dynamical generation of gauge degrees of freedom in analogy with two-dimensional CP^{N-1} models [8]. However, the above considerations of the anomalies imply that this possibility has to be re-examined.

Consider for definiteness $N = 8$ supergravity in $D = 4$ dimensions with local $SU(8)$ hidden symmetry as formulated by Cremmer and Julia [9]. Similar considerations apply, as well, to the gauge-version of de Wit and Nicolai [10]. The only fundamental fields of the theory transforming under $SU(8)$ are, beyond the scalars, the spin-3/2 and spin-1/2 in the 8- and 56-dimensional complex representations of $SU(8)$. Apart from spin factors, the anomaly coefficient of the 56 is five times the anomaly coefficient of the 8: $A(56) = 5A(8)$. Therefore, the anomaly would cancel only if the relative spin factor between spin-3/2 and spin-1/2 would be -5 ! This is not so, since for the Abelian case the spin factor is known to be 3 [11]. If a consistent non-Abelian anomaly for spin-3/2 fields can be found at all, we conclude that it must have the same form as for spin-1/2 with the same spin factor as in the Abelian case.

In the computation of the non-Abelian anomaly in extended supergravity, we have to consider also the effect of additional bilinear fermion terms containing the scalar fields which may also contribute to the anomaly. These terms are of the form

$$\bar{\psi}_\mu \gamma^\nu \gamma^\mu (g^{-1} \partial_\nu g|_{\perp}) \chi = \bar{\psi}_\mu \gamma^\nu \gamma^\mu g^{-1} D_\nu g \chi + 4\text{-fermion terms} . \quad (13)$$

However, we observe that these terms vanish identically for $g = h$, so they can never cancel, for instance, the part of the anomaly induced by the minimal coupling of the spin-3/2 and spin-1/2 fields to the gauge field $A_\mu(g)^*$.

In conclusion, the quantum theory with anomalous $SU(8)$ symmetry differs from the theory formulated in terms of the 70 scalars of the coset $E_7/SU(8)$ by a Wess-Zumino term which depends on extra degrees of freedom of the group $SU(8)$. This Lagrangian does not correspond to $N = 8$ supergravity because of the $SU(8)$ anomaly. It may lead to an inconsistent theory which breaks unitarity. In particular, since $SU(8)$ is a maximal compact subgroup of E_7 , the additional degrees of freedom may leave some non-compact components as propagating degrees of freedom.

An interesting question which arises in the case of supersymmetric σ -models is whether the quantum effects break supersymmetry or not. We can answer this question only partially in the case of an Abelian local $U(1)$ symmetry, as is the case of CP^{N-1} models. This is due to the fact that the non-Abelian extension of the supersymmetric anomaly is slightly more complicated and deserves further investigation [12,13]. In the case of supersymmetric CP^{N-1} models in four dimensions, one can easily derive the supersymmetric extension of the anomalous term found in Ref. 2, using superspace arguments. Let us consider the following functional integral

*) The non-Abelian spin-3/2 anomaly as well as the contribution to the anomaly of the spin-3/2-spin-1/2 interference term have been explicitly computed by N.K. Nielsen (to be published). The first has been shown to be indeed three times the spin-1/2 anomaly. The second has been shown to vanish identically.

$$e^{iF(V)} = \int DS D\bar{S} \exp i(\int d^4x d^4\theta S e^V \bar{S}), \quad (14)$$

S_i being a set of N chiral multiplets coupled to a $U(1)$ vector multiplet. A $U(1)$ gauge transformation on V reads

$$\delta V = i(\Lambda - \bar{\Lambda}), \quad (15)$$

Λ being a chiral superfield. Under this transformation the variation of (14) is

$$\delta F(V) = i \int d^4x (d^2\theta \bar{D}\bar{D}(\delta F/\delta V)\Lambda - i \int d^2\bar{\theta} D D(\delta F/\delta V)\bar{\Lambda}). \quad (16)$$

Using the result for the Abelian anomaly [14],

$$1/4 \bar{D}\bar{D}(\bar{S} e^V S) = c W_\alpha W^\alpha, \quad W_\alpha = \bar{D}\bar{D}D_\alpha V, \quad (17)$$

(where c is a constant computed in Ref. [14]), we get

$$\delta F(V) = c \int d^4x (\int d^2\theta i\Lambda W^2 + \text{h.c.}), \quad (18)$$

Therefore it follows that

$$F[V + i(\Lambda - \bar{\Lambda})] = F(V) + c \int d^4x (\int d^2\theta i\Lambda W^2 + \text{h.c.}). \quad (19)$$

We are now ready to apply this result to the supersymmetric CP^{N-1} model whose action with a hidden $U(1)$ symmetry is

$$\int d^4x d^4\theta (S e^V \bar{S} - V). \quad (20)$$

By choosing the parametrization $S_i = e^{i\Lambda} \hat{S}_i$ [$\hat{S}_i = (1, \hat{S}_i)$] we find that the Lagrangian given by (20) has an additional term, at the quantum level, with respect to the CP^{N-1} model expressed in terms of $N-1$ unconstrained superfields $\hat{S}_i$. The vector superfield V can no longer be integrated to get

$$\int d^4x d^4\theta \log(1 + \hat{S} \cdot \hat{S}). \quad (21)$$

The two Lagrangians differ at the quantum level by the term given by Eq. (18) which is nothing but the supersymmetric extension of the $U(1)$ anomaly. Equation (18) reflects a breakdown of gauge invariance for the model defined by Eq. (20).

As claimed in Ref. [2], the anomalous behaviour of σ -models with respect to the hidden local symmetry may be reflected in an intrinsic sickness of these models. This is not the case in extended supergravity owing to the structure of the non-compact coset space G/H on which the physical scalar fields of the supergravity multiplet live^{*)}. However, in extended supergravities we can have a σ -model in a coset-space smaller than the over-all number of scalar fields. For example, in $N = 8$ supergravity we can have an anomaly-free σ -model on the coset $SL(8, R)/SO(8)$ for the 35 scalars, whilst the 35 pseudoscalars are treated as unconstrained variables. Also, our considerations would not apply to extended supergravities in dimensions different from 4 in which the hidden gauge group is anomaly free.

^{*)} This observation is due to E. Cohen and C. Gomez.

It seems rather obvious that our considerations on the anomalies constitute an obstruction to the hypothesis of dynamical generation of gauge fields in extended supergravity. This is different from the two-dimensional supersymmetric CP^{N-1} models where no gauge anomaly is present [8]. It is worth mentioning that any GUT based on four-dimensional extended supergravity and with chiral fermions would suffer from the problems discussed in this note [7]. The only anomaly-free subgroups of $SU(8)$ which are not vector-like correspond to particular embeddings of $SU(2) \times U(1)$ [15].

Another possible way of staying alive with a hidden chiral symmetry in extended supergravity is to embed this theory in a higher dimensional theory in which this symmetry survives with no anomalies. Then in this case one may envisage, in the reduction to four dimensions, a cancellation of the anomalies between the massless and the infinitely many massive states [16].

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