CERN-TH.4000/84

KALUZA-KLEIN THEORIES IN TWELVE DIMENSIONS

D. Bailin^{*)}

CERN - Geneva

and

A. Love

Physics Department, Bedford College
University of London

We study 8-dimensional compact manifolds which are either products of spheres and complex projective planes, or are coset spaces L_{pqrst} of the type $SU(3) \times SU(2) \times U(1) / SU(2) \times U(1)$. It is shown that for Freund-Rubin compactification, product manifolds are unstable for a wide range of cosmological constants. For the coset spaces L_{pqrst} , we find cases where the harmonic expansion of a spinor or a vector-spinor in twelve dimensions contains the usual quark and lepton representations, and also show that it is possible to choose the metric to obtain an Einstein space.

^{*)} Permanent address: School of Mathematical and Physical Sciences, University of Sussex, Brighton, U.K.

CERN-TH.4000/84

August 1984

I. - INTRODUCTION

There has been a revival of interest¹⁾ in higher dimensional gravity²⁾ as the origin of non-Abelian gauge fields. In such theories (Kaluza-Klein theories) in $4+D$ dimensions, the ground state is assumed to be the product of four-dimensional space-time with some compact D -dimensional manifold B , these extra spatial dimensions being of the order of the Planck length and therefore not directly observable. The gauge fields are a residue of the original $(4+D)$ -dimensional coordinate invariance, arising after this spontaneous compactification as massless modes in an expansion of the metric about the ground state. The gauge group is the isometry group of the manifold B .

The lowest dimensionality in which (at least) an $SU(3) \times SU(2) \times U(1)$ gauge group can arise in this way is eleven³⁾. However, the seven-dimensional compact manifolds M_{pqr} and M_{pqrst} , discussed in Refs. 3) and 4), do not permit leptons or quarks in the usual representations of $SU(3) \times SU(2) \times U(1)$. (The argument is that four-dimensional spinors are always associated with spinors or tensor-spinors on the compact manifold⁵⁾, and a harmonic expansion on M_{pqrst} does not contain the required lepton and quark representations⁴⁾.)

If we wish to maintain the standpoint that gauge fields all arise from the metric, in pure Kaluza-Klein fashion, then we must go to higher dimensionality. In this paper, we shall explore the situation in twelve dimensions, i.e., for eight-dimensional compact manifolds. (Conventional supergravity theories do not appear possible in more than eleven dimensions. However, Kaluza-Klein theories are of interest even if supergravity should not prove to have physical relevance. There are also unconventional possibilities for supergravity theories in more than eleven dimensions⁶⁾.) There is another reason why twelve dimensions might be better than eleven dimensions, namely, that it does not appear possible¹⁾ to obtain chiral fermions in an odd dimensionality. It is sometimes possible in even dimensionality, by introducing some explicit gauge fields in $4+D$ dimensions, to obtain chiral fermions, for compact manifolds of appropriate topology^{1), 7)}. The standpoint we shall adopt here is that we may either tolerate the possibility of as yet undiscovered mirror fermions, or else introduce some explicit gauge fields in $4+D$ dimensions specifically for the purpose of producing chiral fermions. However, we shall insist that we first obtain all the gauge fields of the standard model from the metric, and that the appropriate fermion representations of $SU(3) \times SU(2) \times U(1)$ already occur.

The structure of the paper is as follows. In Section 2, eight-dimensional compact manifolds constructed from (products of) spheres (S^N) and complex projective planes (CP^N) are considered. In Section 3, the stability of product manifolds is discussed in the context of Freund-Rubin compactification⁸⁾, for an arbitrary cosmological constant. Section 4 introduces and parametrizes coset spaces L_{pqrst} of the type $SU(3) \times SU(2) \times U(1) / SU(2) \times U(1)$. In Section 5, the harmonic expansion on L_{pqrst} of spinors and vector-spinors is used to determine the values of the parameters p , q and r for which the usual quark and lepton representations arise. In Section 6, the components of curvature and the Ricci tensor for L_{pqrst} are calculated, and it is shown that there are choices of the metric on these manifolds for which an Einstein space is obtained (as required for Freund-Rubin compactification⁸⁾.) The possibility of calculating gauge field coupling constants is also considered. Section 7 contains our conclusions.

2. - FAMILIAR EIGHT-DIMENSIONAL MANIFOLDS

In this section, we discuss briefly possible eight-dimensional compact manifolds formed from (products of) spheres (S^N) and complex projective planes (CP^N).

N

The possibilities with gauge group (isometry group) at least $SU(3) \times SU(2) \times U(1)$ are listed in the Table. The two cases which are not product manifolds both have problems. The manifold CP^4 does not admit spinors⁹⁾, and the isometry group $SO(9)$ of S^8 has only real spinorial representations. All the examples of product manifolds with CP^2 factors do not admit spinors, and examples with S^1 factors cannot be made into Einstein spaces required for Freund-Rubin compactification⁸⁾. We shall see in the next section that, for Freund-Rubin compactification, product manifolds are unstable for all values of the cosmological constant. Thus, all the manifolds of this section have drawbacks, and in Section 4 we shall consider a more attractive class of "unfamiliar" manifolds which are not (products of) spheres and complex projective planes.

It is possible that there may be compactification mechanisms for which some product manifolds are stable, for a zero value of the (four-dimensional) cosmological constant Λ_4 . In that case, ruling out manifolds which do not admit spinors, and manifolds with only real spinorial representations of the isometry group, we would be left with the possibilities $CP^3 \times S^2$, $S^5 \times S^3$ and $S^5 \times S^2 \times S^1$. (The problem of only real spinorial representations for S^8 and $S^6 \times S^2$ is more severe than the chiral fermion problem endemic to Kaluza-Klein theories, since it has a group

theoretic origin, and could not be remedied by explicit twelve-dimensional gauge fields.) However, a harmonic expansion of spinors or vector-spinors of the tangent space on these manifolds does not contain the usual quark and lepton representations of $SU(3) \times SU(2) \times U(1)$. For $S^5 \times S^3$ and $S^5 \times S^2 \times S^1$, this follows from Section 5 because, as remarked in Section 4, these manifolds are special cases of the manifolds L_{pqrst} . For $CP^3 \times S^2$, it is straightforward to show this using the general procedure of Salam and Strathdee¹⁰⁾.

3. - STABILITY OF PRODUCT MANIFOLDS

It is known¹¹⁾ that in the context of eleven-dimensional supergravity (where compactification is by the Freund-Rubin mechanism,⁸⁾) that product compact manifolds are classically unstable. In that case, the supersymmetry of the theory forbids an eleven-dimensional cosmological constant $\bar{\Lambda}$. Since we are not restricting attention here to supergravity theories, we wish to discuss the stability for arbitrary values of $\bar{\Lambda}$ for which the four-dimensional cosmological constant Λ_4 vanishes.

For a 4+D dimensional Kaluza-Klein theory where the compact manifold is a product of manifolds with dimensions D_1 and D_2 , where

$$D = D_1 + D_2, \quad (1)$$

the ground state metric is of the form

$$\bar{g}_{AB} = \text{diag} \{ 1, \bar{g}_{ij}, \bar{g}_{ab}, \bar{g}_{mn} \} \quad (2)$$

with

$$\bar{g}_{ij} = - R^2(t) \hat{g}_{ij}(x) \quad (3)$$

$$\bar{g}_{ab} = - \tilde{R}_1^2(t) \tilde{g}_{ab}^{(1)}(y) \quad (4)$$

and

$$\bar{g}_{mn} = - \tilde{R}_2^2(t) \tilde{g}_{mn}^{(2)}(z) \quad (5)$$

Here, x , y and z denote respectively the spatial co-ordinates of three-dimensional space and the factors in the product manifold, and $R(t)$, $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are time-dependent radii. For Einstein spaces, the Ricci tensor is given by

$$\hat{R}_{ij} = 2k \hat{g}_{ij} \quad (6)$$

$$\tilde{R}_{ab}^{(1)} = 2\tilde{k}_1 \tilde{g}_{ab}^{(1)} \quad (7)$$

and

$$\tilde{R}_{mn}^{(2)} = 2\tilde{k}_2 \tilde{g}_{mn}^{(2)} \quad (8)$$

(As usual, k is 0, 1 or -1 for flat, closed or open cosmology for the ordinary dimensions. If for example, the D_1 -dimensional factor of the compact manifold is a D_1 -dimensional sphere, then $2\tilde{k}_1$ is D_1-1 .) The 4+D dimensional Einstein field equations are

$$\bar{R}_{AB} - \frac{1}{2} (\bar{R} + \bar{\Lambda}) \bar{g}_{AB} = 8\pi \bar{G} \bar{T}_{AB}, \quad (9)$$

where A and B range over all 4+D dimensions and $\bar{\Lambda}$ is the 4+D dimensional cosmological constant. We assume that the energy momentum tensor $\bar{T}_{AB}$ is of the form

$$\bar{T}_{\mu\nu} = H g_{\mu\nu}, \quad \bar{T}_{ab} = \tilde{H}_1 \tilde{g}_{ab}, \quad \bar{T}_{mn} = \tilde{H}_2 \tilde{g}_{mn} \quad (10)$$

where μ, ν are the values 0,1,2,3 and H , $\tilde{H}_1$ and $\tilde{H}_2$ are functions only of $\tilde{R}_1$ and $\tilde{R}_2$. For Freund-Rubin compactification,

$$H = -\tilde{H}_1 = -\tilde{H}_2 = \frac{1}{2} F^2(t), \quad (11)$$

where $F(t)$ is determined by the expectation value of the rank four antisymmetric tensor field strength,

$$F^{\mu\nu\rho\sigma} = |g_4|^{-1/2} \epsilon^{\mu\nu\rho\sigma} F(t). \quad (12)$$

The field equation for this tensor field gives

$$F^2(t) = \tau \tilde{R}_1^{-2D_1} \tilde{R}_2^{-2D_2}, \quad (13)$$

where τ is a constant.

Stability of the product compact manifold can be studied by making small perturbations in the Einstein field equation (9), about the equilibrium values of R , $\tilde{R}_1$ and $\tilde{R}_2$. (A similar calculation has been done for the case of a compact manifold where only a single radius $\tilde{R}$ is varied¹²⁾, so we omit the details.). The equilibrium solution is given by

$$2\tilde{k}_1 \tilde{R}_1^{-2} = f_1 + \frac{\bar{\Lambda}}{D+2}, \quad 2\tilde{k}_2 \tilde{R}_2^{-2} = f_2 + \frac{\bar{\Lambda}}{D+2} \quad (14)$$

and we find the criteria necessary for stability are

$$2 \frac{\bar{\Lambda}}{D+2} + f_1 + f_2 + \frac{1}{2} \left(\tilde{R}_1 \frac{\partial f_1}{\partial \tilde{R}_1} + \tilde{R}_2 \frac{\partial f_2}{\partial \tilde{R}_2} \right) < 0 \quad (15)$$

and

$$\begin{aligned} & \left[\bar{\Lambda}/(D+2) + f_1 + \frac{1}{2} \tilde{R}_1 \frac{\partial f_1}{\partial \tilde{R}_1} \right] \left[\bar{\Lambda}/(D+2) + f_2 + \frac{1}{2} \tilde{R}_2 \frac{\partial f_2}{\partial \tilde{R}_2} \right] \\ & - \frac{1}{4} \tilde{R}_1 \frac{\partial f_2}{\partial \tilde{R}_1} \tilde{R}_2 \frac{\partial f_1}{\partial \tilde{R}_2} < 0, \end{aligned} \quad (16)$$

where

$$(D+2) f_1 \equiv 8\pi \bar{G} \left[4H - (D_2+2) \tilde{H}_1 + D_2 \tilde{H}_2 \right] \quad (17)$$

and

$$(D+2) f_2 \equiv 8\pi \bar{G} \left[4H - (D_1+2) H_2 + D_1 \tilde{H}_1 \right]. \quad (18)$$

For Freund-Rubin compactification where H , $\tilde{H}_1$ and $\tilde{H}_2$ are given by (11) and (123), the stability condition (15) gives

$$\bar{\Lambda}/(D+2) + f_1 < \frac{1}{2} D f_1 \quad (19)$$

and the stability condition (16) gives

$$\left[\bar{\Lambda}/(D+2) + f_1 \right] \left[\bar{\Lambda}/(D+2) + (1-D)f_1 \right] > 0. \quad (20)$$

Since, from (14), $\bar{\Lambda}/(D+2)+f_1$ must be positive for a solution to exist, (19) and (20) cannot both be satisfied, and the product manifold is never stable for Freund-Rubin compactification.

4. - MANIFOLDS L_{pqrst}

It has been found in Sections 2 and 3 that familiar eight-dimensional compact manifolds, constructed from (products of) spheres S^N and complex projective planes CP^N , have problems with stability, with a failure to admit spinors, or with non-existence of complex spinor representations. In this section, we construct eight-dimensional coset spaces of the type $SU(3) \times SU(2) \times U(1)/SU(2) \times U(1)$. We shall find in Section 5 that, for some values of their parameters, these manifolds will allow the known quark and lepton representations of $SU(3) \times SU(2) \times U(1)$ to arise.

The coset spaces G/H with

$$G \equiv SU(3) \times SU(2) \times U(1) \quad (21)$$

and

$$H \equiv SU(2) \times U(1) \quad (22)$$

may be parametrized by first choosing an explicit representation of G in terms of block diagonal 6×6 matrices

$$\tilde{\lambda}_a = \text{diag}(\lambda_a, 0, 0) \quad (a = 1, \dots, 8) \quad (23)$$

$$\tilde{\varrho} = \text{diag}(0, \varrho, 0) \quad (24)$$

and

$$Y = \text{diag}(0, 0, 1). \quad (25)$$

Next, it is necessary to specify the embedding of H in G . We take the $U(1)$ factor to be

$$Z = \frac{1}{2}(\sqrt{3} p \tilde{\lambda}_8 + q \tilde{\sigma}_3 + 2r Y) \quad (26)$$

with p , q and r integers, to obtain a compact $U(1)$. Generically¹³⁾ the isometry group G_I is

$$G_I = G \times N'(H)/H \quad (27)$$

where $N'(H)/H$ is the normalizer of H in G with any $U(1)$ factors which also occur explicitly in G divided out. The factor G is associated with the left action of G on the coset-space, and the factor $N'(H)/H$ is associated with the right action of G on the coset space. For the present manifolds, the extra $U(1)$ factor over and above the standard model is not particularly welcome. However, its effects may be suppressed by spontaneous symmetry breaking at a scale well above the electroweak scale but below the Planck scale. It is in any case not possible to do better. Clearly, we cannot obtain just $SU(3) \times SU(2) \times U(1)$ by considering a manifold with G larger than $SU(3) \times SU(2) \times U(1)$. Nor can we obtain a manifold of dimensionality greater than seven with isometry group just $SU(3) \times SU(2) \times U(1)$ by considering manifolds G/H with G' a subgroup of $SU(3) \times SU(2) \times U(1)$ and H' an arbitrary subgroup of H . (This is easily seen by exhausting the possibilities.)

There are a few exceptional cases for which the manifolds introduced here reduce to "familiar" manifolds of the Table, with a larger isometry group. For $p = 0$, $r \neq 0$ and also for $q = 0$, $r \neq 0$ we obtain $S^5 \times S^3$. For $p = r = 0$, we obtain $S^5 \times S^2 \times S^1$, and $q = r = 0$ yields $Cp^2 \times S^3 \times S^1$. (The only cases in the Table which are candidates to arise in this way are those with an isometry group of rank > 5 .)

To complete the specification of the manifold (rather than just the coset space) it is necessary to say which combinations of $\tilde{\lambda}_8$, $\tilde{\sigma}_3$ and Y are in the tangent space. These must be independent of Z , but there is no reason for them to be orthogonal to Z . (This has been made in the eleven-dimensional context by Randjbar-Daemi et al.⁴⁾) For p , q and r non-zero, we may take the combinations lying in the tangent space to be

$$X' = \frac{1}{2} (\sqrt{3} \tilde{\lambda}_3 + 2s Y) \quad (28)$$

and

$$X'' = \frac{1}{2} (\tilde{\sigma}_3 + 2t Y) \quad (29)$$

where s and t are free parameters. These are clearly independent of each other, and are independent of Z provided

$$ps + qt - r \neq 0 \quad (30)$$

Making use of stereographic co-ordinates¹⁴⁾ we can parametrize the general element of the coset space in terms of co-ordinates z^A ($A=4,5,6,7$), z^m ($m=1,2$), z^3 and z^8 as

$$L(z) = M_3(z^A) M_2(z^m) \exp(i z_3 X') \exp(i z_8 X'') \quad (31)$$

where

$$M_3(z^A) \equiv \text{diag}(N_3, I, 1) \quad (32)$$

and

$$M_2(z^m) \equiv \text{diag}(I, N_2, 1) \quad (33)$$

In (33),

$$(1 + |w_1|^2 + |w_2|^2) N_3 = \begin{pmatrix} 1 - |w_1|^2 + |w_2|^2 & -2w_1 w_2^* & -2w_1 \\ -2w_2 w_1^* & 1 + |w_1|^2 - |w_2|^2 & -2w_2 \\ 2w_1^* & 2w_2^* & 1 - |w_1|^2 - |w_2|^2 \end{pmatrix} \quad (34)$$

with

$$w_1 \equiv z^5 + i z^4, \quad w_2 \equiv z^7 + i z^6 \quad (35)$$

and

$$|w_1|^2 + |w_2|^2 \leq 1. \quad (36)$$

In (34),

$$(1 + |u|^2) N_2 = \begin{pmatrix} 1 - |u|^2 & -2u \\ 2u^* & 1 - |u|^2 \end{pmatrix} \quad (37)$$

with

$$u = z^2 + iz' \quad (38)$$

and

$$|u|^2 \leq 1 \quad (39)$$

The vielbein one-forms B^α , ($\alpha=1, \dots, 8$) for the manifold now follow in standard fashion¹⁰⁾ as the components of $L^+(z)dL(z)$ along the generators $\tilde{\lambda}_A$, ($A=4, 5, 6, 7$). $\tilde{\sigma}^m$ ($m=1, 2$), X' and X'' . We find,

$$\begin{aligned} B^4 &= a^{-1} \left[K_4 \cos \frac{3}{2} z^3 - K_5 \sin \frac{3}{2} z^3 \right] \\ B^5 &= a^{-1} \left[K_4 \sin \frac{3}{2} z^3 + K_5 \cos \frac{3}{2} z^3 \right] \\ B^6 &= a^{-1} \left[K_6 \cos \frac{3}{2} z^3 - K_7 \sin \frac{3}{2} z^3 \right] \\ B^7 &= a^{-1} \left[K_6 \sin \frac{3}{2} z^3 + K_7 \cos \frac{3}{2} z^3 \right] \\ B^1 &= -b^{-1} \left[K_1 \cos z^8 - K_2 \sin z^8 \right] \\ B^2 &= -b^{-1} \left[K_1 \sin z^8 + K_2 \cos z^8 \right] \\ B^3 &= c^{-1} \left[(r-ps-qt) dz^3 + pt K_3 + \frac{1}{\sqrt{3}} (r-qt) K_8 \right] \\ B^8 &= d^{-1} \left[(r-ps-qt) dz^8 + (r-ps) K_3 + \frac{1}{\sqrt{3}} qs K_8 \right], \quad (40) \end{aligned}$$

where the one-forms K_α , ($\alpha=1, \dots, 8$) are the same combinations as occur¹⁴⁾ for M_{pqr} , and are listed in Ref. 15). The most general rescaling of the vielbein consistent with the $SU(3) \times SU(2) \times U(1)$ symmetry is contained in the scale factors a , b , c and d .

5. - HARMONIC EXPANSIONS OF SPINORS AND VECTOR-SPINORS ON L_{pqrst}

If we assume that there are no fields with spin greater than two in the twelve-dimensional theory, then we know fermions must transform as spinors or vector-spinors⁵⁾ of the tangent space group $SO(8)$ of L_{pqrst} . We shall show in this section that there are choices of the parameters p , q and r (all non-zero) for which the known quark and lepton representations of $SU(32) \times SU(2) \times U(1)$ occur in the harmonic expansion of such a spinor or vector-spinor on the manifold. This is in contrast to the situation⁴⁾ for the eleven-dimensional manifolds M_{pqrst} .

Salam and Strathdee¹⁰⁾ have shown how to expand a field transforming as a particular representation of the tangent space group in terms of irreducible representations of G , for the coset space G/H . (For this procedure, G need not be the isometry group of G/H .) To discover which representations of G occur in this expansion, it is necessary to compare the H content of a representation of G with the H content of the representation of the tangent space group to which the field belongs^{10),4),5)}. [H is always a subgroup of the tangent space group $SO(8)$.] The representations of G which occur in the harmonic expansion are those which overlap with the tangent space group representation in at least one representation of H .

To carry out this procedure, we need to know the embedding of H both in G and in the tangent space group $SO(8)$. The embedding of H in G is as in (26) and the preceding remarks. The embedding of H in $SO(8)$ can be obtained using (3.17) of Salam and Strathdee¹⁰⁾. We find

$$\begin{aligned}\tilde{\lambda}_1 &= M^{56} - M^{47} \\ \tilde{\lambda}_2 &= M^{64} - M^{57} \\ \tilde{\lambda}_3 &= M^{67} - M^{45}\end{aligned}\tag{41}$$

and

$$Z = -q M^{12} - \frac{3}{2} p (M^{45} + M^{67})\tag{42}$$

where the M^{ij} are the standard generators of $SO(8)$ such that

$$[M^{ij}, M^{kl}] = i(\delta^{ik} M^{jl} + \delta^{jl} M^{ik} - \delta^{il} M^{jk} - \delta^{jk} M^{il}) \quad (43)$$

An eight-spinor of $SO(8)$ (obtained by using a projection operator to split the 16-spinor into two pieces of opposite chirality) has the H content

$$\begin{aligned} |8\text{-spinor}\rangle = & |2, \frac{1}{2}q\rangle + |2, -\frac{1}{2}q\rangle + |1, \frac{3}{2}p + \frac{1}{2}q\rangle \\ & + |1, \frac{3}{2}p - \frac{1}{2}q\rangle + |1, -\frac{3}{2}p + \frac{1}{2}q\rangle + |1, -\frac{3}{2}p - \frac{1}{2}q\rangle, \end{aligned} \quad (44)$$

where the notation is $|SU(2), Z\rangle$. The content of the usual fermion representations of the standard model (the same for each generation) is

$$\begin{pmatrix} \nu \\ e \end{pmatrix}_L = |1, \frac{1}{2}q - \frac{1}{2}r\rangle + |1, -\frac{1}{2}q - \frac{1}{2}r\rangle \quad (45)$$

$$e_R = |1, -r\rangle \quad (46)$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L = |2, \frac{1}{2}p + \frac{1}{2}q + \frac{1}{6}r\rangle + |2, \frac{1}{2}p - \frac{1}{2}q + \frac{1}{6}r\rangle \quad (47)$$

$$+ |1, -p + \frac{1}{2}q + \frac{1}{6}r\rangle + |1, -p - \frac{1}{2}q + \frac{1}{6}r\rangle$$

$$u_R = |2, \frac{1}{2}p + \frac{2}{3}r\rangle + |1, -p + \frac{2}{3}r\rangle \quad (48)$$

and

$$d_R = |2, \frac{1}{2}p - \frac{1}{3}r\rangle + |1, -p - \frac{1}{3}r\rangle. \quad (49)$$

The known quarks and leptons then occur in the harmonic expansion of the eight-spinor for

$$-3p = \pm q = r. \quad (50)$$

If we also allow the vector-spinor representation of $SO(8)$, there are the additional representations of H,

$$\begin{aligned}
 |\text{vector-spinor}\rangle \supset & |1, \frac{3}{2}p + \frac{3}{2}q\rangle + |1, -\frac{3}{2}p + \frac{3}{2}q\rangle + |1, \frac{3}{2}p - \frac{3}{2}q\rangle \\
 & + |1, -\frac{3}{2}p - \frac{3}{2}q\rangle + |2, 3p + \frac{1}{2}q\rangle + |2, 3p - \frac{1}{2}q\rangle + |2, -3p + \frac{1}{2}q\rangle \\
 & + |2, -3p - \frac{1}{2}q\rangle + |2, \frac{3}{2}q\rangle + |2, -\frac{3}{2}q\rangle + |3, \frac{3}{2}p + \frac{1}{2}q\rangle \\
 & + |3, \frac{3}{2}p - \frac{1}{2}q\rangle + |3, -\frac{3}{2}p + \frac{1}{2}q\rangle + |3, -\frac{3}{2}p - \frac{1}{2}q\rangle \quad (51)
 \end{aligned}$$

The known quarks and leptons can then also occur for

$$+3p = \pm q = r, \quad 3p = \pm 3q = \pm r \quad (52)$$

(where all combinations of signs are intended). Thus, manifolds L_{pqrs} with parameters p, q and r satisfying (50) or (52) are consistent with the known quark and lepton representations of the standard model.

6. - EINSTEIN SPACE CONDITION FOR L_{pqrs}

For Freund-Rubin compactification⁸⁾, the compact manifold is required to be an Einstein space. In this section, we investigate whether L_{pqrs} admits an Einstein metric. The curvature and the Ricci tensor are most easily calculated in the language of modern differential geometry^{16), 14)}. With the aid of the Maurer-Cartan equations the spin-connection $B^{\alpha\beta}$ ($\alpha, \beta=1, \dots, 8$) may be calculated from the vielbein of (40). The curvature is then given by

$$R^{\alpha\beta} = dB^{\alpha\beta} - B^{\alpha\gamma} \wedge B_{\gamma}{}^{\beta} = R^{\alpha\beta}_{\gamma\delta} B^{\gamma} \wedge B^{\delta}, \quad (53)$$

where d is an exterior derivative, and the Ricci tensor is given by

$$R^{\alpha}_{\beta} = R^{\alpha\gamma}_{\beta\gamma} \quad (54)$$

In the case of L_{pqrs} , we find

$$\begin{aligned}
 R^A_B = -\frac{3}{4}a^2 \left\{ 1 - \frac{1}{2}a^2 \left[\left(\frac{qs}{d} \right)^2 + \left(\frac{r-qt}{c} \right)^2 \right] \right\} \delta^A_B \quad (55) \\
 (A, B = 4, 5, 6, 7)
 \end{aligned}$$

$$R_n^m = -\frac{1}{2} b^2 \left\{ 1 - \frac{1}{2} b^2 \left[\left(\frac{pt}{c} \right)^2 + \left(\frac{r-ps}{d} \right)^2 \right] \right\} \delta_n^m \quad (m, n = 1, 2) \quad (56)$$

$$R_3^3 = -\frac{1}{8} a^4 \left(\frac{r-qt}{c} \right)^2 - \frac{1}{4} b^4 \left(\frac{pt}{c} \right)^2 \quad (57)$$

$$R_8^8 = -\frac{1}{8} a^4 \left(\frac{qs}{d} \right)^2 - \frac{1}{4} b^4 \left(\frac{r-ps}{d} \right)^2 \quad (58)$$

$$R_8^3 = R_3^8 = -\frac{1}{8} a^4 \frac{qs(r-qt)}{cd} - \frac{1}{4} b^4 \frac{pt(r-ps)}{cd} \quad (59)$$

and all other components zero. For an Einstein space,

$$R_\beta^\alpha = \rho \delta_\beta^\alpha \quad (60)$$

for some constant ρ . This condition can be satisfied for

$$-qs(r-qt) = 8pt(r-ps) \quad (61)$$

with the choice of scale factors

$$b^2/a^2 = 2 \quad (62)$$

$$c^2/d^2 = 8(pt/qs)^2 \quad (63)$$

$$c^2/a^2 = \frac{1}{4} \left[(r-qt)^2 + 8(pt)^2 \right] \quad (64)$$

and consistently

$$d^2/a^2 = \frac{1}{4} \left[8(r-ps)^2 + (qs)^2 \right] \quad (65)$$

Thus, the manifolds L_{pqrst} admit an Einstein metric.

Having fixed the scale factors, it is now possible to calculate ratios of gauge coupling constants ^{(15), (17)-(19)}, under the assumption that upon dimensional reduction the complete Yang-Mills Lagrangian derives from the twelve-dimensional

Einstein Lagrangian. Identifying the standard model $SU(3) \times SU(2) \times U(1)$ with the isometries associated with the left-action of G on H (as we have been doing throughout) we find (in standard normalization of the coupling constants) that

$$(66)$$

where n is an undetermined integer. [The result of g_1^2 depends on the requirement that $SU(2)$ quark doublets should have $Y = 1/6$.] The value of g_2^2/g_3^2 is in tolerable agreement with renormalization group extrapolations of the gauge coupling constants to the Planck scale¹⁸⁾, but the value of g_1^2 is hopelessly large for the values (50) and (52) of p , q and r which allow the usual quark and lepton representations. Though this is disappointing, it is not disastrous for these models since it is entirely possible that a complete theory contains matter multiplets (such as explicit vector fields in twelve dimensions) which contribute to the four-dimensional Yang-Mills Lagrangian and alter the gauge coupling constant ratios. (This phenomenon occurs in ten-dimensional supergravity²⁰⁾.) It is also possible that compactification is not by the Freund-Rubin mechanism, in which case the manifold need not be an Einstein space, the scale factors need not be given by (62), (63) and (64), and the gauge coupling constant ratios could change.

7. - CONCLUSIONS

The seven-dimensional manifolds M_{pqrst} of Witten^{3),4)}, with isometry group $SU(3) \times SU(2) \times U(1)$, do not have the usual quark and lepton representations in the appropriate harmonic expansions⁴⁾. We have considered here eight-dimensional manifolds with isometry group at least $SU(3) \times SU(2) \times U(1)$, and have observed first that manifolds made from (products of) spheres S^N and complex projective planes CP^N have difficulties. In particular, product manifolds are unstable for arbitrary cosmological constant when compactification is by the Freund-Rubin mechanism.

As has been remarked in Section 4, it is not possible in more than seven compact dimensions to obtain just the $SU(3) \times SU(2) \times U(1)$ gauge group as the isometry group of the manifold. We have considered coset spaces L_{pqrst} , of the type $SU(3) \times SU(2) \times U(1)/SU(2) \times U(1)$, where the isometry group $SU(3) \times SU(2) \times U(1) \times U(1)$ is a

minimal increase on the standard model. [The extra $U(1)$ comes from the right action of the numerator on the coset space.] The effects of the extra gauge field are presumably suppressed by spontaneous symmetry breaking below the Planck scale but well above the electroweak scale. We find that the known quark and lepton representations of the standard model are able to occur in the harmonic expansion of spinor and vector-spinor representations on the manifold, for values of p , q and r given by (50) and (52). The manifolds L_{pqrs}^{pqrst} also admit an Einstein metric, as required for Freund-Rubin compactification⁸⁾.

The chiral fermion problem¹⁾ remains, though it may possibly be resolved by introducing suitable explicit gauge fields in twelve dimensions⁷⁾, or by embedding the theory in a still higher dimensional theory with unconventional tangent space group⁶⁾. (It would certainly be necessary to take this latter course if we wanted to incorporate supersymmetry, since conventional supergravity theories are not possible in more than eleven dimensions.) In any case, these twelve-dimensional Kaluza-Klein theories have the advantage over eleven-dimensional theories that the known quark and lepton representations of the standard model can be obtained.

ACKNOWLEDGEMENTS

We would like to thank Jean-Pierre Derendinger, Martin Dunwoody, Phil Elliott, Jim Foster, Paul Frampton and Peter Rosen for helpful discussions. We are particularly grateful to Robin Tucker for explaining some of the mysteries of differential geometry. One of us (D.B.) thanks the Theory Division at CERN for generous hospitality during the period when most of this work was completed. This research was supported in part by SERC under grant NG13140.

MANIFOLD	ISOMETRY GROUP
CP^4	$SU(5)$
$CP^3 \times S^2$	$SU(4) \times SU(2)$
$CP^2 \times CP^2$	$SU(3) \times SU(3)$
$CP^2 \times S^4$	$SU(3) \times SO(5)$
$CP^2 \times S^3 \times S^1$	$SU(3) \times SU(2) \times SU(2) \times U(1)$
$CP^2 \times S^2 \times S^2$	$SU(3) \times SU(2) \times SU(2)$
$CP^2 \times S^2 \times S^1 \times S^1$	$SU(3) \times SU(2) \times U(1) \times U(1)$
S^8	$SO(9)$
$S^6 \times S^2$	$SO(7) \times SU(2)$
$S^5 \times S^3$	$SU(4) \times SU(2) \times SU(2)$
$S^5 \times S^2 \times S^1$	$SU(4) \times SU(2) \times I(1)$

TABLE - "Familiar" eight-dimensional compact manifolds
with isometry group at least $SU(3) \times SU(2) \times U(1)$.

REFERENCES

- 1) Some of the literature may be traced from:
E. Witten - Princeton Preprint (October 1983).
- 2) Th. Kaluza - Sitzungsber.Previs.Wiss.Phys.Math. Kl (1921) 966;
O. Klein - Z.Phys. 37 (1926) 895.
- 3) E. Witten - Nucl.Phys. B186 (1981) 412.
- 4) S. Randjbar-Daemi, A. Salam and J. Strathdee - ICTP Preprint IC/84/15 (1984).
- 5) See, for example: S.P. de Alwis and I.G. Koh - Imperial College Preprint TP/83-84/42 (1984).
- 6) S. Weinberg - Phys.Lett. 138B (1984) 47.
- 7) S. Randjbar-Daemi, A. Salam and J. Strathdee - Phys.Lett. 132B (1983) 56, and references therein.
- 8) P.G.O. Freund and M.A. Rubin - Phys.Lett. 97B (1980) 233.
- 9) S.W. Hawking and C.N. Pope - Phys.Lett. 73B (1978) 42;
C.J. Isham and C.N. Pope - Phys.Lett. 114B (1982) 137;
S. Watanabe - Phys.Lett. 136B (1984) 250.
- 10) A. Salam and J. Strathdee - Ann.Phys. 141 (1982) 316.
- 11) M.J. Duff, B.E.W. Nilsson and C.N. Pope - Phys.Lett. 139B (1984) 154.
- 12) D. Bailin, A. Love and C.E. Vayonakis - Phys.Lett. 142B (1984) 344.
- 13) L. Castellani, L.J. Romans and N.P. Warner - Caltech Preprint CALT-68-1057 (1983).
- 14) L. Castellani, R. D'Auria and P. Fré - Torino Preprint IFTT-427 (1983).
- 15) D. Bailin and A. Love - Phys.Lett. 144B (1984) 359.
- 16) T. Eguchi, P.B. Gilkey and A.J. Hanson - Physics Reports 66 (1980) 213.
- 17) S. Weinberg - Phys.Lett. 125B (1983) 265.
- 18) D. Bailin and A. Love - Phys.Lett. 137B (1984) 348.
- 19) Z.F. Ezawa and I.G. Koh - Phys.Lett. 142B (1984) 153.
- 20) Z.F. Ezawa and I.G. Koh - Phys.Lett. 142B (1984) 157.