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FORM FACTOR EFFECTS ON NUCLEUS-MAGNETIC MONOPOLE BINDING

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ABSTRACT

Magnetic monopoles may form bound states with nuclei having a sufficiently large magnetic moment. In previous analyses this magnetic moment has mainly been considered to arise from a pointlike structure. We consider the effects due to a spatial distribution, as reflected by a form factor. The binding will continue to be present, but the binding energies are rather sensitive to the shape and scale of the form factor. Also, the ambiguity with boundary conditions at the origin, which is resolved when the (spin- $\frac{1}{2}$) nucleus has a pointlike magnetic moment, reappears when the latter has a spatial distribution.

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1. - INTRODUCTION

The possibility that magnetic monopoles form bound states with nuclei has been the subject of a considerable amount of study during the last years¹⁾⁻⁵⁾. It is important for the planning of experimental searches to know whether monopoles can be expected to appear in the free state, or as a bound system with ordinary matter. In the latter case, the nucleus is very likely to be stripped of some or all of its bound electrons. The bound state would thus carry a net charge and thereby ionize more strongly and produce more radiation damage than a free (i.e., neutral) monopole. Under the assumption of this scenario, Price et al.⁶⁾ recently presented a very low upper bound on the possible flux of magnetic monopoles.

It is relatively easy to decide when bound states can be formed. At large relative distances, $r \gg \Lambda_{\text{QCD}}^{-1}$, and non-relativistic velocities, the monopole-nucleus system can be described by an equation of the Pauli type^{*}),

$$\left[-\frac{1}{2M} (\vec{\nabla} - ieZ\vec{A})^2 - \vec{\mu} \cdot \vec{B} \right] \psi = E \psi \quad (1.1)$$

where $\vec{A}$ and $\vec{B}$ are the potential and magnetic fields of the monopole, eZ the charge of the nucleus, $\vec{\mu}$ its magnetic moment, and M is the reduced mass of the monopole-nucleus system. (If the monopole is superheavy, as predicted by grand unified theories, the latter is essentially equal to the mass of the nucleus.) When the nucleus is of spin S , the wave function, ψ , is a $(2S+1)$ -component spinor, and

$$\vec{\mu} = g_M \frac{Ze}{2M} \vec{S} \quad (1.2)$$

is an operator acting on the spinor indices. (g_M is the gyromagnetic ratio of the nucleus.)

If the monopole has a magnetic charge

$$g = \frac{n}{2e}, \quad n = \pm 1, \pm 2, \dots,$$

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^{*}) We choose units so that $\hbar = c = 1$.

the electromagnetic field around the monopole-nucleus system carries an angular momentum,

$$|q| = \left| \frac{1}{2} n Z \right|$$

which can combine with the orbital angular momentum to values

$$l = |q|, |q|+1, \dots$$

These in turn can be coupled to spin S to give a total angular momentum

$$j = |q| - S, |q| + 1 - S, \dots,$$

provided $|q| > S$ *). In this case there are

$$N(j) = \begin{cases} S + j + 1 - |q|, & |q| - S \leq j \leq |q| + S \\ 2S + 1, & j \geq |q| + S \end{cases} \quad (1.3)$$

different ways to couple spins to a total angular momentum j . Thus, if we separate Eq. (1.1) into eigenstates of angular momentum, the wave function can be written:

$$\psi_{j j_z}(r, \Omega) = \sum_{\alpha=S+1-N}^S f_{\alpha}(r) \phi_{j j_z}^{(\alpha)}(\Omega), \quad (1.4)$$

where

$$\phi_{j j_z}^{(\alpha)}(\Omega) = \sum_{S_z} \langle S S_z j + \alpha j_z - S_z | j j_z \rangle Y_{j + \alpha, j_z - S_z}^{(q)}(\Omega) \chi_{S, S_z}. \quad (1.5)$$

Here $Y_{\ell \ell_z}^{(q)}$ are the monopole harmonics of Wu and Yang⁷⁾, and χ_{S, S_z} are basis vectors for a spin S representation of the rotation group. Inserting Eqs. (1.4) and (1.5) into Eq. (1.1), one finds the radial dependence to be determined by the

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*) We have found that the only violations of this inequality that can occur in nature are the cases of a monopole of minimal magnetic charge ($|n| = 1$) coupled to H^2 ($S = 1$, $|q| = \frac{1}{2}$), Li^8 ($S = 2$, $|q| = 3/2$) or B^{10} ($S = 3$, $|q| = 5/2$). In all three cases, j can take the values $\frac{1}{2}$, $3/2, \dots$. We shall not consider these exceptions further.

equation

$$-\left[\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{\Lambda}{r^2}\right] f(r) = 2ME f(r) \quad (1.6)$$

where f is an N -component vector and Λ is a $N \times N$ Hermitian matrix

$$\Lambda^{ab} = [q^2 - (j+a)(j+a+1)] \delta^{ab} + q g_M M^{ab}, \quad (1.7)$$

with

$$M^{ab} = \int d\Omega \phi_{jjz}^{(a)\dagger} \vec{S} \cdot \hat{n} \phi_{jjz}^{(b)}. \quad (1.8)$$

Λ can always be diagonalized by a unitary transformation. This decouples Eq. (1.6) into N Bessel equations,

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{\lambda_n - \frac{1}{4}}{r^2} + 2ME \right] \sqrt{r} \hat{f}_n(r) = 0, \quad (1.9)$$

where λ_n , $n = 1, \dots, N$ are the eigenvalues of Λ . As it stands, Eq. (1.9) does not constitute a well-posed eigenvalue problem, but if we assume the $1/r^2$ potential to be regulated at the origin (due to physics we have not yet taken into account), this equation predicts an infinite number of bound states whenever $\lambda_n > 1/4$ ⁸⁾. Since these states become arbitrarily loosely bound as $E \rightarrow 0^-$, the conclusion of their existence is independent of the detailed monopole-nucleus interaction at short distances (and thus of any particular model of the magnetic monopole).

However, to obtain definite predictions for the energy levels and wave functions, we need some knowledge of the physics which regulates Eq. (1.9) at small r .

Several different approaches have been taken to regulate the short-distance behaviour. In Ref. 1), the interaction was regulated by smearing it over the

magnetic moment distribution of the nucleus, thus taking its extension somewhat into account. In Ref. 2), the nucleus was treated as being pointlike, but was prevented from reaching the origin by a hard core repulsion. In Ref. 3), the theory was regulated, following Kazama and Yang⁹⁾, by using the Dirac Hamiltonian for a particle with an anomalous magnetic moment. The main feature of all these approaches is that a length scale, which is lacking in Eq. (1.9), is introduced into the problem. In the case of the proton this will obviously be of the same order of magnitude, but the problem is sufficiently sensitive to details to give rather different results for the binding energies: Ref. 1) finds a binding energy $E_B \approx 190$ keV for the ground state, Ref. 2) gives a lower bound $E_B > 15$ keV, while Ref. 3) obtains^{*)} $E_B \approx 263$ keV. Since we must expect these approaches to give even more diverging results for the heavier nuclei, we would like to decide which one is physically the most reasonable.

Here we shall study the competition between relativistic and finite size effects, by introducing form factors into the Dirac equation. For real nuclei, we always find it consistent to perform the reduction to a non-relativistic equation. However, the equation we obtain differs in an essential way from the one used by Makino et al¹⁾. In our opinion, the latter has some unphysical features (to be spelled out later). Without a pointlike anomalous magnetic moment, we find that the nucleus is no longer prevented from reaching the origin, with the usual ambiguity in imposing a boundary condition here as a result. Thus, a whole family of theories, parametrized by a θ angle, can be defined. This ambiguity remains after the non-relativistic reduction, and leads to an uncertainty in the binding energies which cannot be avoided as long as the nucleus is treated as a one-particle system. We shall argue, however, that the θ -dependence disappears for the heavier nuclei, and present a simplified effective description of the nucleus-monopole system which permits analytical treatment.

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*) Except for a state of binding energy 938 MeV (i.e., exactly zero total energy) which is absent in the non-relativistic treatments.

2. - DIRAC EQUATION FOR PARTICLES WITH FORM FACTORS

The electromagnetic structure of a spin- $\frac{1}{2}$ nucleus can be described by two form factors

$$\langle p' | J^\mu(0) | p \rangle = Ze \bar{u}(p') \left[\gamma^\mu F_1(q^2) + \frac{i}{2M} \sigma^{\mu\nu} q_\nu \kappa F_2(q^2) \right] u(p) \quad (2.1)$$

where $q = (p-p')$. These are normalized so that

$$F_1(0) = F_2(0) = 1,$$

and κ is related to the gyromagnetic ratio by

$$g_M = 2(1 + \kappa).$$

In Eq. (2.1), the nucleus is assumed to satisfy the mass shell condition,

$$p^2 = p'^2 = M^2,$$

but for a bound state this condition is violated. This makes it problematic to calculate with a non-constant form factor $F_1(q^2)$.

Restricting ourselves to the case when the nucleus is a proton, the form factors have been fitted¹⁰⁾ to the expressions

$$\begin{aligned} F_e(q^2) &\equiv F_1(q^2) + \left(\frac{q}{2M}\right)^2 \kappa F_2(q^2) = \left(\frac{\Lambda^2}{\Lambda^2 - q^2}\right)^2, \\ F_m(q^2) &\equiv F_1(q^2) + \kappa F_2(q^2) = (1 + \kappa) \left(\frac{\Lambda^2}{\Lambda^2 - q^2}\right)^2, \end{aligned} \quad (2.2)$$

where

$$\Lambda = 840 \text{ MeV} \simeq 0.9 M. \quad (2.3)$$

Solving for the F_i 's, we find

$$\begin{aligned} F_1(q^2) &= \left[1 - (1 + \alpha) \left(\frac{q}{2M} \right)^2 \right] \frac{4M^2}{4M^2 - q^2} \left(\frac{\Lambda^2}{\Lambda^2 - q^2} \right)^2, \\ F_2(q^2) &= \frac{4M^2}{4M^2 - q^2} \left(\frac{\Lambda^2}{\Lambda^2 - q^2} \right)^2. \end{aligned} \quad (2.4)$$

These expressions differ somewhat from those used by Makino et al¹⁾.

Now consider the propagation of a particle with the above form factors in an external field. In Fourier space the multiple scattering expansion for the propagator sums into the equation

$$G(p, p') = \frac{p + M}{p^2 - M^2} \left\{ \delta(p - p') + \int \frac{d^4 q}{(2\pi)^4} T(q) \tilde{A}_\mu(q) G(p - q, p') \right\}, \quad (2.5)$$

where $\tilde{A}_\mu(q)$ is the Fourier transform of the external electromagnetic potential, and

$$T(q) = eZ \left[\gamma^\mu F_1(q^2) + \frac{i}{2M} \sigma^{\mu\nu} q_\nu \alpha F_2(q^2) \right]. \quad (2.6)$$

Transformed back to real space, we obtain the differential equation

$$\left\{ i\gamma^\mu [\partial_\mu + ieZ A_\mu^{\text{eff}}] - M + \frac{Ze\alpha}{2M} \sigma^{\mu\nu} F_{\mu\nu}^{\text{eff}} \right\} G(x, x') = \delta(x - x'), \quad (2.7)$$

where

$$\begin{aligned} F_{\mu\nu}^{\text{eff}}(x) &= i \int \frac{d^4 q}{(2\pi)^4} e^{iqx} F_2(q^2) [q_\nu \tilde{A}_\mu(q) - q_\mu \tilde{A}_\nu(q)], \\ A_\mu^{\text{eff}}(x) &= \int \frac{d^4 q}{(2\pi)^4} e^{iqx} F_1(q^2) \tilde{A}_\mu(q). \end{aligned} \quad (2.8)$$

$F_{\mu\nu}^{\text{eff}}$ (resp. A_μ^{eff}) is seen to be the electromagnetic field strength (resp. potential) averaged over the magnetic moment (resp. electric charge) distribution of the nucleus.

In a static field, the bound states can be found from the eigenvalue problem corresponding to Eq. (2.7). There is, however, a problem with gauge invariance if we attempt to use a smeared potential, $A_{\mu}^{\text{eff}} = A_{\mu}$. This arises from the fact that the first term in the electromagnetic current (2.1) is conserved only when the external momenta of the nucleus satisfy $p^2 = p'^2$, - a condition which cannot be enforced on Eq. (2.7). This problem is particularly critical in the case of a magnetic monopole field, since the form factor $F_1(q^2)$ is equivalent to having a distance-dependent charge, $Ze(r)$, on the nucleus. (A radial motion of such a charge thus violates charge conservation.) To maintain rotational invariance of the theory, the Dirac condition

$$eg = \frac{1}{2} n, \quad (2.9)$$

must be fulfilled. This prevents us from using a "running" charge, $e(r)^*$. A complete treatment would require us to take into account that the nucleus is composed of (fractionally charged) quarks, and that a monopole of minimal n must also carry a colour magnetic charge. We shall here ignore such complications and simply replace

$$A_{\mu}^{\text{eff}} \rightarrow A_{\mu},$$

as required for rotational and gauge invariance in a one-body treatment. A similar approximation was made by Makino et al¹⁾.

If we insert the magnetic monopole field

$$\vec{B} = \frac{g \hat{r}}{r^2}, \quad \text{i.e.} \quad \vec{B} = \frac{4\pi i g \hat{q}}{q}, \quad (2.10)$$

into Eq. (2.8), we find

$$\vec{B}^{\text{eff}} = \frac{g \hat{r}}{r^2} \Phi(r) \quad (2.11a)$$

where

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*) Unless the magnetic charge, g , runs in the opposite direction so as to maintain the quantization condition, (2.9). Charge renormalizations, due to vacuum polarization, have been shown to behave in this manner¹¹⁾. However, for a nucleus, as here, the form factors are due to its composite structure rather than vacuum polarization.

$$\Phi(r) = 1 - \frac{1}{(1-\varepsilon)^2} \left\{ (1-2\varepsilon)(1+\Lambda r) + \frac{1}{2}(1-\varepsilon)\Lambda^2 r^2 \right\} e^{-\Lambda r} + \varepsilon^2 (1+2Mr) e^{-2Mr} \quad (2.11b)$$

Here $\varepsilon = (\Lambda/2M)^2 \approx 0.2$. Makino et al.¹⁾ used a different form factor,

$$F_2(q^2) = \left(\frac{\Lambda^2}{\Lambda^2 - q^2} \right)^2, \quad (2.12)$$

which leads to a simpler expression for Φ , obtained by setting $\varepsilon = 0$ in Eq. (2.11b). This results in a more attractive potential (cf. Fig. 1). The effect of the form factor is thus to modify the Kazama-Yang Hamiltonian⁹⁾ to

$$H^{\text{eff}} = \vec{\alpha} \cdot (\vec{p} - Ze\vec{A}) + \beta M - \frac{\kappa e \beta}{2M} \vec{\sigma} \cdot \vec{B}^{\text{eff}}, \quad (2.13)$$

where $\vec{\alpha}, \beta$ and $\vec{\sigma}$ are Dirac matrices.

To solve the eigenvalue problem

$$H^{\text{eff}} \psi = E \psi, \quad (2.14)$$

we separate out the angular dependence in the usual way. For the states of minimal angular momentum, $j = 0$ (which is the only partial wave in which binding can occur),

$$\psi(\vec{r}) = \frac{1}{r} \begin{bmatrix} \frac{eg}{1eg_1} F(r) \eta_0(\hat{r}) \\ -i G(r) \eta_0(\hat{r}) \end{bmatrix}, \quad (2.15)$$

where the two-component spinor, η , has been defined in Ref. 12). Inserted into Eq. (2.14), we find the coupled equations

$$\begin{aligned} \frac{d}{dr} G(r) &= \left[M - E - \frac{\kappa |eg|}{2M} \frac{\Phi(r)}{r^2} \right] F(r), \\ \frac{d}{dr} F(r) &= \left[M + E - \frac{\kappa |eg|}{2M} \frac{\Phi(r)}{r^2} \right] G(r), \end{aligned} \quad (2.16)$$

which have to be solved numerically. Assuming weak binding and weak potential,

$$M - E \ll M, \quad (2.17)$$

$$\frac{\alpha |eg|}{2M} \frac{\Phi(r)}{r^2} \ll M,$$

we may perform the non-relativistic reduction to an ordinary Schrödinger equation,

$$\left\{ -\frac{1}{2M} \frac{d^2}{dr^2} - \frac{\alpha |eg|}{2M} \frac{\Phi(r)}{r^2} \right\} F(r) = E_B F(r), \quad (2.18)$$

with $E_B = E - M$. The effective binding potential is plotted in Fig. 1 for the form factors (2.4) resp. (2.14). We have assumed a monopole of minimal magnetic charge, $|eg| = \frac{1}{2}$. We note that the minimum potential energy is only a few per cent of the rest mass energy. Thus, the non-relativistic approximation appears to be consistent (we have nevertheless used the full Dirac equation in our numerical solution, since this is just as easy).

Our Eq. (2.18) is not equivalent to the one obtained by Makino et al.¹⁾. They use a potential with an additional term,

$$\Delta V = \frac{|eg|}{2Mr^2} [1 - \Phi(r)], \quad (2.19)$$

which becomes repulsive at the origin. The difference arises from the fact that they do not set $F_1(q^2) = 1$ from the beginning. In our opinion, the term (2.19) is unphysical: at short distances the monopole does not interact with the nucleus as a collective system, but with individual quarks having no pointlike anomalous magnetic moments. Thus the interaction should vanish at short distances, as reflected by the potentials in Fig. 1. In fact, these potentials are qualitatively similar to the effective potential between a magnetic monopole and a hydrogen atom, calculated in the Born-Oppenheimer approximation¹³⁾.

However, since the potential in Eq. (2.18) does not force the wave function to vanish at the origin, we encounter an ambiguity in imposing boundary conditions here. As has been much discussed in the literature¹⁴⁾⁻¹⁷⁾, we can define a

one-parameter family of self-adjoint Hamiltonians, specified by the boundary conditions

$$\frac{F'(0)}{G(0)} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right), \quad (2.20)$$

where θ is the free parameter. $\theta = 0$ or $\theta = \pm\pi$ leads to a CP invariant theory. These are also the boundary conditions which are dynamically enforced if the fermion has a pointlike anomalous magnetic moment, which is either negative (implying $\theta = 0$) or positive (implying $\theta = \pm\pi$). In the non-relativistic reduction, Eq. (2.20) translates into

$$F'(0) = \frac{1}{2M} \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \frac{dF}{dr}(0). \quad (2.21)$$

Clearly, beyond the (doubtful) requirement of CP symmetry, there is no way to determine θ at this level.

3. - NUMERICAL SOLUTIONS

We have solved the eigenvalue problem (2.16) numerically for a range of boundary conditions (2.20). The result is depicted in Fig. 2. The two CP-conserving boundary conditions lead to binding energies of 557 keV ($\theta = 0$) and 236 keV ($\theta = -\pi$). The latter corresponds to the dynamically-imposed boundary condition of a positive pointlike anomalous magnetic moment, and does not differ much from the binding energy of 263 keV found in that case³⁾. This agreement seems to be somewhat accidental, however. If we investigate the one-parameter class of potentials,

$$V_{\xi}(r) = \frac{\kappa}{4Mr^2} \Phi(\xi r), \quad (3.1)$$

(with $\xi \rightarrow \infty$ corresponding to the pointlike limit, and $\xi = 1$ to the physical proton), the binding energy behaves as in Fig. 3 (using $\theta = -\pi$). The behaviour at small ξ is easily understood from Fig. 1; as ξ decreases, the binding potential becomes more and more shallow, with $V_{\min} \sim \xi^2$. The behaviour as $\xi \rightarrow \infty$ can be found analytically using lowest order perturbation theory around the pointlike

limit. We find (to leading order):

$$\frac{\Delta E_B}{|E_B^\infty|} = -k (2\pi^2)^{1/4} \frac{1+4\beta^2}{8\beta} \sinh(\pi\beta) \tilde{\xi}^{5/4} e^{-\sqrt{8}\tilde{\xi}}, \quad (3.2)$$

where

$$\begin{aligned} \tilde{\xi} &= \frac{1}{4} \kappa \xi \frac{\Lambda}{M} \simeq 0.4 \xi, \\ \beta &= \frac{1}{2} \sqrt{2\kappa - 1} \simeq 0.8, \end{aligned} \quad (3.3)$$

and

$$k = \begin{cases} \frac{1}{1-\epsilon} \simeq 1.25 & , \quad \text{for the form factor (2.4)} \\ 1 & , \quad \text{for the form factor (2.12)} \end{cases} \quad (3.4)$$

This verifies the large- ξ behaviour observed in Fig. 3.

4. - SIMPLIFIED DESCRIPTION

The uncertainty as to how the form factor F_1 should be treated, as well as the uncertainty in the boundary condition (2.20) severely limits the accuracy of the predictions one can make when treating the nucleus as a single particle. One can thus just as well mimic the effect of a form factor by using a simpler potential which permits an analytic treatment, and can be generalized to nuclei of higher spins as well.

Looking at Fig. 1, a reasonable approximation is to replace the potential in Eq. (2.18) by

$$V(r) \simeq \begin{cases} -\frac{\kappa |Zeg|}{2Mr^2} & , \quad r \geq r_0 = f\Lambda^{-1}, \\ \text{const.} & , \quad r \leq r_0, \end{cases} \quad (4.1)$$

where f is a fudge factor which is about 5. For nuclei of higher spin and states of non-minimal angular momentum, the factor $\kappa |Zeg|$ should be replaced by the appropriate eigenvalue λ_n entering Eq. (1.9). To check the non-relativistic

approximation in the case of heavier nuclei, we expect $\kappa \sim A^0$, $(Z, M) \sim A^1$ and $r_0 \sim A^{1/3}$, where A is the nucleon number. Thus,

$$\frac{V_{\min}}{M} \sim A^{-2/3}, \quad (4.2)$$

and the non-relativistic approximation becomes progressively better for heavier nuclei. In the limit

$$\left| \frac{2E_B}{M} \right| \ll \frac{\lambda_n}{M^2 r_0^2} \quad (4.3)$$

the solution to Eq. (2.18), with the replaced potential (4.1), becomes in the inner region, $r < r_0$:

$$F(r) \sim \sin(\sqrt{\lambda_n} \frac{r}{r_0} + \phi), \quad (4.4)$$

where ϕ is determined from the boundary condition (2.21),

$$\tan \phi = \frac{1}{2} \frac{\sqrt{\lambda_n}}{M r_0} \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right). \quad (4.5)$$

As the nucleon number A increases, the right-hand side of Eq. (4.5) is expected to scale like $A^{-5/6}$. Thus, the dependence on θ becomes less pronounced for heavier nuclei.

In the external region, $r > r_0$, the solution is given by a Bessel function

$$F(r) \sim \sqrt{Mr} K_{i\beta}(\sqrt{2M|E_B|}r), \quad (4.6)$$

where $\beta = (\lambda_n - 1/4)^{1/2}$. This must match continuously and differentiably with the inner solution, which leads to the eigenvalue condition

$$E_B^{(N)} = -\frac{1}{2Mr_0^2} x_N^2 \quad (4.7)$$

where x_N is the N 'th root of the equation

$$\frac{x \frac{d}{dx} (\sqrt{x} K_{i\beta}(x))}{\sqrt{x} K_{i\beta}(x)} = \sqrt{\lambda_n} \cot(\sqrt{\lambda_n} + \varphi) \quad (4.8)$$

(counting from above, $x_N \rightarrow 0$ as $N \rightarrow \infty$). For small values of x ,

$$K_{i\beta}(x) \sim \sin\{\beta \ln(\frac{1}{2}x) - \arg \Gamma(1+i\beta)\}, \quad (4.9)$$

and Eq. (4.8) simplifies to

$$\sin[\beta \ln(\frac{1}{2}x) - \chi] = 0, \quad (4.10)$$

where we have defined

$$\chi = \arg \Gamma(1+i\beta) + \arccot\left[\sqrt{1+\frac{1}{4}\beta^2} \cot(\sqrt{\frac{1}{4}+\beta^2} + \varphi) - \frac{1}{2\beta}\right], \quad (4.11)$$

and used the relation $\beta = (\lambda_n - 1/4)^{1/2}$. Thus

$$E_B^{(N)} = - \left(\frac{2 e^{2\chi/\beta}}{M r_0^2} \right) e^{-\frac{2\pi}{\beta} N}, \quad (4.12)$$

which agrees with a parametrization used by Bracci and Fiorentini¹⁸⁾, but with a more specific model for their scale parameter, a . The binding energy of the lowest (non-relativistic) monopole-proton state, as a function of the parameter θ , is plotted in Fig. 2. It agrees with the more elaborate form factor calculation within the intrinsic accuracy of such one-particle models.

For the lower bound states of heavier nuclei, neither the inequality (4.3) nor the small x approximation (4.9) are valid. Instead, we expect the kinetic energy to be negligible compared to the potential energy, since their ratio will behave like

$$\frac{E^{\text{kin}}}{E^{\text{pot}}} \sim \frac{1/M r_0^2}{\lambda_n/M r_0^2} \sim \lambda_n^{-1} \sim A^{-1}. \quad (4.13)$$

Thus, we estimate the binding energy to be

$$E_B \simeq -\frac{\lambda_n}{2Mr_0^2} \sim A^{-2/3}, \quad (4.14)$$

where $r_0 = A^{1/3}$ fm, and λ_n can be calculated explicitly as outlined in Section 1.

An evaluation of Eq. (1.8), performed in the Appendix, gives

$$M^{a,a} = \frac{(j+a)(j+a+1) + S(S+1) - j(j+1)}{2(j+a)(j+a+1)} q \quad (4.15)$$

$$M^{a,a+1} = M^{a+1,a} = \left[\frac{(j+a+1+q)(j+a+1-q)(2j+a+S+2)(2j+a-S+1)(a+S+1)(S-a)}{4(j+a+1)^2(2j+2a+1)(2j+2a+3)} \right]^{1/2}$$

with all the other matrix elements being zero.

For the states of lowest angular momentum, $j = |q| - S$, there is only a single eigenvalue,

$$\lambda_0 = SX - |q| \quad (4.16a)$$

(corresponding to the strongest attraction if $g_M > 1$) and for $j = |q| + 1 - S$ there is the pair

$$\lambda_{\pm} = (S - \frac{1}{2})X - 2|q| - 1 \pm \sqrt{(|q|+1)^2 + (2S-1-|q|)X + \frac{1}{4}X^2}. \quad (4.16b)$$

Here $X = g_M |q|$, $q = Zeg$. g_M is related to the magnetic moment, μ , tabulated in nuclear magnetons, by the expression

$$g_M = \frac{A}{SZ} \mu. \quad (4.17)$$

The Table shows numerical values for λ and estimated binding energies for a sample of nuclei. These binding energies are somewhat larger than those estimated by Goebel⁵⁾, since he considers the "worst case" of a hard core repulsion.

5. - CONCLUDING REMARKS

The binding between nuclei and magnetic monopole is rather sensitive to short-distance effects, due to the singular nature of the $1/r^2$ -potential, even when the rms radius of the bound system is large (as is the case for the monopole-proton system). To account for the short-distance behaviour in a fundamentally correct way, one has to consider the nucleus as a many-body system composed of quarks. This, of course, will push the ambiguity at the origin to the quark level, where it might be resolved by taking the finite size structure of the monopole into account. A calculation within the Born-Oppenheimer approximation, analogous to the one performed by Ruijgrok et al.¹³⁾ for the monopole-hydrogen atom system, does not seem impossible to repeat for a three-quark nuclei. Such an undertaking might also give some additional insight into the Rubakov effect¹⁹⁾, and its influence upon the binding.

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APPENDIX

If we replace the Clebsch-Gordan coefficients in Eq. (1.5) by 3-j symbols and write

$$\vec{S} \cdot \hat{r} = \sum_m (-1)^m \sqrt{\frac{4\pi}{3}} Y_{1,m}^{(0)} S_m \quad (\text{A.1})$$

where

$$\begin{aligned} S_1 &= -\frac{1}{\sqrt{2}} (S_x + i S_y) \\ S_0 &= S_z \\ S_{-1} &= \frac{1}{\sqrt{2}} (S_x - i S_y) \end{aligned} \quad (\text{A.2})$$

the matrix elements (1.8) may be rewritten

$$\begin{aligned} M^{ab} &= (-1)^{q-a+b} \sqrt{\frac{4\pi}{3}} (2j+1) * \\ &* \sum_{S_z S'_z m} (-1)^{j_z - S'_z} \begin{pmatrix} S & j+b & j \\ S'_z & j_z - S'_z & -j_z \end{pmatrix} \begin{pmatrix} S & j+a & j \\ S_z & j_z - S_z & -j_z \end{pmatrix} * \\ &* \chi_{SS'_z}^+ (-1)^m S_m \chi_{SS_z} * \\ &* \int d\Omega Y_{j+b, -j_z + S'_z}^{(-q)} Y_{1, -m}^{(0)} Y_{j+a, j_z - S_z}^{(q)} , \end{aligned} \quad (\text{A.3})$$

where $Y_{j,m}^{(q)}$ are monopole harmonics, and we have used theorem 1 of Ref. 20). We now evaluate

$$\chi_{SS'_z}^+ (-1)^m S_m \chi_{SS_z} = (-1)^{S-S_z} \sqrt{S(S+1)(2S+1)} \begin{pmatrix} S & S & 1 \\ S_z & -S'_z & m \end{pmatrix} \quad (\text{A.4})$$

and, from theorem 3 of Ref. 20),

$$\begin{aligned} \int d\Omega Y_{j+b, -j_z + S'_z}^{(-q)} Y_{1, -m}^{(0)} Y_{j+a, j_z - S'_z}^{(q)} &= \\ = (-1)^{2j+a+b+1} \sqrt{\frac{3}{4\pi}} \sqrt{(2j+2a+1)(2j+2b+1)} * \\ * \begin{pmatrix} j+a & j+b & 1 \\ q & -q & 0 \end{pmatrix} \begin{pmatrix} j+a & j+b & 1 \\ j_z - S'_z & -j_z + S'_z & -m \end{pmatrix} \end{aligned} \quad (A.5)$$

Inserting this into Eq. (A.3)

$$\begin{aligned} M^{ab} &= (-1)^{q+2b+2j+S+1} (2j+1) \sqrt{(2j+2a+1)(2j+2b+1)S(S+1)(2S+1)} * \\ * \begin{pmatrix} j+a & j+b & 1 \\ q & -q & 0 \end{pmatrix} \sum_{S'_z m} (-1)^{j_z - S'_z - S'_z} \begin{pmatrix} S & j+a & j \\ S'_z & j_z - S'_z & j_z \end{pmatrix} * \\ * \begin{pmatrix} S & j+b & j \\ S'_z & j_z - S'_z & j_z \end{pmatrix} \begin{pmatrix} S & S & 1 \\ S'_z & -S'_z & m \end{pmatrix} \begin{pmatrix} j+a & j+b & 1 \\ j_z - S'_z & -j_z + S'_z & -m \end{pmatrix} = \\ = (-1)^{a+b+j-S+q+1} \sqrt{(2j+2a+1)(2j+2b+1)S(S+1)(2S+1)} \\ * \begin{pmatrix} j+a & j+b & 1 \\ q & -q & 0 \end{pmatrix} \left\{ \begin{matrix} j+a & j+b & 1 \\ S & S & j \end{matrix} \right\}, \end{aligned} \quad (A.6)$$

where the sum over the four 3-j symbols is easily evaluated with the aid of the graphical rules described in Ref. 21). The object in curly brackets is a 6-j symbol. Inserting explicit formulae for the 3-j and 6-j symbols into Eq. (A.16) it can be further reduced to the elementary expressions given in Eq. (4.15).

	S	μ	$ q $	X	λ_0	λ_+	λ_-	E_B (MeV)
$^{23}_{11}\text{Na}$	3/2	2.2175	5.5	17.0	20.0	12.4	-2.4	1.6
$^{27}_{13}\text{Al}$	5/2	3.6414	6.5	19.7	42.8	35.6	15.2	2.6
$^{55}_{25}\text{Mn}$	5/2	3.444	12.5	37.9	82.3	64.6	35.0	1.6
$^{85}_{37}\text{Rb}$	5/2	1.3524	18.5	23.0	39.0	21.4	-5.4	0.4
$^{135}_{56}\text{Ba}$	3/2	0.8365	28.0	37.6	28.4	-4.7	-34.1	0.1
$^{138}_{57}\text{La}$	5	3.707	28.5	51.2	227.5	195.4	149.4	0.9

Table : Relevant parameters and estimated largest binding energy for a sample of nuclei. The magnetic monopole is assumed to be of minimal strength, $|eg| = \frac{1}{2}$.

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FIGURE CAPTIONS

- Fig. 1 : Effective potential for the form factors given by (a) Eq. (2.4), (b) Eq. (2.12) and (c) a simplified potential introduced in Section 4 which mimics the overall behaviour.
- Fig. 2 : Binding energy for the lowest non-relativistic state as a function of the boundary condition parameter θ ; (a) for the form factor (2.4), (b) for the simplified potential (4.1).
- Fig. 3 : Variation of the binding energy as the form factor is scaled to the pointlike limit ($\xi \rightarrow \infty$), $\theta = -\pi$; (a) for the form factor (2.4), (b) for the form factor (2.12).

Fig. 1

Fig. 2

Fig. 3