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TWO-LOOP β -FUNCTION FOR THE GROSS-NEVEU MODEL

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ABSTRACT

The β -function for the Gross-Neveu model is calculated to two loops.

The result is:

$$\beta = -\frac{N-1}{2\pi} g^3 + \frac{N-1}{(2\pi)^2} g^5 + \dots$$

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The Gross-Neveu model¹⁾ belongs to the interesting class of renormalizable field theories for which asymptotic freedom is accompanied by non-trivial dynamical behaviour like the spontaneous breakdown of a symmetry. As a general feature of asymptotic freedom, the elimination of the cut-off in terms of the renormalization group invariant Λ parameter requires that we know the universal coefficients β_0, β_1 of the β -function for the running coupling constant. Apparently for the Gross-Neveu model the two-loop coefficient β_1 has not been calculated so far. It is the purpose of the present letter to fill in this gap in the existing literature.

The theory is defined in the σ -field formulation by the Euclidean action:

$$L = \int d^2x \left[-\bar{\psi} \not{\partial} \psi - \frac{1}{2} \sigma^2 - g \sigma \bar{\psi} \psi \right] \quad (1)$$

The Feynman rules and the superficially divergent vertex functions are shown in Fig. 1 and Fig. 2 respectively. In Fig. 2 the flavour index a ($a=1, \dots, N$) has been suppressed. As indicated, the σ -fermion and four-fermion vertex (propagator for external legs factored out) will only be considered for external momenta proportional to a common four-momentum p (proportionality parametrized by η). The symbol $\mathbb{1}$ denotes the unit matrix in Dirac space. The terms in the four-fermion vertex which are indicated by dots are supposed to vanish when the trace is taken separately along each of the two fermion lines, i.e. their Dirac structure is of the kind $\gamma^\mu \otimes \gamma^\mu$ or $\gamma^5 \otimes \gamma^5$. For the vertex renormalization constants Z_1 to Z_4 , calculated in the following by dimensional regularization, we define dimensionless expansion coefficients $\zeta_{.,\lambda}$ according to

$$Z_{.,\lambda}^{-1} = 1 + g^2 \zeta_{.,1} (p^2)^{\frac{d}{2}-1} + g^4 \zeta_{.,2} (p^2)^{d-2} + \dots \quad (2)$$

The diagrams to be considered to two loops are shown in Fig. 3. The insertions refer to self-energies and σ -fermion vertex corrections. Since the fermion self-energy vanishes to one loop, the corresponding diagrams have not been drawn. Diagrammatically the situation is identical to QED when treated in the Landau gauge. However, the calculation is much simpler here because the σ -field does not propagate.

To one loop we get:

$$\begin{aligned}\xi_{3,1} &= -N \text{Tr} \Pi u(d) \\ \xi_{1,1} &= -|1-\eta|^{d-2} u(d) \\ u(d) &= (d-1) \frac{B(d/2, d/2)}{(4\pi)^{d/2}} \Gamma(1 - \frac{d}{2})\end{aligned}\quad (3)$$

where Tr denotes the trace in Dirac space; B and Γ are Euler's Beta- and Gamma-functions.

The calculation of the two-loop contributions to Z_1, Z_2, Z_3 essentially reduces to the calculation of the diagrams in Fig. 4 because the one-loop vertex (1b) only depends on the momentum of the σ line.

For $\tilde{\xi}_{2,1}, \tilde{\xi}_{1,1}$ from Fig. 4 we get:

$$\begin{aligned}\tilde{\xi}_{2,1} &= \frac{1}{2} \frac{B(d/2, d-1)}{(4\pi)^{d/2}} \cdot \frac{\Gamma(3-d)}{\Gamma(2-d/2)} \\ \tilde{\xi}_{1,1}(\eta=0) &= -(3d-4) \frac{1}{2-d} \tilde{\xi}_{2,1}\end{aligned}\quad (4)$$

The reduction of the two-loop coefficients of Z_1, Z_2, Z_3 to products of the one-loop coefficients from Eq. (2) and $\tilde{\xi}_{2,1}, \tilde{\xi}_{1,1}$ takes the form:

$$\begin{aligned}\xi_{2,2}^{(2b)} &= -\xi_{3,1} \cdot \tilde{\xi}_{2,1} \\ \xi_{2,2}^{(2c)} &= \xi_{1,1}(\eta=0) \tilde{\xi}_{2,1} \\ \xi_{3,2}^{(3c)} &= \xi_{1,1}(\eta=0) \xi_{2,1}\end{aligned}$$

$$\begin{aligned}
 \xi_{1,2}^{(1c)} &= - \xi_{3,1} \tilde{\xi}_{1,1} \\
 \xi_{1,2}^{(1d)} &= \xi_{1,1}(\eta=0) \tilde{\xi}_{1,1} \\
 \xi_{1,2}^{(1e)} &= (\xi_{1,1}(\eta=0))^2 \\
 \xi_{1,2}^{(1f)}(\eta=0) &= - \xi_{1,1}(\eta=0) \tilde{\xi}_{1,1}(\eta=0)
 \end{aligned} \tag{5}$$

For the four-fermion vertex in Fig. 3, it is to be noted that only the triple exchange diagrams (4d) to (4i) can lead to a new type of divergence. The u.v. divergencies of the diagrams (4b) and (4b') cancel each other so as to ensure renormalizability to the one-loop level. This cancellation is also effective between the diagrams (4c) and (4c'). Notwithstanding these facts, it is obvious that due to the trace projection Z_4 can receive contributions only from the insertions to (4a) and the triple exchange diagrams, i.e.:

$$\begin{aligned}
 Z_4^{-1}(p^2, \eta) &= Z_1^{-1}(p^2, \eta) Z_1^{-1}((\eta-1)^2 p^2, 0) Z_3((\eta-1)^2 p^2) \\
 &+ g^6 \xi_{4,2}^{\text{III}}(\eta) (p^2)^{d-2} + O(g^8)
 \end{aligned} \tag{6}$$

The calculation of $\xi_{4,2}^{\text{III}}$ is easiest to perform for $\eta=1$. We find:

$$\zeta_{4,2}^{\text{II}}(\eta=1) = -\frac{1}{(4\pi)^d} B\left(\frac{d}{2}, \frac{d}{2}\right) (1-d) \Gamma\left(1-\frac{d}{2}\right). \quad (7)$$

$$\left\{ \frac{d}{2} B\left(\frac{d}{2}, \frac{d}{2}\right) \Gamma\left(1-\frac{d}{2}\right) + (4-3d) \frac{B(d/2, d-1)}{\Gamma(2-d/2)} \Gamma(2-d) \right\}$$

The pole terms in the curly bracket cancel each other. However, a finite contribution remains, leading to the conclusion:

$$\zeta_{4,2}^{\text{IV}} = \frac{1}{2} \cdot \frac{1}{(2\pi)^2} \cdot \frac{1}{2-d} + O(2-d)^0 \quad (8)$$

The divergence in $\zeta_{4,2}^{\text{III}}$ implies that a QED-like multiplicative renormalization with Z_2 and Z_3 as wave function renormalization constants for, respectively, the fermion and σ -fields, and a coupling constant renormalization based on the σ -fermion vertex, i.e. on Z_1 , would not render the four-fermion vertex finite to order g^6 . Obviously a quartic counter-term of this order is required in addition.

To understand this observation let us consider the σ -field formulation for a moment as a mere bookkeeping device for the various contractions of internal Dirac and flavour indices, i.e. restrict the discussion to Green functions with only fermionic legs. From the $O(2N)$ symmetry of the model²⁾ one expects the renormalization of these Green functions to be accomplished by a fermion wave function renormalization Z_2 and a coupling constant renormalization

$$g_R^2 = g^2 Z_{g^2} = g^2 Z_2^2 Z_4^{-1} \quad (9)$$

Eliminating the bare coupling and bare fermion field in terms of renormalized ones, the action becomes

$$L = \int d^2x \left[-Z_2 \bar{\Psi}_R \not{\partial} \Psi_R - \frac{1}{2} \sigma^2 - g_R Z_4^{1/2} \sigma \bar{\Psi}_R \Psi_R \right] \quad (10)$$

which is supposed to yield finite fermion Green functions order by order in g_R if Z_2 , Z_4 are suitably chosen.

To extend the renormalization to Green functions involving external σ -lines, we should allow for the possibility that σ_R mixes with $\bar{\Psi}_R \Psi_R$ which has the same quantum numbers. Instead of a purely multiplicative renormalization we then have

$$G = Z_3^{1/2} G_R + g_R \delta Z \bar{\psi}_R \psi_R \quad (11)$$

where Z_3 and δZ have to be chosen such that the two-point function for the σ -field and the σ -fermion vertex become finite. Inserting (11) into (10) obviously produces a quartic counter-term as required by our analysis of the divergencies to two loops.

The appearance of an additional renormalization constant can be understood also from the following point of view. According to the Feynman rules in Fig. 1, external σ -lines are equivalent to insertions of the composite operator $-g\bar{\psi}\psi$. In general we have to expect different renormalization constants if the operator is inserted once (σ -fermion vertex) or twice (two-point function for σ).

We now derive the two-loop β -function for which we adopt the convention ($p^2 = \mu^2$):

$$\mu \frac{d}{d\mu} g_R = \beta(g_R) = \beta_0 g_R^3 + \beta_1 g_R^5 + \dots \quad (12)$$

The coefficients β_0, β_1 are related to the expansion coefficients $\zeta_{g^2, \dots}$ of Z_2^{-1} according to:

$$\beta_0 = \lim_{d \rightarrow 2} \left(1 - \frac{d}{2}\right) \zeta_{g^2, 1} \quad (13)$$

$$\beta_1 = \lim_{d \rightarrow 2} (2-d) \zeta_{g^2, 2}$$

Since β_0 and β_1 are universal, they are not affected by the particular value for η chosen in the calculation of Z_{g^2} . Combining our results from Eqs. (3) to (8), we find:

$$\begin{aligned} \beta_0 &= - \frac{N-1}{2\pi} \\ \beta_1 &= \frac{N-1}{(2\pi)^2} \end{aligned} \quad (14)$$

For $N=1$ the Gross-Neveu model is identical to the massless Thirring model whose β -function is known to be identically zero³⁾. The vanishing of β_0, β_1 for $N=1$ therefore was to be expected.

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FIGURE CAPTIONS

- Fig. 1 Feynman rules for the inverse propagators and the σ -fermion vertex.
- Fig. 2 The vertex renormalization constants Z_2 , Z_3 , Z_1 and Z_4 defined in terms of the inverse fermion propagator, the inverse σ propagator, the σ -fermion vertex and the four-fermion vertex respectively.
- Fig. 3 The various diagrams contributing to two loops to the fermion and σ -self-energy, the σ -fermion vertex and the four-fermion vertex.
- Fig. 4 Subdiagrams occurring in the evaluation of the two-loop diagrams.

$$\frac{a \rightarrow a'}{p} = \delta^{aa'} i \not{p}$$

$$----- = 1$$

$$\begin{array}{c} a' \\ \swarrow \\ --- \\ \searrow \\ a \end{array} = -g \delta^{aa'} \not{1}$$

- Figure 1 -

$$\begin{array}{c} p \rightarrow \text{---} \circ \text{---} \end{array} = Z_2^{-1}(p^2) i \not{p}$$

$$\begin{array}{c} p \text{---} \text{---} \circ \text{---} \end{array} = Z_3^{-1}(p^2)$$

$$\begin{array}{c} \eta \cdot p \\ \swarrow \\ \text{---} \circ \text{---} \\ \searrow \\ p \end{array} = -g Z_1^{-1}(p^2, \eta) \not{1}$$

$$\begin{array}{c} (\eta-1)p \rightarrow \text{---} \circ \text{---} \\ \swarrow \quad \searrow \\ p \quad \eta \cdot p \end{array} = g^2 Z_4^{-1}(p^2, \eta) \not{1} \otimes \not{1} + \dots$$

- Figure 2 -

$$\begin{array}{c} \rightarrow \bigcirc \rightarrow \\ \hline \end{array} = \begin{array}{c} \rightarrow \\ \hline \end{array} - \begin{array}{c} \rightarrow \\ \hline \text{---} \text{---} \end{array} - \begin{array}{c} \rightarrow \\ \hline \text{---} \text{---} \end{array}$$

(2a) (2b) (2c)

$$\begin{array}{c} \text{---} \bigcirc \text{---} \\ \hline \end{array} = \begin{array}{c} \text{---} \\ \hline \end{array} - \begin{array}{c} \text{---} \bigcirc \text{---} \\ \hline \end{array} - \begin{array}{c} \text{---} \bigcirc \text{---} \\ \hline \end{array}$$

(3a) (3b) (3c)

$$\begin{array}{c} \rightarrow \bigcirc \leftarrow \\ \hline \end{array} = \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + 2 \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array}$$

(1a) (1b) (1c) (1d) (1e) (1f)

$$\begin{array}{c} \rightarrow \bigcirc \leftarrow \\ \hline \end{array} = \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + \text{insertions} \\ \begin{array}{c} \rightarrow \\ \hline \end{array} + 2 \begin{array}{c} \rightarrow \\ \hline \end{array} + 2 \begin{array}{c} \rightarrow \\ \hline \end{array} \\ \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array} + \begin{array}{c} \rightarrow \\ \hline \end{array}$$

(4a) (4b) (4b') (4c) (4c') (4d) (4e) (4f) (4g) (4h) (4i)

- Figure 3 -

$$p \text{ --- } \bullet \text{ ---} = g^2 (p^2)^{d/2-1}$$

$$p \text{ --- } \overset{\curvearrowright}{\bullet} \text{ ---} = -g^4 i \not{p} (p^2)^{d-2} \tilde{\zeta}_{2,1}$$

$$\text{---} \bullet \begin{array}{l} \nearrow \eta \cdot p \\ \searrow p \end{array} = -g^5 \not{\eta} (p^2)^{d-2} \tilde{\zeta}_{1,1}(\eta)$$

- Figure 4 -