ON THE $SU(3)$ TOPOLOGICAL CHARGE IN
LATTICE GAUGE THEORIES

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A B S T R A C T

We propose a practical method to define an integer-valued topological charge for $SU(3)$ lattice gauge theory.

It is well known that it is interesting to measure the expectation value of $\langle Q^2 \rangle / V$ in lattice gauge theories, Q being the topological charge and V the volume of the lattice.

Arguments based on $1/N$ expansion and PCAC [1-2] suggest that

$$\langle Q^2 \rangle / V \approx (180 \text{ MeV})^4 \quad (1)$$

in pure gauge theories. After the first attempts, Woit has shown that there is a relatively simple method to define the topological charge in $SU(2)$ gauge theory [3].

We consider a lattice $L^3 \times T$ and by a gauge transformation we go to the Coulomb gauge (in this gauge $\sum_i \text{Re Tr}(U_s(i))$ attains its maximal value, U_s being the links pointing in the space direction). We now turn our attention to the links in the time direction; we have T lattices of L^3 elements. If we are able to define a topological charge for each of these lattices, $Q(t)$ ($t=1, \dots, T$), then the total topological charge will be:

$$Q = \sum_t Q(t) \quad (2)$$

The definition of $Q(t)$ is more or less standard. We consider our lattice embedded in real space (with periodic boundary conditions) and we divide the space into non-overlapping tetraheds having the lattice points as vertices; contiguous tetraheds must have overlapping edges. The simplest way to perform this operation on a cubic lattice, consists in representing the cube as the union of six tetraheda. Now at each vertex i of the tetrahed we have a group element $g(i)$, i.e. the gauge link in the time direction (in the Coulomb gauge). We can now associate to each tetrahed in real space (k) a tetrahed in the group space $t(k)$; if $v(k)$ is the volume of the k^{th} tetrahed in the group space, we have

$$Q(t) = \sum_k v(k) \quad (3)$$

where the sum runs over all the tetraheds at time t . This construction is the natural generalization of the construction of the topological charge in $O(3)$ in two dimensions. Equation (3) is easy to implement in $SU(2)$ theories but not in $SU(3)$ where a closed formula for $v(k)$ is lacking.

The simplest method of computing the topological charge in SU(2) is the following: consider a fixed, arbitrary, element g_0 in the group space; for a given tetrahed $t(k)$ we have two choices, either it does or it does not contain g_0 ; we thus define a function $f(k)$ such that $|f(k)|=1$ in the first case, $f(k)=0$ in the second case. The sign of f is fixed by the orientation of the tetrahed. We then have:

$$Q(t) = \sum_k f(k) \quad (4)$$

The extension to SU(3) is straightforward, in principle; we must consider a 5-dimensional manifold V_5 in SU(3) with special topological properties, such that eq.(4) holds if $|f(k)| \neq 0$ when V_5 and the tetrahed intersect and $f(k)=0$ elsewhere. Even in this form, the algorithm is not simple to implement.

We present a variation of this scheme which we believe is simpler to use. We first discuss the continuum case. We have to compute the topological charge of a function $g(x)$ and we notice that if $g(x)=g_1(x) \cdot g_2(x)$, then

$$Q[g] = Q[g_1] + Q[g_2] \quad (5)$$

If we succeed in decomposing $g(x)$ in such a way that $g_1(x)$ is topologically trivial, ($Q[g_1]=0$) and g_2 belongs to a SU(2) subgroup of SU(3) our problem is obviously solved.

We suggest the following decomposition:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta^* & \alpha^* \end{pmatrix} \cdot B(\omega) \quad (6)$$

where:

$$\omega=(a,b,c) \quad , \quad B(\omega) = \begin{pmatrix} a^* & -b & -c \\ b^* & (a \cdot b \cdot b^* + c \cdot c^*)/f^2 & (a-1) \cdot b^* \cdot c/f^2 \\ c^* & (a-1) \cdot b \cdot c^*/f^2 & (a^* \cdot c \cdot c^* + b \cdot b^*)/f^2 \end{pmatrix}$$

and $f = \sqrt{(b \cdot b^* + c \cdot c^*)}$, whose geometrical meaning is quite clear. The decomposition is discontinuous on the four-dimensional manifold $b=c=0$,

$a \neq 1$. Now $SU(3)$ is an eight-dimensional manifold so that both g_1 and g_2 will be continuous for generic g . We have now to prove that $Q[g_1] = 0$ in the generic case. In other words, that we can deform g_1 to a constant g_0 ; if g_1 were a continuous function of a, b and c that would be evident as $\Pi_3(S_5) = 0$; but if we try to deform $g_1(x)$ to g_0 , sticking to the form given in equation (6), we face the problem of the point of discontinuity. In principle if we have an auxiliary function $\omega(x, \tau)$, with $\sum_i |\omega_i|^3 = 1$,

$$\begin{aligned}\omega(x, 0) &= (a, b, c) \\ \omega(x, 1) &= (a_0, b_0, c_0)\end{aligned}\tag{7}$$

(such a function can be constructed because $\Pi_3(S_5) = 0$), and it is possible that this function must always intersect the one dimensional (inside S_5) manifold $|a|=1$, recall that $\Pi_3(S_3) \neq 0$; however we can choose the point of intersection of $\omega(x, \tau)$ with the one-dimensional manifold $|a|=1$, at $a=1$ where $g(\omega)$ is a continuous function. With this choice we can construct a contour function $g(x, \tau)$ such that:

$$\begin{aligned}g(x, 0) &= g_1(x) \\ g(x, 1) &= g_0\end{aligned}\tag{8}$$

the function being obviously given by:

$$g(x, \tau) = g(\omega(x, \tau))\tag{9}$$

What happens on the lattice? If we apply eq (6) to produce an $SU(2)$ configuration on the lattice, it is easy to see that the result is not topologically invariant, the danger comes by those configurations which have only one element near $|a|=1$. In this case, one element of $SU(2)$ will be arbitrarily producing a mini-instanton or a dislocation (a topological charge concentrated in one point). This problem cannot arise if the $g(x)$ configuration is smooth.

There are many possible solutions to this problem:

- a) Smoothing $g(x)$ by averaging over the near points (e.g. the 27 points such that $|\Delta x| \leq 1$, $|\Delta y| \leq 1$, $|\Delta z| \leq 1$).
- b) Smoothing the $SU(2)$ field in the same way to cancel the contribution of the mini instanton.
- c) Interpolating $g(x)$ on a finer lattice.
- d) A combination of a, b and c.

Operations a,b and c may remove small size instantons, but they should not destroy the large scale ones and alter the topological charge coming from the infra-red region.

As an exercise, we have run this algorithm in the version (a) on a 6^4 lattice at $\beta=5.6$. We have checked the consistency of our approach by considering gauge configurations of the form:

$$g_i(k,k') = R_p \cdot g_i \cdot R_{p'}, \quad p,p'=1..3 \quad (10)$$

where:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

In most cases the topological charge was either independent from p and p' or there was a value which appeared five or more times over the nine possible choices of p and p' . The existence of configurations for which the computed topological charge was dependent on p and p' indicates that a further smoothing is necessary (maybe (b) and (c) are more efficient).

The estimated value of the topological charge (with some real small instantons lost) is given by:

$$\langle Q^2 \rangle / V = 0.25 \pm 0.07 \quad (11)$$

where the error does not reflect the systematic deviation induced by our procedure.

Summarizing, we have an efficient way to compute the topological charge for relatively smooth configurations; one should investigate the most appropriate smoothing procedure. A more radical solution than (a)..(c) consists in looking for the minimum of :

$$\sum_{i,\mu} (1 - \text{Re Tr}[g(i) \cdot g^+(i+\mu)])^2 \quad (12)$$

by using the steepest descent method starting from the configuration whose topological charge is needed. Only those configurations with non-

trivial topological charge should be out of the domain of attraction of the trivial configuration. For those configurations, the method we propose should be suitable. Of course we could also smooth directly the gauge fields, by minimizing:

$$\sum_{i,\mu} (1 - \text{Re Tr } \square)^2 \quad (13)$$

the sum runs over all the plaquettes, and computing the topological charge using as g 's the gauge fields which wind around the lattice. For such smooth configurations, the topological charge should also be proportional to:

$$\sum \square (1 - \text{Re Tr } \square) \quad (14)$$

It is clear that different methods deal differently with mini-instantons so that the result should not be the same far away from the continuous limit.

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