CERN-TH.4061/84

DISCRETE APPROXIMATIONS TO THE 1+1 DIMENSIONAL $O(3)$

SYMMETRIC LATTICE THEORY

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A B S T R A C T

Discretized versions of the $O(3)$ symmetric quantum chain are considered with the unit vector variables pointing to vertices of regular polyhedra. Critical data are found by analyzing mass gaps of finite chains, determined numerically. Although g_c and ν move with the increase of the number of allowed directions towards the behaviour expected for the continuous group, the β function of any of the models considered does not reveal asymptotic freedom.

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1. - INTRODUCTION

Discrete spin models of statistical physics are very important in testing the theory of critical phenomena and also appear to be successful abstractions of several realistic physical systems. Ising models with all kinds of interactions, $Z(p)$ symmetric planar vector models and Potts-type systems are the best known examples.

In field theoretic models, most of the symmetries of interest are continuous. Nonetheless in some cases, theories defined on discrete submanifolds provide quite accurate approximations to them. At the same time, efficiency of their numerical solution can be largely increased.

The icosahedron subgroup (Y_{120}) of $SU(2)$ proved enough accurate for the calculation of the expectation value of the action density in Monte Carlo simulations of the $SU(2)$ lattice gauge theory¹⁾. Attempts based on the largest subgroup of $SU(3)$ [$G(1060)$] failed²⁾, however, the use of a larger matrix manifold³⁾ seems to reproduce the heavy quark potential to 1% accuracy in the crossover region⁴⁾. Similar experience was also gained by approximating compact electrodynamics on the lattice by $Z(p)$ gauge theories⁵⁾.

In this paper we propose to investigate global symmetric vector models defined on regular polyhedra as natural discrete approximations to the lattice version of the three-component non-linear σ model, in two dimensions. From this point of view, models like the Ising model⁶⁾, the four-state Potts model⁷⁾ and the six-state cubic spin model⁸⁾ appear to belong to the same family. They are approximations to the $O(3)$ model defined on smaller regular polyhedra. Vector models defined on the larger polyhedra (icosahedron, dodecahedron) are studied for the first time here.

All these models undergo a "freezing" phase transition at some value of the coupling. One expects the critical couplings, once they are mapped onto the $O(3)$ coupling scale, to be pushed towards zero as the approximation becomes more and more dense.

We find that their β functions behave close to the critical point as

$$\beta_d(g) \sim \frac{1}{2} (g - g_{d,c}) + O((g - g_{d,c})^2) + \dots \quad (1a)$$

(the index d refers to the discrete models). With $g_{d,c}$ approaching zero on the

$O(3)$ scale, one should also observe the increase of ν so that the quadratic term could dominate, giving a chance to see in a certain coupling range ($g > g_{d,c}$) the true $O(3)$ behaviour. We were able to verify the increasing tendency in $g_{d,c}^{-1}$ and ν as the models better approximate $O(3)$. This view offers a new way to relate critical data of different models (in form of inequalities).

A magnetic transition in the $O(3)$ model (which can happen only at $g = 0$ in two dimensions) would break down the $O(3)$ symmetry to $U(1)$. In most of the approximating models, the polyhedral symmetry could be broken to different $Z(p)$ axial symmetries. Again, in a model approximating the $O(3)$ system one expects the gaps belonging to different breakdown patterns to be degenerate at least close to $g_{d,c}$. We have verified this partial restoration of the $O(3)$ symmetry in our numerically established gap spectra.

A different scenario seems to be realized in two-dimensional planar spin systems, where the leading term in (1a) is absent also in the approximating models. The β function of this class starts as

$$\beta_{Z(p \geq 5-6)} \sim C(g-g_c)^{1+\sigma} + \dots \quad (1b)$$

Here σ seems to increase with p ⁹⁾, eventually reaching the Kosterlitz-Thouless value, 0.5 for $p \rightarrow \infty$. For the densest approximations to $O(3)$, we have investigated and excluded this scenario explicitly in our case.

The ν indices of the models considered in our paper range between $\sim 2/3$ and < 3 . This implies that we do not find the trace of the asymptotically free regime. The conclusion therefore is that these models are not appropriate for a quantitative study of the $O(3)$ vector model, at least in the neighbourhood of their respective critical points.

In Section 2, one introduces the Hamiltonian of the set of models we are going to investigate. The approximate relationship between the actual couplings and that of the $O(3)$ model is established. The discrete vector models built on smaller subgroups are related to models known in statistical physics. The tetrahedron vector model is equivalent to a four-state Potts, the octahedron to a six-state cubic spin and the cube to the eight-state $Z(2)^3$ Ising model. The two higher vector models are new. The numerical technique which is a variant of the Lanczos algorithm¹⁰⁾ for finding the mas gap of finite quantum systems is reviewed in Section 3. Details of constructing Hilbert subspaces corresponding to states with different symmetries are given.

In Section 4 two different (related) methods are applied to extract critical data for the present class of models. The Roomany-Wyld approximants¹¹⁾ to the β function and an approximant, devised in the same style, for the finite scale change renormalization function $\Delta g(g)$ ¹²⁾, yield mutually consistent descriptions. In the final section 5, prospective ways to improve the reliability of the results of our investigation are discussed.

2. - THE MODELS

Models defined on discrete manifolds will be introduced with the aim of approximating $O(3)$ symmetric quantum chains:

$$\hat{H} = \hat{K} - x \hat{V} = \frac{g}{2\alpha} \left(\sum_{i=1}^N \hat{L}_i^2 - x \sum_{i=1}^N \underline{n}_i \underline{n}_{i+1} \right), \quad \underline{n}_i^2 = 1, \quad x = \frac{2}{g^2}. \quad (2)$$

In Eq. (2), the $O(3)$ symmetry is realized by the action of the group elements on the sphere S_2 ; $\hat{L}_i$ denotes the generators of the rotations to be performed at site i on the unit vector variables $\underline{n}_i$.

2.1. - The manifolds

The possible orientations of $\underline{n}_i$ are discretized by restricting the choice to directions which point towards the vertices of some regular polyhedron (P) drawn into S_2 . Discrete subgroups of $O(3)$ (G_n , where n gives the number of the group elements) are realized by their action on one of these manifolds. This group action is the natural action of G_n on one of its factor spaces G_n/H , H being the stability group of $\underline{n}$ [typically $Z(p)$]. In this way, one can identify the polyhedron with $G_n/Z(p)$. The universality class of different discrete vector models depends also on the factor space. The polyhedra and some of their geometric data relevant to our calculations are listed in Table 1.

By geometrical face vertex duality, the models in Table 1 also cover those discretizations where the unit vectors would point towards the middle of the faces. For example, the cubic group can act both on a cube and its dual, the octahedron. In the first case the orbit of $\underline{n}$ is isomorphic to G_{24}/Z_3 , while in the latter, to G_{24}/Z_4 . However, models defined on manifolds which consist of

vectors pointing to the middle of edges (G_{12}/Z_2 , G_{24}/Z_2 , G_{60}/Z_2) were not considered in the present study.

2.2 - The Hamiltonian

The form of the potential term V of Eq. (2) is unchanged, only the vectors $\underline{n}_i$ are restricted to some P .

The kinetic term of Eq. (2) will be approximated by a conveniently chosen finite rotation difference ratio. The first term of its numerator is the sum of rotations transforming a given direction into its neighbours $\{\underline{n}^{(\alpha)}\}$: $T_{\underline{n}, \underline{n}^{(\alpha)}}$. The rotations of minimal angle (ϕ), with axis parallel to $\underline{n} \times \underline{n}^{(\alpha)}$ are used. As it was shown in Ref. 13) in several examples, the same shift operators would also appear when performing the Hamiltonian limit of the Lagrangian discrete vector models. The rotations are the same relative to each vertex direction, the kinetic term is therefore explicitly G_n symmetric. The Hermitian conjugates to these rotations appear as contributions of rotations from the neighbouring directions. By subtracting the unity from each shift operator, one ensures that the Taylor series of the numerator starts with terms proportional to $\hat{L}_1^2$. Intuitively, the best approximation to $O(3)$ arises by putting into the denominator a function $f(\phi)$ whose Taylor series starts with the coefficient of $\hat{L}_1^2$ in the numerator, which is $N^{(\alpha)}(\phi^2/4)$ ($N^{(\alpha)}$ being the number of nearest neighbour directions).

With this prescription, the Hamiltonian (2) is approximated as follows:

$$\hat{H} \sim \frac{g}{2\alpha} \left(-\sum_i \left(\frac{1}{N^{(\omega)}} \sum_{(\omega)} T_i^{\underline{n}_i, \underline{n}^{(\omega)}} - 1 \right) \frac{1}{\phi^2/4} - x \sum_i \underline{n}_i \cdot \underline{n}_{i+1} \right) \quad (3)$$

From Table 1, comparing $1-\cos\phi$ (by using the cosine for the first neighbour) to $\phi^2/2$, we see that the uncertainty related to the choice of the scale factor decreases as the approximating polyhedron gets denser.

For the computer programme, it is more convenient to use

$$\begin{aligned} \hat{H}_d &= \frac{g_d}{2\alpha} \hat{W}_d, \quad \hat{W}_d = -\sum_i \left(\frac{1}{N^{(\omega)}} \sum_{(\omega)} T_i^{\underline{n}_i, \underline{n}^{(\omega)}} - 1 \right) - x_d \sum_i \underline{n}_i \cdot \underline{n}_{i+1}, \\ x_d &= \frac{2}{g_d^2}, \quad g_d = \frac{g}{\phi/2}, \quad \hat{H}_d \sim \frac{2}{\phi} \hat{H}. \end{aligned} \quad (4)$$

2.3. - Relations to known models

Here we discuss the relationship of the models defined on smaller polyhedra to models appearing in a different context in statistical physics.

The tetrahedron vector model contains two energetically different relative orientations for two neighbouring spins. By appropriate normalization of the potential energy, one recognizes that it is the four-state Potts model. Its critical index ν was conjectured to be $2/3$ in two dimensions¹⁴⁾, however, most theoretical analyses performed to date give somewhat higher values. In particular, the analysis of the mass gap in the Hamiltonian version of the model yields $\nu = 0.76$ ¹⁵⁾.

The Lagrangian version of the octahedron vector model (called six-state cubic spin model) was introduced by Kim and Lévy⁸⁾ as a model of phase transitions in some cubic rare earths, which have six-fold degenerate ground states. Recently, Badke et al.¹⁶⁾ have performed a Monte Carlo simulation of this version of the model and exploiting the hyperscaling relation have obtained $\nu = 0.66$.

Finally, co-ordinates of the vertices in the cube being $(\pm a/2, \pm a/2, \pm a/2)$, $\underline{n}$ can be represented with the help of three Ising variables. The action of any of the shift operators $T_{\underline{n}, \underline{n}^{(\alpha)}}$ can be understood as flipping one of these variables. Therefore, the cubic vector model is the assembly of three identical, non-interacting, transverse Ising models. Its exact solution was constructed by Pfeuty¹⁷⁾ yielding $g_c = \sqrt{2}$, $\nu = 1$.

These results serve as checks of the numerical methods which will be described in the next two sections.

3. - THE NUMERICAL METHOD

3.1. - The Hilbert space in the computer

For calculations intended in the weak coupling regime, the configurational basis is most useful. One of its vectors is given as a list of the actual directions in which the unit vectors are pointing along the chain. Using a

standard labelling of the vertices of a given P, the vector is characterized by a code number, $C(\ell_1 \dots \ell_N)$. Throughout this subsection, we shall consider for concreteness the example of the icosahedron (Fig. 1).

The ground state of a finite quantum mechanical system is invariant under both translations and global polyhedral transformations. The use of a symmetrized basis reduces the dimensionality of the relevant subspace by about a factor $N.n$, where N is the number of sites, n being the number of elements of the symmetry group. (This crude estimate seems to work quite well for the relatively short chains we have studied.) A normalized basis vector is represented by the code of a specifically chosen configuration (our rule was to take the configuration with the smallest code number). Then a basis vector b is written as

$$b = \frac{1}{\sqrt{m_b}} \sum'_{\text{transf}} t_f [(\ell_1 \dots \ell_N)_{\text{rep}}] \quad (5)$$

Here t_f denotes the possible symmetry transformations and the prime on the summation means that each different configuration connected to $(\ell_1 \dots \ell_N)_{\text{rep}}$ by some t_f 's should appear only once. That is, m_b denotes the number of different configurations appearing in the linear combination (5).

A truncated finite basis is given in the computer by the basis vector list containing the representative codes B (B -list) and an associated multiplicity list (m -list).

Assume now that we have a starting subspace H_0 . An extension step consists of adding all those new basis vectors to the (B,m) -lists, which occur by acting once on the elements of B with the Hamiltonian $\hat{W}_d$ of (4). Since the potential is diagonal in this representation, the new elements are created by the action of the shift operators $T^{\underline{n}, \underline{n}}_{(\alpha)}$. As a result, configurations in which a vector at some site is shifted to one of its neighbouring directions will occur. The realisation of this operation requires a list of the nearest neighbours associated with each direction (a 12×5 block for the icosahedron).

The representative code of the newly created element is found by performing all the symmetry transformations (its multiplicity can be counted at the same time). Comparing it to the previous B -list, one decides whether it should be added or not.

The $x_d = \infty$ ground state of the N-site icosahedron vector model is

$$b_1 = \frac{1}{\sqrt{12}} \sum_{\ell=1}^{12} \frac{(\ell \dots \ell)}{N},$$

which is the single element we choose for H_0 . On the list it appears as

$$B_1 = C(1 \dots 1), \quad m_1 = 12. \quad (6)$$

The action of the Hamiltonian commuting with translations and discrete rotations extends its symmetry to the whole sector built on it.

The systems we have studied are expected to undergo a transition(s) of magnetic nature, that is, in the large x phase of the infinite system, the discrete rotational symmetry is broken to a $Z(p)$ axial symmetry (the singled out axis is that of the magnetization). In the case of the icosahedron, the axes connecting opposite vertices have an unbroken $Z(5)$ symmetry. The corresponding sector of the Hilbert space was generated by the action of the Hamiltonian on the starting (B,m)-list containing four elements:

$$B = [C(1, \dots, 1), C(12, \dots, 12), C(2, \dots, 2), C(7, \dots, 7)],$$

$$m = [1, 1, 5, 5]. \quad (7)$$

The lowest energy state of the sector at $x_d = \infty$ was given as the starting eigenstate candidate. Its components are put on the ϕ_0 -list as the starting wave function in H_0 :

$$\psi_0^{(0)} = (2(1+z^2))^{-1/2} [1, -1, z, -z], \quad (8)$$

where z is the cosine of the angle between nearest neighbours in the icosahedron. In this sector, for the symmetrization of the basis, only combinations of translations and $Z(5)$ rotations around the "1-12" axis should be applied.

We have also computed the energy of gap excitations breaking the icosahedron symmetry to $Z(3)$, by choosing the magnetization to point towards the middle of the (1,2,3)-triangle, for instance. The corresponding starting (B,m)-lists and the wave function $\psi_0^{(0)}$ are the following:

$$B = [C(1, \dots, 1), C(9, \dots, 9), C(4, \dots, 4), C(5, \dots, 5)],$$

$$m = [3, 3, 3, 3] \quad (9)$$

and

$$\psi_o^{(6)} = (2(z_1^2 + z_2^2))^{-1/2} [z_1, -z_1, z_2, -z_2]. \quad (10)$$

In Eq. (10), z_1 is the cosine of the angle between the direction 1 and the middle direction of (1,2,3), while z_2 is the same for 4 and (1,2,3).

One could also study states realizing the breakdown pattern $G_n \rightarrow Z_2$. For the illustration of the question of (partial) restoration of the $O(3)$ symmetry in the spectrum, we thought it is sufficient to restrict our attention to Eqs. (7) and (8), and (9) and (10). Similarly, we have constructed subspaces which correspond to the breakdown patterns $G_{24} \rightarrow Z(4)$ and $G_{24} \rightarrow Z(3)$ in the cubic vector model.

In the other discrete models, we have studied the sector where the maximal $Z(p)$ symmetry is kept unbroken. Although our findings in the case of the cube and the icosahedron give some justification for this, the more prosaic reason was that this gap can be evaluated with a given accuracy from the smallest Hilbert space among all the possibilities.

3.2. - The modified Lanczos algorithm

The mass gaps we want to compute are energy differences of lowest lying states in sectors of the Hilbert space with different symmetry. Our task is therefore reduced to finding these lowest eigenvalues and then forming

$$m = \frac{g_d}{2a} M \equiv \frac{g_d}{2a} (E_o(\text{sym } G_d) - E_o(\text{sym } Z_p)). \quad (11)$$

For this purpose, a simplified and modified form of the Lanczos algorithm¹⁸⁾ can be used more efficiently than the original one. Its most important features are reviewed below for completeness.

The sequence of Hilbert subspaces arising under the consecutive action of the Hamiltonian on the previous one is denoted by $\{H_n\}$.

The suggested first modification is that the best lowest eigenvalue and lowest vector ($E_0^{(n)}$ and $\psi_0^{(n)}$) are determined in each H_n , not just in the maximal available vector space as one does with the usual algorithm (there, only E_0 is determined, ψ_0 is lost). For this, we apply to $\psi_0^{(n-1)}$ a Lanczos construction truncated promptly at the 2×2 matrix stage:

$$\begin{pmatrix} c_0 & b_1 \\ b_1 & c_1 \end{pmatrix} = \begin{pmatrix} \langle \hat{W} \rangle & (\langle \hat{W}^2 \rangle - \langle \hat{W} \rangle^2)^{1/2} \\ (\langle \hat{W}^2 \rangle - \langle \hat{W} \rangle^2)^{1/2} & \frac{\langle \hat{W}^3 \rangle - 2 \langle \hat{W}^2 \rangle \langle \hat{W} \rangle + \langle \hat{W} \rangle^3}{\langle \hat{W}^2 \rangle - \langle \hat{W} \rangle^2} \end{pmatrix} \quad (12)$$

where $\langle W^n \rangle \equiv \langle \psi_0^{(n-1)} | W^n | \psi_0^{(n-1)} \rangle_{H_n}$. (All the matrix elements of W^n are computed within the fixed truncated subspace H_n). Having found the lower eigenvalue of (12), one easily reconstructs the corresponding improved eigenvector in H_n :

$$\psi_{0,1}^{(n)} = \left[\left(\frac{c_0 - E_{0,1}^{(n)}}{b_1} \right)^2 + 1 \right]^{-1/2} \left(1 + c_0 \frac{c_0 - E_{0,1}^{(n)}}{b_1^2} - \frac{c_0 - E_{0,1}^{(n)}}{b_1^2} \hat{W} \right) \psi_0^{(n-1)}. \quad (13)$$

Next, one forms the matrix (12) with the help of $\psi_{0,1}^{(n)}$ and continues the procedure within H_n until the prescribed convergence is reached, yielding $E_0^{(n)}$ and $\psi_0^{(n)}$. Then one performs the next extension step and repeats the above algorithm in H_{n+1} .

The explicit knowledge of $\psi_0^{(n)}$ at the beginning of the $(n+1)^{th}$ extension step gives the possibility of classifying the basis vectors according to their importance in $\psi_0^{(n)}$. A crude realization of this is the second modification we have proposed. A small control parameter r is introduced, whose value is chosen empirically [for $0(10^{-5})$ accurate eigenvalues in the models studied until now $r \sim (10^{-3} - 10^{-4})$ was an appropriate choice]. If the absolute value of a component of $\psi_0^{(n)}$ is found smaller than r , the corresponding basis vector is left out of the actual extension step [but not erased from the (B,m)-lists]. As a result, we have observed in the weak coupling region a considerable reduction in the dimensionality of the Hilbert space necessary to find E_0 with a prescribed accuracy.

For the evaluation of the moments appearing in (12) one has to compute the vector ψ' resulting from the application of W on some $\psi \in H_n$. Denoting the components of ψ by $\psi(\{l\}_{rep})$ in the symmetrized basis, the result of the application

of W can be written as:

$$\begin{aligned} \psi'(\{l'\}_{rep}) = & - \sum_{\{l\}_{rep} \in B} R(\{l'\}_{rep} \leftarrow \{l\}_{rep}) \frac{1}{N^{(\alpha)}} \psi(\{l\}_{rep}) + \\ & + (N - x) V(\{l'\}_{rep}) \psi(\{l'\}_{rep}). \end{aligned} \quad (14)$$

The quantities $R(\{l'\}_{rep} \leftarrow \{l\}_{rep})$ are the matrix elements of $\Sigma_1 \Sigma_{(\alpha)} T^{\underline{n}, \underline{n}}(\alpha)$ between two symmetrized basis vectors. They form a sparse matrix, so it is advised to store only its non-zero elements. In practice, we associate two more auxiliary lists with each element of the B-list: (L_1, L_2) . L_1 tells us where the configurations $\{l\}_{rep}$ having access to $\{l'\}_{rep}$ are on the B-list; L_2 gives the actual values of $R(\{l'\}_{rep} \leftarrow \{l\}_{rep})$. Then the summation on the right-hand side of (14) goes only over the elements of L_1 : these lists are constructed in the extension step for each new element of the B-list as well as the quantity $V(\{l\}_{rep}) \equiv \Sigma_i \frac{n_i n_{i+1}}{i+1}$.

The mass gap was calculated to $O(10^{-6})$ accuracy for chains of length $N = 4-7$ (tetrahedron), $N = (3-6)$ (octahedron), $N = 3-5$ (icosahedron) and $N = 2-4$ (dodecahedron). The accuracy of $N.H(g, N)$ appearing in our analysis is therefore not worse than $O(10^{-4})$. The limitation in the size N comes from the dimensionality of the excited state subspace necessary for finding the lowest eigenvalue with the above accuracy. The maximal lengths listed correspond to $\sim 10^4$ dimensional subspaces. To the present stage, the calculations used less than five hours on the IBM-3081 computer at CERN^{*)}.

4. - ANALYSIS AND RESULTS

The invariance of the physics of the limiting, finite length, continuum theory ($N \rightarrow \infty$, $L \equiv Na = \text{fixed}$, a being the lattice spacing) under the change in the resolution of its approximations, discretized in space, is reflected in the mass gap data through the equality¹¹⁾:

*) Trial runs of our "scalar" FORTRAN programme were performed also on the CYBER-205 machine at SARA-Amsterdam. The Lanczos-algorithm described above was fully vectorized automatically. We are indebted to Dr. W. van Leeuwen for his generous help making possible these test runs.

$$gNM(g,N) = g'N'M(g',N') \quad , \quad N, N' \gg 1. \quad (15)$$

Equation (15) was the starting point in Ref. 11) for the "derivation" of the Roomany-Wyld approximants to the β function of the theory:

$$\frac{\beta_d(g)}{g} = \frac{\ln(NM(g,N)/(N-1)M(g,N-1))}{\ln \frac{N}{N-1} \left(1 - x \frac{\partial}{\partial x} \ln[NM(g,N) \cdot (N-1)M(g,N-1)] \right)} \quad (16)$$

The estimate of the critical point is found through the zero of this function

$$R(g_{d,c}, N) \equiv \frac{NM(g_{d,c}, N)}{(N-1)M(g_{d,c}, N-1)} = 1. \quad (17)$$

Close to $g_{d,c}$, one uses for $\beta_d(g)$ the linear part of (1) and determines the critical exponent ν from the slope of the fit.

In order to get some information about the systematic errors involved in the above procedure, when applied with moderate N , we propose to undertake a slightly different analysis too, which, for $N \rightarrow \infty$, would give an identical result to (16). Writing the solution of (15) when $N' = N-1$ as

$$g(N-1) = g(N) + \Delta g(g(N)) \quad (18)$$

and assuming $\Delta g \ll g$ close to $g_{d,c}$ (which has to be verified a posteriori) one expands (15) up to terms linear in $\Delta g(g)$. When expressing $\Delta g(g)/g$, a symmetrization in N and $N-1$ similar to what appears in the denominator of (16) is used:

$$\frac{\Delta g(g)}{g} = \frac{R(g,N) - 1}{1 - x \frac{\partial}{\partial x} \ln[NM(g,N) \cdot (N-1)M(g,N-1)]} \quad (19)$$

Finally, by integrating the definition of the β_d function between $g(N)$ and $g(N-1)$ ($\approx g_{d,c}$), one finds the estimate:

$$\nu = \frac{\ln \frac{N}{N-1}}{\ln \left(1 + \frac{\Delta g/g}{1 - g_{d,c}/g} \right)} \quad (20)$$

The critical couplings were determined by linearly interpolating between the closest g values to them $[R(g_1, N) < 1 < R(g_2, N)]$. The values displayed in

column 2 of Table 2 [entries of column 1 mapped onto the $O(3)$ scale with the help of (4)] satisfy our expectation by showing clearly the decreasing tendency of $g_{d,c}^{(0(3))}$.

Next $\beta_d(g)/g$ and $\Delta g/g$ were calculated according to (16) and (19) from chains of e maximal lengths. They are displayed on Figs. 2 and 3 as functions of $g_{d,c}/g$ for all the models investigated, but for the cube. One observes the tendency that the β function becomes flatter as one goes to denser approximations, again in accordance with our qualitative expectations. The same tendency is evident also for $\Delta g/g$. Also its order of magnitude justifies stopping the expansion of (15) at the term linear in Δg .

By using the closest points to $g_{d,c}$ ($g > g_{d,c}$), in which the gap was calculated, one finds the ν values, displayed in columns 3 and 4 of Table 2. The errors are the mean quadratic deviations of the ν values, computed from a single g -point, from the linear fits. The systematic error, being of similar magnitude, makes it senseless to find the quadratic term in (1) by statistically better fits.

The critical data obtained for the cubic vector model ("Ising model") can be compared also to the finite size scaling results for the transverse Ising model¹⁹⁾. Both ν and $g_{d,c}$ agree to 1% accuracy with values obtained from the analytic solution of $N = 5$ and 4 sites long chains in Ref. 19). It is also remarkable that the ν value obtained for the octahedron model with $\Delta g(g)$ based analysis is very close to the value coming from the Lagrangian simulation¹⁶⁾. The ν index of the tetrahedron model differs from that conjectured for the four-state Potts model by the same amount as the values found with other theoretical analyses.

The excitation energy belonging to different symmetry breakdown patterns was calculated for short ($N=3,4$) chains in the case of the cubic and icosahedron vector models. To the accuracy of our calculations, the degeneracy of these excitations could be verified in the regions $0.85 < x_{\text{cube}} < 1.15$ and $0.6 < x_{\text{icosa}} < 1.2$. This means that the degeneracy of the excitation spectrum seems to be valid beyond the critical region. It is plausible that all denser approximations to the $O(3)$ model would show this degeneracy too.

Finally, we mention two additional tests of the stability of our conclusions. The ν values determined for the icosahedron and dodecahedron models from chains shorter than the maximal by one site were found to agree with those of Table 2 within errors. Also, in these models, fits were attempted by assuming the

$Z(p) \rightarrow U(1)$ scenario, (1b). The values obtained for σ are in absolute value smaller than 0.05 and compatible with zero within errors.

5. - CONCLUSIONS

In this paper we presented a systematic study of the discrete vector models in two dimensions. In case of the tetrahedron, octahedron and cubic models, we have verified known results in the Hamiltonian formulation. The critical data of the icosahedron and dodecahedron models were determined here for the first time.

For the smaller discrete subgroups, the second-order transition takes place on the $O(3)$ coupling scale in the strong coupling regime, therefore we do not expect any correction to their β functions reflecting asymptotic freedom.

The systematic deviations between the results of different analyses become important first for the icosahedron. However, none of the fits in themselves seem to reveal non-negligible corrections to the linear β function. The series of the critical point estimates ($x_c[3/2] = 1.04$, $x_c[4/3] = 1.14$, $x_c[5/4] = 1.17$) hints to the 1% accuracy of the best $g_{d,c}$ estimate. Longer chains should reduce also the systematic errors in ν .

It is, however, the dodecahedron model which would merit a more detailed investigation. Its critical point on the $O(3)$ scale is very close to the $g < 0.5$ region where finite size calculations with the full group approached the asymptotic free regime¹¹⁾. $x_{\text{dodeca},c}$ is expected to be pushed to still larger x for longer chains ($x_{d,c}[3/2] = 0.87$, $x_{d,c}[4/3] = 0.96$). The large mean square deviation of the ν values indicates important corrections to the linear fit to the β function, which could be quantified once the systematic errors are reduced. However, even in that case, the probable value of ν will be smaller than 3. Therefore we doubt if any coupling interval could be found, where a pure asymptotically free regime would be observed.

For longer chains, the dimensionality reduction which results from basis vector importance sampling might not be sufficient and the application of stochastic methods of finding ground state eigenvalues of Hamilton operators²⁰⁾ might be unavoidable.

Finally, we notice that the comparison of β functions of theories whose physics is expected to be close to each other, seems to us the most systematic and unambiguous characterization for any class of field theoretical models, including lattice gauge theories.

ACKNOWLEDGEMENTS

Discussions with U.M. Heller and K. Szlachányi at different stages of this work are gratefully acknowledged. We thank R. Petronzio for reading the manuscript. A.P. is indebted to the Theory Division of CERN for the hospitality extended to him.

Polyhedron	Factor Space	cos (n,n _{neighbour})					$\phi^2/2$
		1st	2nd	3rd	4th	5th	
Tetrahedron	$G_{12}/Z(3)$	-1/3	-	-	-	-	1.83
Octahedron	$G_{24}/Z(4)$	0	-1	-	-	-	1.23
Cube	$G_{24}/Z(3)$	1/3	-1/3	-1	-	-	0.76
Icosahedron	$G_{60}/Z(5)$	0.447214	-0.447214	-1	-	-	0.61
Dodecahedron	$G_{60}/Z(3)$	0.745356	1/3	-0.745356	-1/3	-1	0.27

Table 1 - List of the discrete vector models. The classification in terms of factor spaces, the cosines of the angles between different directions and $\phi^2/2$ for the angle between nearest neighbour directions are displayed.

Vector model	$g_{d,c}$	g_c (0(3)-scale)	$v(\Delta g/g)$	$v(\beta/g)$	v, g_c (other sources)
Tetrahedron	1.417	1.355	0.75 ± 0.02	0.72 ± 0.01	$v=2/3^{14}), v=0.76^{15})$ $v=0.66^{17})$
Octahedron	1.305	1.023	0.65 ± 0.01	0.62 ± 0.01	
Cube	1.420	0.875	0.982 ± 0.001	-	$v=1, g_c=\sqrt{2}^{18})$
Icosahedron	1.309	0.723	1.41 ± 0.01	1.32 ± 0.02	-
Dodecahedron	1.440	0.529	2.58 ± 0.09	2.47 ± 0.07	-

Table 2 - The critical data of the discrete vector models. Critical couplings on the intrinsic and the 0(3) scale are given. For the critical exponent v two estimates are shown. The last column contains critical data from other sources, for comparison.

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FIGURE CAPTIONS

Fig. 1 The standard labelling of vertices of the icosahedron.

Fig. 2 The Roomany-Wyld approximants to $\beta_d(g)/g$ for various models as a function of $g_{d,c}/g$.

Fig. 3 Approximants to $\Delta g(g)/g$ for various models as a function of $g_{d,c}/g$ [Eq. (19)].

- Figure 1 -

- Figure 2 -

- Figure 3 -