# LOCALITY IN STRONG COUPLING LIOUVILLE FIELD THEORY AND STRING MODELS FOR DIMENSIONS 7, 13 AND 19

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ABSTRACT

We construct a local field in Liouville string field theory for space-time dimensions $\mathcal{D} = 7, 13$, and 19. The spectrum of the corresponding string models has neither ghosts nor tachyons.

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In ref. [1], the one space-time quantum Liouville field theory in a box was solved in order to construct a consistent string model in $\mathcal{D} < 26$ space-time dimensions. We started from the Lagrangian density

$$\mathcal{L} = \frac{1}{16\pi\hbar}\left[\frac{1}{2}\left(\varphi_\tau^2 - \varphi_\sigma^2\right) - e^{\varphi} + 2\varphi_{\sigma\sigma}\right] \tag{1}$$

where $0 \leq \sigma \leq \pi$ is the space-like coordinate along the open string. The coupling constant $\hbar$ is related to $\mathcal{D}$ by

$$\hbar = \frac{3}{25-\mathcal{D}} \tag{2}$$

We separated the modes of (1) in terms of the modes $p_n$ of a field $p(\sigma)$ such that

$$p(\sigma) = \sum_{n=-\infty}^{+\infty} p_n \, e^{-in\sigma} \tag{3}$$

$$\left[p_n , p_m\right] = n\hbar\, \delta_{n,-m} \tag{4}$$

and of another equivalent set $\tilde{p}_n$ expressible as a power series in $p_m$

$$\left[\tilde{p}_n , \tilde{p}_m\right] = n\hbar\, \delta_{n,-m} \tag{5}$$

$$\tilde{p}_0 = -p_0 \equiv \frac{i\omega}{2\pi} \tag{6}$$

The conformal generators $L_n$ can be constructed from either $p_n$ or $\tilde{p}_n$, by normal ordering

$$L_n = \frac{1}{2\hbar}\left(\sum_r N(p_r p_{n-r}) - in\, p_n\right) + \frac{1}{8\hbar}\delta_{n,0} \tag{7}$$

$$= \frac{1}{2\hbar}\left(\sum_r \tilde{N}(\tilde{p}_r \tilde{p}_{n-r}) - in\,\tilde{p}_n\right) + \frac{1}{8\hbar}\delta_{n,0} \tag{8}$$

The conformally covariant operators

$$\psi_1^\eta(\sigma) = K^\eta e^{\eta p_0 \sigma} e^{-i\frac{\eta^2}{2}\hbar\sigma} N\left(\exp\left[i\eta \sum_{m\neq 0} \frac{p_m}{m} e^{-im\sigma}\right]\right) \qquad (9)$$

$$\psi_2^\eta(\sigma) = \tilde{K}^\eta e^{-\eta p_0 \sigma} e^{-i\frac{\eta^2}{2}\hbar\sigma} \tilde{N}\left(\exp\left[-i\eta \sum_{m\neq 0} \frac{\tilde{p}_m}{m} e^{-im\sigma}\right]\right) \qquad (10)$$

where

$$p_0 K^\eta = K^\eta (p_0 - i\hbar\eta) \quad , \quad [p_m, K] = 0 \quad m \neq 0 \qquad (11)$$

$$p_0 \tilde{K}^\eta = \tilde{K}^\eta (p_0 + i\hbar\eta) \quad , \quad [\tilde{p}_m, \tilde{K}] = 0 \quad m \neq 0 \qquad (12)$$

were proved in ref. [1] to obey a simple algebra

$$\psi_i^\eta(\sigma)\, \psi_j^\eta(\sigma') = \sum_{k,\ell} S_{i\ell}^{jk}(\omega, \sigma-\sigma', \eta)\, \psi_k^\eta(\sigma')\, \psi_\ell^\eta(\sigma) \qquad (13)$$

provided that $\eta$ is such that

$$2\hbar\eta^2 - \eta + 1 = 0 \qquad (14)$$

Using (13), we constructed a local Liouville field $e^{-\varphi/2}$ as a bilinear form in $\psi_1^\eta$ and $\psi_2^\eta$ in close connection with its classical form, so that

$$\left[e^{-\varphi(\sigma)/2}, e^{-\varphi(\sigma')/2}\right] = 0 \qquad (15)$$

We also determined the set of values of $p_0$, or $\omega$. As seen from (11) and (12), $e^{-\varphi/2}$ shifts $p_0$ by $0$ or $\pm 2i\hbar\eta$. It turned out that the commutation relation (16) can be realized on a finite set of real values of $\omega$ as the semi-classical analysis of ref. [2] suggested, and furthermore that the Hilbert space of $p_m$, $m>0$ excitations contains only positive norm states.

In refs. [1-3], we have applied these results to the string model in $\mathcal{D}$ space-time dimensions. From eqs. (2) and (14), it follows that this is possible only for $-\infty < \mathcal{D} < 1$, $0 < \hbar < 1/8$, when $\eta$ is real, and $e^{-\varphi/2}$ hermitian. When $\mathcal{D} > 1$,



$\hbar > 1/8$, eq. (14) has complex solutions. The construction of the local field $e^{-\varphi/2}$ is still possible using one of the complex solutions, but the values of $p_0$ on which $e^{-\varphi/2}$ acts consistently become neither purely imaginary nor real, and the hamiltonian $L_0$ of the Liouville theory becomes complex. One should then look for another local field which will act consistently on a positive norm Hilbert space. In this paper, we show that this is indeed possible for the special values 7, 13 and 19 of $\mathcal{D}$, corresponding to $\hbar$ = 1/6, 1/4 and 1/2. Using this Liouville Hilbert space together with the orbital modes of the string yields a string spectrum which has neither ghosts nor tachyons, the lowest state being massless[f1)].

Let us first examine which set of values of the zero mode $p_0$ is desirable for $\hbar > 1/8$ to yield a reasonable spectrum for the string theory.

In ref. [3], we found, in particular, that for the special values $\hbar = \frac{2n-1}{8n^2}$, $n$ = 1,2,3,..., the set of values $\omega_m$ of $\omega$ was given by

$$\omega_m = 2m h\eta \;;\; m = 0, 1, \ldots, n-1 \;;\quad h = 2\pi\hbar \qquad (16)$$

Each value of $\omega$ gives, for the string model, a set of Regge trajectories spaced by integers and with leading intercept

$$\alpha(\omega) = 1 - \frac{1}{8\hbar}\left(1 - \frac{\omega^2}{\pi^2}\right) \qquad (17)$$

Specializing in the above mentioned values of $\omega$ and $\hbar$, we obtain

$$\alpha_m = -\frac{(m-1)^2}{2n-1} + \frac{m^2}{2n-1} \;;\; m = 0, 1, \ldots, n-1 \qquad (18)$$

From this equation, we see one of the main results of ref. [3], namely that the string model contains no tachyon for $n$ = 1,2,..., and that the highest intercept, obtained for $m = n-1$, is always zero. The lowest intercept ($\omega = 0$) has the value $1 - 1/8\hbar$. The highest intercept ($\alpha_{n-1} = 0$) is obtained for $\omega = \pi(1 - 8\hbar)^{1/2}$. Of all the values $\omega_m$ of eq. (16), these are the only two to have a reasonable continuation in the case $\hbar > 1/8$. However, the value $\omega = 0$ then leads to a tachyon, and should

be avoided ; the value $\omega = \pi(1-8\hbar)^{1/2}$ for the massless state becomes purely imaginary. Hence, for $\hbar > 1/8$, we are led to take $\omega$ purely imaginary, i.e. $p_0$ real,[f2] avoiding the tachyonic values $|p_0| < 1/2\,(8\hbar-1)^{1/2}$. Taking $p_0$ real brings two simplifications : first, the ordinary hermiticity of the $p_m$ oscillators is recovered : $p_m^+ = p_{-m}$, rather than $p_m^+ = \tilde{p}_{-m}$ for imaginary $p_0$ [4]. Second, the Liouville hamiltonian $L_0$ is manifestly positive for real $p_0$, while for imaginary $p_0$, an upper limit for $|p_0|$ had to be set to achieve positivity.

The algebra (13) of $\psi_1^{\eta}$ and $\psi_2^{\eta}$ remains valid for $\hbar > 1/8$, for each of the two complex conjugate solutions $\eta_\pm = \frac{1}{4\hbar}\left(1 \pm i\sqrt{8\hbar-1}\right)$ of eq. (14). It is natural to supplement it with the commutation properties of $\psi_1^{\eta_+}$ and $\psi_2^{\eta_-}$ with different $\eta$'s. The same method as that used in ref. [1] for the derivation of eq. (13) shows that $\psi_1^{\eta_+}$ and $\psi_2^{\eta_-}$ commute simply :

$$\psi_1^{\eta_+}(\sigma)\,\psi_2^{\eta_-}(\sigma') = \psi_2^{\eta_-}(\sigma')\,\psi_1^{\eta_+}(\sigma)\left(\frac{\omega-\eta_-\hbar}{\omega+\eta_+\hbar}\right) e^{i\frac{\pi}{2}\varepsilon(\sigma-\sigma')} \tag{19}$$

and a similar expression is valid for $\psi_1^{\eta_-}(\sigma)\,\psi_2^{\eta_+}(\sigma)$ by exchanging $\eta_+$ and $\eta_-$.

For $\hbar > 1/8$ the individual operators $\psi_1^{\eta_\pm}$ and $\psi_2^{\eta_\pm}$ shift $p_0$ by complex values, according to eqs. (11) and (12). Hence, only appropriate combinations can be used, such that $p_0$ remains real. We define $(\psi_1^{\eta_+}\psi_2^{\eta_-})(\sigma)$ and $(\psi_2^{\eta_+}\psi_1^{\eta_-})(\sigma)$ by removing the short-distance singular behaviour of the products, as explained in ref. [5]. These operators shift $p_0$ by a real value :

$$p_0\,\psi_1^{\eta_+}\psi_2^{\eta_-} = \psi_1^{\eta_+}\psi_2^{\eta_-}\left(p_0 + \frac{1}{2}\sqrt{8\hbar-1}\right) \tag{20}$$

$$p_0\,\psi_2^{\eta_+}\psi_1^{\eta_-} = \psi_2^{\eta_+}\psi_1^{\eta_-}\left(p_0 - \frac{1}{2}\sqrt{8\hbar-1}\right) \tag{21}$$

The commutation properties of each of these two operators with itself at a different $\sigma$ are trivial. For example,

$$(\psi_1^{\eta_+}\psi_2^{\eta_-})(\sigma)\,(\psi_1^{\eta_+}\psi_2^{\eta_-})(\sigma') = (\psi_1^{\eta_+}\psi_2^{\eta_-})(\sigma')\,(\psi_1^{\eta_+}\psi_2^{\eta_-})(\sigma)\, e^{-i\frac{\pi}{4\hbar}\varepsilon(\sigma-\sigma')} \tag{22}$$



and the same expression is valid for $\psi_2^{\eta_+}\psi_1^{\eta_-}$. On the other hand, the commutation of $(\psi_2^{\eta_+}\psi_1^{\eta_-})(\sigma)$ with $(\psi_1^{\eta_+}\psi_2^{\eta_-})(\sigma')$ involves four terms. Two of these terms contain the same operators at $\sigma$ and at $\sigma'$, while the other two contain the products $(\psi_2^{\eta_+}\psi_2^{\eta_-})(\sigma)(\psi_1^{\eta_+}\psi_1^{\eta_-})(\sigma)$ and $(\psi_1^{\eta_+}\psi_1^{\eta_-})(\sigma')(\psi_2^{\eta_+}\psi_2^{\eta_-})(\sigma)$ with coefficients which depend on the values of $S_{ie}^{jk}$ in eq. (13). These terms are unwelcome because $\psi_1^{\eta_+}\psi_1^{\eta_-}$ and $\psi_2^{\eta_+}\psi_2^{\eta_-}$ shift $p_0$ by a purely imaginary value. To keep a real $p_0$ we try to cancel those terms with those coming from the commutation of $(\psi_1^{\eta_+}\psi_2^{\eta_-})(\sigma)$ with $(\psi_2^{\eta_+}\psi_1^{\eta_-})(\sigma')$. Thus, we consider the operator

$$\Phi(\sigma) = \alpha(p_0)(\psi_1^{\eta_+}\psi_2^{\eta_-})(\sigma) + \beta(p_0)(\psi_2^{\eta_+}\psi_1^{\eta_-})(\sigma) \tag{23}$$

and look for $\alpha$ and $\beta$ such that $\Phi(\sigma)$ has a simple commutation property with $\Phi(\sigma')$. Using the explicit expressions for $S_{ie}^{jk}$ given in ref. [1], one finds that this is possible if one has

$$\sin(\pi\eta_+) = \pm\sin(\pi\eta_-) \tag{24}$$

and

$$\cos(4i\pi\eta_+ p_0) = \cos(4i\pi\eta_- p_0) \tag{25}$$

The first condition, (24), gives the three values of $\hbar$ mentioned at the beginning of the paper : 1/6, 1/4 and 1/2 corresponding to space-time dimensions 7, 13 and 19 for the string model. The second condition, (25), is automatically satisfied when $p_0$ is an integer times $1/2\,(8\hbar-1)^{-1/2}$, consistently with eqs. (20)-(22). The resulting commutation property of $\Phi(\sigma)$ and $\Phi(\sigma')$ is then simply

$$\Phi(\sigma)\,\Phi(\sigma') = \Phi(\sigma')\,\Phi(\sigma)\,\exp\left(-\frac{i\pi}{4\hbar}\varepsilon(\sigma-\sigma')\right) \tag{26}$$

The coefficients $\alpha$ and $\beta$ must also satisfy

$$\alpha(p_0)\,\beta(p_0 + i\hbar(\eta_+-\eta_-))\,\frac{\Gamma(1+2i\eta_+p_0)\,\Gamma(2i\eta_+p_0)}{\Gamma(1-2i\eta_+p_0-\eta_+)\,\Gamma(\eta_+-2i\eta_+p_0)}$$
$$= \alpha(p_0 - i\hbar(\eta_+-\eta_-))\,\beta(p_0)\left(\frac{-\sin(\pi\eta_+)}{\sin(\pi\eta_-)}\right)\frac{\Gamma(1+2i\eta_-p_0)\,\Gamma(2i\eta_-p_0)}{\Gamma(1-\eta_--2i\eta_-p_0)\,\Gamma(\eta_--2i\eta_-p_0)} \tag{27}$$

This equation is a recursion relation for the product $\alpha(p_0)\,\beta(p_0+i\hbar(\eta_+-\eta_-))$. The relative normalization of $\alpha$ and $\beta$ can be fixed for example by asking that $\Phi$ be hermitian, which means

$$\alpha(p_0) = \beta^+(p_0 - i(\eta_+-\eta_-)\hbar) \tag{28}$$

Eq. (27) then simply becomes a recursion relation for $|\alpha(p_0)|^2$:

$$|\alpha(p_0)|^2 = -|\alpha(p_0 - i\hbar(\eta_+-\eta_-))|^2 \frac{\sin\pi\eta_+}{\sin\pi\eta_-}\,\frac{\Gamma(1+2i\eta_-p_0)\,\Gamma(2i\eta_-p_0)}{\Gamma(1-\eta_- - 2i\eta_-p_0)\,\Gamma(\eta_- - 2i\eta_-p_0)}\,\frac{\Gamma(1-2i\eta_+p_0-\eta_+)\,\Gamma(\eta_+-2i\eta_+p_0)}{\Gamma(1+2i\eta_+p_0)\,\Gamma(2i\eta_+p_0)} \tag{29}$$

Using the consistency conditions (24) and (25), together with the elementary relation $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, it is simple to verify that the complicated coefficient on the right-hand side is real and positive for $\hbar = 1/4$.

It is most interesting to note that this coefficient blows up for $p_0 = i\hbar(\eta_+-\eta_-) \equiv -\frac{1}{2}(8\hbar-1)^{-1/2}$ and vanishes for $p_0 = i\hbar(\eta_- - \eta_+) \equiv 1/2\,(8\hbar-1)^{1/2}$. This means that $\alpha\left(\frac{1}{2}(8\hbar-1)^{1/2}\right) = \alpha(0) = 0$. $\Phi$ is thus naturally restricted to acting only on states with $p_0 = \frac{n}{2}(8\hbar-1)^{1/2}$; $n = 1,2,3\ldots$ or only on states with $p_0 = -\frac{n}{2}(8\hbar-1)^{1/2}$, $n = 1,2,3,\ldots$, without connecting the two sets, or reaching states with $p_0 = 0$. Thus we can choose to work only with the states $p_0 = \frac{n}{2}(8\hbar-1)^{1/2}$, $n = 1,2,3,\ldots$, and still have a consistent algebra for $\Phi$. Eq. (26) is to be used next in order to construct a local field. For $\hbar = 1/4$, $\Phi$ is a two-dimensional fermion, and locality is directly satisfied : one finds a fermionization which is analogous to the sine-Gordon-Thirring correspondence. States with arbitrary positive $n$ are present, fermionic or bosonic according to the parity of $n$. For $\hbar = 1/2$ and $1/6$, (26) gives a factor $\pm i$. Local fields must be quadratic in $\Phi$, and are bosonic. The set of values of $n$ is therefore restricted to the odd integers unless parastatistics is introduced.



Translating in terms of the string model, the states which we have just determined correspond to a spectrum without tachyons. The highest intercept is zero according to eq. (17), and is obtained for $p_0 = \frac{1}{2}(8\hbar-1)^{1/2}$. For interacting strings, however, the situation is more delicate than for the cases $\mathcal{D} < 1$ of ref. [1], [3] : the operators with dimensions corresponding to the emission of one of the states of the spectrum are not built only in terms of $\Phi$.

The implications of the above results to two-dimensional statistical models have already been discussed in ref. [5]. In the similar discussion of supersymmetric Liouville field theory [6], [7], the allowed values are $\mathcal{D}$ = 3,5,7.

FOOTNOTES

f1) We cannot rule out the existence of solutions for other values of $\hbar$, but they must be much more complicated than those of this paper and we have not found any at the present time.

f2) We remark that going from real to imaginary $p_0$ is reminiscent of going from plane waves to solitons in the inverse scattering data of the sine-Gordon equation.

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