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DIMENSION SHIFTING OPERATORS AND NULL STATES IN
2-D CONFORMALLY INVARIANT FIELD THEORIES

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ABSTRACT

We discuss the existence and properties of differential operators which transform covariant operators into covariant operators of different weights in two-dimensional conformally invariant field theories. We relate them to null states and the vanishing of the Kac determinant in representations of the conformal algebra, and to the existence of differential equations for Green functions of covariant operators. In this framework, we rederive the essential features of our earlier work on dual models with shifted intercept, which in Euclidean space-time gives explicit solutions of the conformal bootstrap equations where all operators are marginal.

1. - INTRODUCTION

There is a growing interest both in physics and in mathematics for representations of the 2-D conformal group and for the related properties of conformally invariant theories. The basic objects are, on the one hand, the algebra of conformal transformations which preserve the real axis:

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m}, \quad (1.1)$$

and on the other hand, the set of conformally covariant operators $\mathcal{A}^{(\alpha)}$ such that $[\delta^{(\alpha)}, \mathcal{A}^{(\alpha)}]$ is the conformal weight]

$$[\mathcal{L}_n, \mathcal{A}^{(\alpha)}(z)] = z^n \left[z \frac{\partial}{\partial z} + n \delta^{(\alpha)} \right] \mathcal{A}^{(\alpha)}(z). \quad (1.2)$$

In the following, it will be important to distinguish irreducible representations and for them we shall use the symbols L_n and C instead of $\mathcal{L}_n$ and $\mathcal{C}$. Except, possibly, when Kac's determinant vanishes, the quantum Liouville theory provides such an irreducible representation with

$$C = 1 + \frac{3}{n}. \quad (1.3)$$

This is one of the powerful tools which we developed starting from our quantum solution of Liouville field theory¹⁾⁻⁴⁾. This representation is generated by a single set of harmonic mode operators p_n , n integer. It is irreducible since the corresponding Fock space is isomorphic to the space generated by all L_n 's with negative indices applied to the ground state (highest weight) vector. We explicitly built the set of covariant operators^{3),4)} and showed that a subset satisfies linear differential equations^{4),5)}. Later on, in our unpublished attempts to derive explicit expressions for relevant Liouville's Green functions, we came across differential operators, which we shall call shift operators, which transform covariant operators into covariant operators of different conformal weights. The differential equations mentioned above are particular cases where the shift operators give zero.

There is another direction where one can look for general hints. The orbital modes of a string, which are simple free fields, lead to a reducible representation of (1.1) with

$$\mathcal{C} = \mathcal{D} , \quad (1.4)$$

where $\mathcal{D}$ is the number of space-time components. Long ago we tried to generalize the Veneziano model by shifting the leading Regge intercept α_0 away from $\alpha_0 = 1$ while keeping the infinite family of ghost killing conditions⁵⁾. We worked entirely in the space of orbital modes and constructed shift operators in the corresponding reducible representation. We noticed that the existence of null states is the key factor⁵⁾.

In the recent independent developments of Belavin, Polyakov and Zamolodchikov⁶⁾, one precisely takes as a starting point that null states should decouple from arbitrary Green functions of covariant operators. On the other hand, one knows from the example of dual models⁷⁾ that null states only decouple from physical Green functions in general.

In the present paper, we study systematically the properties of shift operators, differential equations and null states. Our main motivation is the frustrating situation we encounter in the conformally invariant models for $c > 1$, where differential equations can only be obtained for Green functions of unphysical operators with negative dimension. We want to relate by a shift operator a physical operator to an unphysical one. We show in Section 2 how this can be done at the classical level. Using only the Lie algebra (1.1), it is shown that there is a one-to-one correspondence between shift operators and null states in any representation. Irreducible representations are discussed using the explicit construction suggested by the Liouville theory^{2),4)}; there all null states are shown to vanish identically. This is expected since null states would span invariant subspaces, thereby contradicting the irreducibility. The corresponding shift operators give zero on a subset of covariant operators. This gives back the above-mentioned differential equations^{3),4)}. The vanishing of null states trivially shows that they decouple from all Green functions in the irreducible case. For illustration, we discuss, in more detail than before¹⁾, the realization of (1.1) by Poisson brackets which follows from the canonical structure of the classical Liouville field theory (Section 1). In the last section, we also make explicit use of shift operators by rederiving the essential features of our earlier work on dual models with shifted intercept⁵⁾.

2. - CLASSICAL STRUCTURE IN THE IRREDUCIBLE CASE

Here we develop the general properties suggested by the canonical structure of classical Liouville field theory¹⁾. Introduce

$$U(\sigma) = 2\hbar \sum_{n=-\infty}^{+\infty} L_n e^{-in\sigma} . \quad (2.1)$$

The basic objects are the two independent solutions of the Schrödinger equation

$$\left[-\partial_\sigma^2 + U(\sigma) \right] \psi_{1,2} = \frac{1}{4} \psi_{1,2} , \quad (2.2)$$

and their logarithmic derivatives

$$P = \partial_\sigma \ln \psi_1 , \quad \tilde{P} = \partial_\sigma \ln \psi_2 , \quad (2.3)$$

which satisfy the Riccati equation

$$U - \frac{1}{4} = P^2 + \partial_\sigma P = \tilde{P}^2 + \partial_\sigma \tilde{P} . \quad (2.4)$$

In (2.1), the coefficient is chosen such that the quadratic and linear terms in (2.4) both have coefficients equal to one. It is written as $2\hbar$ in order to have a simple transition to the quantum case.

In general, the conformally covariant observables are linear combinations of powers of ψ_1 and ψ_2 . We now proceed to derive differential identities relating these monomials. As it is well known, if the Wronskian of ψ_1 and ψ_2 is taken to be one, one has

$$\psi_2(\sigma) = \psi_1(\sigma) \int^\sigma d\sigma' \psi_1^{-2}(\sigma') , \quad (2.5)$$

and, therefore,

$$\psi_1^2(\sigma) \partial_\sigma \left(\frac{\psi_2}{\psi_1} \right) = 1 . \quad (2.6)$$

More generally, for all positive integer r ,

$$\left(\psi_1^2 \partial_\sigma\right)^2 \left(\frac{\psi_2}{\psi_1}\right)^n = \frac{\Gamma(n+1)}{\Gamma(n-2+1)} \left(\frac{\psi_2}{\psi_1}\right)^{n-2}. \quad (2.7)$$

Consider now the monomial

$$\psi_1^m \psi_2^n = \psi_1^{m+n} \left(\frac{\psi_2}{\psi_1}\right)^n.$$

It follows from (2.7) that

$$\begin{aligned} \left[\psi_1^{m+n-2\ell} \left(\psi_1^2 \partial_\sigma\right)^2 \psi_1^{-m-n} \right] \psi_1^m \psi_2^n \\ = \frac{\Gamma(n+1)}{\Gamma(n-2+1)} \psi_1^{m-2} \psi_2^{n-2}. \end{aligned} \quad (2.8)$$

The bracket can be simplified by means of (2.3):

$$\prod_{\ell=1,2,\dots,0} \left(\partial_\sigma - (m+n-2\ell)P \right) \psi_1^m \psi_2^n = \frac{\Gamma(n+1)}{\Gamma(n-2+1)} \psi_1^{m-2} \psi_2^{n-2}. \quad (2.9)$$

In this formula and hereafter, the differential operators are to be ordered as indicated by the order of the indices shown under the Π symbol.

In general, that is if $\Gamma(n-r+1)$ is not infinite, (2.9) displays the existence of a shift operator, in the family of monomials. It relates monomials $\psi_1^m \psi_2^n$ and $\psi_1^{m'} \psi_2^{n'}$ such that

$$m-n = m'-n'. \quad (2.10)$$

If, on the contrary, $n-r+1 \leq 0$, the right member of (2.9) vanishes. One obtains differential equations satisfied by $\psi_1^m \psi_2^n$.

Obviously, a similar discussion can be carried out with the roles of ψ_1 and ψ_2 interchanged and, in this connection, the important operators are those which are invariant in the exchange of p and $\tilde{p}$. They are polynomials in U , as we shall see, and have an intrinsic structure related to the conformal group. The

simplest example is

$$(\partial_\sigma + P)(\partial_\sigma - P) = (\partial_\sigma + \tilde{P})(\partial_\sigma - \tilde{P}) = \partial_\sigma^2 - \mathcal{O} + \frac{1}{4} . \quad (2.11)$$

As we already noticed²⁾, Eq. (2.4) can be rewritten as

$$\partial_\sigma(P - \tilde{P}) + (P + \tilde{P})(P - \tilde{P}) = 0 . \quad (2.12)$$

Define $\chi(\sigma)$ such that

$$P + \tilde{P} = 2 \partial_\sigma \chi ; \quad (2.13)$$

one obtains (λ is an integration constant)

$$P - \tilde{P} = 2 \lambda e^{-2\chi} , \quad (2.14)$$

$$\begin{aligned} P &= \partial_\sigma \chi + \lambda e^{-2\chi} , \\ \tilde{P} &= \partial_\sigma \chi - \lambda e^{-2\chi} . \end{aligned} \quad (2.15)$$

By means of (2.15) the shift operators can be expanded in λ . They are invariant under the exchange of P and $\tilde{P}$ if all odd powers vanish. This is easily seen to be the case if the differential operators involve a product over values which are symmetric with respect to the origin. There are two cases (ν is an arbitrary positive integer)

$$\begin{aligned} \Delta^{(2\nu-1)} &= \prod_{\ell=-\nu, \dots, \nu} (\partial_\sigma - 2\ell P) = \prod_{\ell=-\nu, \dots, \nu} (\partial_\sigma - 2\ell \tilde{P}) , \\ \Delta^{(2\nu)} &= \prod_{\ell=-\nu+1, \dots, \nu} (\partial_\sigma - (2\ell-1)P) = \prod_{\ell=-\nu+1, \dots, \nu} (\partial_\sigma - (2\ell-1)\tilde{P}) . \end{aligned} \quad (2.16)$$

For instance,

$$\begin{aligned}
 \Delta^{(2)} &= \partial_\sigma^2 - U + \frac{1}{4}, \\
 \Delta^{(3)} &= (\partial_\sigma + 2P) \partial_\sigma (\partial_\sigma - 2P) = \partial_\sigma^3 - 2(\partial_\sigma U + U \partial_\sigma) + \partial_\sigma, \\
 \Delta^{(4)} &= \partial_\sigma^4 - 3(\partial_\sigma^2 U + U \partial_\sigma^2) + \frac{3}{2} \partial_\sigma^2 + 4U(\partial_\sigma U) - (\partial_\sigma U).
 \end{aligned}
 \tag{2.17}$$

Next we recall the Poisson bracket structure which was suggested by the classical Liouville theory, namely

$$\begin{aligned}
 \{U(\sigma), U(\sigma')\} &= 2\pi \hbar \left[2\partial_\sigma U(\sigma) + 2U(\sigma)\partial_\sigma - \partial_\sigma^3 \right] \delta(\sigma - \sigma') \\
 &= -2\pi \hbar \Delta^{(3)}(\sigma) \delta(\sigma - \sigma').
 \end{aligned}
 \tag{2.18}$$

The Fourier decomposition (1.1) leads to

$$i \{L_n, L_m\} = (n-m) L_{n+m} + \frac{1}{4\pi} (n^3 - n) \delta_{n, -m}. \tag{2.19}$$

The conformal Lie algebra is thus realized by Poisson brackets with $C = 3/\hbar$. As we showed earlier^{1), 2)}, Eq. (2.18) is equivalent to

$$\{P(\sigma), P(\sigma')\} = \{\tilde{P}(\sigma), \tilde{P}(\sigma')\} = 2\pi \hbar \delta(\sigma - \sigma'), \tag{2.20}$$

and, L_n being quadratic in p or $\tilde{p}$, we have a simple harmonic oscillator representation. An easy computation shows that

$$i \{L_n, \psi_1^m \psi_2^n\} = -ie^{i\sigma} \left\{ \partial_\sigma + \frac{i}{2}(n+m) \right\} \psi_1^m \psi_2^n, \tag{2.21}$$

and the monomial $\psi_1^m \psi_2^n$ has conformal weight

$$\delta = -\frac{1}{2}(n+m). \tag{2.22}$$

For a fixed value of $m+n$, we have a family of monomials with the same conformal weight which we call isobars. According to (2.9)

$$\Delta^{(N+1)} \left(\tau_1^{N-2} \tau_2^2 \right) = 0, \quad 0 \leq r \leq N. \quad (2.23)$$

All isobars with positive m and n satisfy the same differential equation.

When it does not give zero, $\Delta^{(N)}$ shifts the conformal weight by N which is the engineering dimension of ∂_σ^N .

In Liouville theory, σ is the space variable and τ the time variable. The above discussion can be regarded as describing the canonical structure at time $\tau = 0$. It is trivial to see that

$$p_n(\tau) = p_n e^{-in\tau}, \quad L_n(\tau) = L_n e^{-in\tau}.$$

Going to Euclidean time, one simply replaces $e^{i\sigma}$ by $z = e^{\tau+i\sigma}$. Writing $z = x + iy$ displays the space properties of the associated 2-D statistical system.

Coming back to Eq. (2.7), we make the trivial remark that if $r > n + 1$, we can replace it by the equation

$$\left(\tau_1^2 \partial_\sigma \right)^n \left[\left(\frac{\tau_2}{\tau_1} \right)^n L_n \frac{\tau_2}{\tau_1} \right] = \Gamma(n+1) \left[\sum_{p=0}^n \frac{\Gamma(n-p+1)}{\Gamma(n-p+1)} (-1)^p \right] (-1)^{n-1} \left(\frac{\tau_2}{\tau_1} \right)^{n-2}. \quad (2.24)$$

Following the same procedure as above, one finds that the shift operator (2.16) applied to $\phi_1^m, \phi_2^n \ln \phi_2 / \phi_1$, with m and n both positive gives a non-zero result, proportional to $\phi_1^{-n-1} \phi_2^{-m-1}$. According to Eq. (2.22), this relates a family of isobars with negative weight to another family with positive weight. Finally, $\ln \phi_2 / \phi_1$ can be expressed in terms of U . First, one writes

$$\partial_\sigma L_n \frac{\tau_2}{\tau_1} = \frac{1}{\tau_1 \tau_2}, \quad (2.25)$$

and uses the equation

$$\tau_1(\sigma) \tau_2(\sigma) \propto G(\sigma, \sigma),$$

with

$$\left(-\partial_\sigma^2 + \mathcal{U} - \frac{1}{4}\right) G(\sigma, \sigma') = \delta(\sigma - \sigma').$$

This can be used to compute $1/\psi_1\psi_2$ in perturbation in $\mathcal{U}$. The integration constant to recover $\ln \psi_2/\psi_1$ does not contribute in Eq. (2.24).

We thus have a complete picture of the properties which we shall discuss later in quantum theories.

3. - SHIFT OPERATORS AND NULL STATES

In our previous study of the quantum Liouville theory, we already derived quantum versions of the differential equations (2.22)^{3),4)}. The quantum structure is very similar to the classical one. In this section, we rather concentrate on the existence of shift operators $\Delta^{(N)}$ in a theory where the general algebra (1.1) is realized. Consider a family of isobar operators $\mathcal{A}^{(N,\alpha)}$ satisfying

$$[\mathcal{L}_n, \mathcal{A}^{(N,\alpha)}(z)] = z^n [z\partial_z + n\delta^{(N)}] \mathcal{A}^{(N,\alpha)}(z), \quad (3.1)$$

α distinguishes the different isobars. We look for an operator $\Delta^{(N)}$ acting on $\mathcal{A}^{(N,\alpha)}$ such that, for values of $\delta^{(N)}$ to be determined, $\Delta^{(N)}(\mathcal{A}^{(N,\alpha)})$ is also covariant but with weight $\delta^{(N)} + N$:

$$[\mathcal{L}_n, \Delta^{(N)} \mathcal{A}^{(N,\alpha)}] = z^n [z\partial_z + n(\delta^{(N)} + N)] \Delta^{(N)} \mathcal{A}^{(N,\alpha)}. \quad (3.2)$$

It is not obvious a priori that such an object can be constructed, but we already know that $\Delta^{(N)}$ exists at the classical level of Liouville. Consider first the case $N = 2$. The classical expression (2.17) suggests the ansatz ($\partial \equiv z\partial_z$)

$$\Delta^{(2)} \mathcal{A} = -\partial^2 \mathcal{A} + \alpha \left\{ \left(\sum_n \mathcal{L}_n z^{-n} \right), \mathcal{A} \right\}_+ + \beta \mathcal{A}, \quad (3.3)$$

$$\left\{ \left(\sum_n \mathcal{L}_n z^{-n} \right), \mathcal{A} \right\}_+ \equiv \left(\sum_{n < 0} \mathcal{L}_n z^{-n} + \frac{1}{2} \mathcal{L}_0 \right) \mathcal{A} + \mathcal{A} \left(\sum_{n \geq 0} \mathcal{L}_n z^{-n} + \frac{1}{2} \mathcal{L}_0 \right).$$

Imposing (3.1) and (3.2) leads to

$$\alpha = \frac{2}{3} (2\delta^{(2)} + 1), \quad (3.4)$$

$$\beta = -\frac{1}{36} (2\delta^{(2)} + 1)(2\delta^{(2)} + 5), \quad (3.5)$$

$$\delta_{\pm}^{(2)} = \delta_{\pm}^{(2)} \equiv \frac{1}{16} [5 - 5 \pm \sqrt{(5-1)(5-25)}]. \quad (3.6)$$

Comparing with Kac's formula

$$\begin{aligned} \varepsilon_{m,n} = \frac{1}{48} [(13 - 5)(m^2 + n^2) - 24mn - 2(1-5) \\ + (m^2 - n^2)\sqrt{(5-1)(5-25)}] , \end{aligned} \quad (3.7)$$

one sees that

$$\delta_{+}^{(2)} = \varepsilon_{1,2} \quad , \quad \delta_{-}^{(2)} = \varepsilon_{2,1} .$$

For $N = 3$, one similarly obtains

$$\begin{aligned} \Delta^{(3)} \mathcal{A} = -i \partial^3 \mathcal{A} - \delta^{(3)} (\delta^{(3)} + 1) \partial \{ (\sum_n z^{-n} L_n), \mathcal{A} \}_+ \\ - (\delta^{(3)} + 1)(\delta^{(3)} + 2) \{ (\sum_n L_n z^n), \partial \mathcal{A} \}_+ \\ + \frac{1}{12} (\delta^{(3)} + 1)(3\delta^{(3)} + 5) \partial \mathcal{A} , \end{aligned} \quad (3.8)$$

which works if

$$\delta_{\pm}^{(3)} = \delta_{\pm}^{(3)} ; \quad \delta_{+}^{(3)} = \varepsilon_{1,3} \quad , \quad \delta_{-}^{(3)} = \varepsilon_{3,1} .$$

By explicit computation, one can go to higher values of N and obtain an infinite set of shift operators similar to the classical set of Section 2. The complexity rapidly increases and it is more useful to next display the general connection between $\delta^{(N)}$ and the zeros of Kac's determinant.

Assume there exists a vacuum state $|0\rangle$ which is such that

$$\mathcal{L}_n |0\rangle = 0, \quad n \geq -1. \quad (3.10)$$

For any covariant operator $\mathcal{A}$ with weight δ one has, according to (3.1)

$$\partial \mathcal{A}(z) |0\rangle = (z \mathcal{L}_- + \delta) \mathcal{A}(z) |0\rangle,$$

and, therefore

$$\mathcal{A}(z) |0\rangle = z^\delta e^{z \mathcal{L}_-} | \delta \rangle, \quad (3.11)$$

$$| \delta \rangle = \lim_{z \rightarrow 0} z^{-\delta} \mathcal{A}(z) |0\rangle, \quad (3.12)$$

$$\mathcal{L}_0 | \delta \rangle = \delta | \delta \rangle. \quad (3.13)$$

From (3.1) and (3.11), it follows that for $r > 0$

$$\mathcal{L}_2 | \delta \rangle = \lim_{z \rightarrow 0} (1+1) z^2 \delta | \delta \rangle = 0. \quad (3.14)$$

We now turn back to the shift operator. $\Delta^{(N)}$ is made up with powers of ∂ combined with polynomials in $\mathcal{L}_n$'s. Hence, if we define

$$| \delta^{(N)}, \alpha \rangle = \lim_{z \rightarrow 0} z^{-\delta^{(N)}} \mathcal{A}^{(N, \alpha)}(z) |0\rangle, \quad (3.15)$$

$$| \delta^{(N)} + N, \alpha \rangle = \lim_{z \rightarrow 0} z^{-\delta^{(N)} - N} \Delta^{(N, \alpha)} \mathcal{A}^{(N, \alpha)} |0\rangle, \quad (3.16)$$

we have

$$| \delta^{(N)} + N, \alpha \rangle = \mathcal{G}^{(N)}(\alpha) | \delta^{(N)}, \alpha \rangle, \quad (3.17)$$

where $p_N^{(-)}$ is a polynomial in ℓ 's with negative indices such that

$$\mathcal{L}_0 \mathcal{G}_N^{(-)} = \mathcal{G}_N^{(-)} (\mathcal{L}_0 + N). \quad (3.18)$$

Formulae (3.13) and (3.14) lead to

$$\mathcal{L}_2 |\delta^{(N)}, \alpha\rangle = \mathcal{L}_2 |\delta^{(N)} + N, \alpha\rangle = 0, \quad \alpha > 0, \quad (3.19)$$

$$(\mathcal{L}_0 - \delta^{(N)}) |\delta^{(N)}, \alpha\rangle = 0, \quad (3.20)$$

$$(\mathcal{L}_0 - \delta^{(N)} - N) |\delta^{(N)} + N, \alpha\rangle = 0. \quad (3.21)$$

As in dual models, we call physical states any state $|\varepsilon, \beta\rangle$ such that

$$\mathcal{L}_2 |\varepsilon, \beta\rangle = 0, \quad \alpha > 0; \quad (\mathcal{L}_0 - \varepsilon) |\varepsilon, \beta\rangle = 0. \quad (3.22)$$

One sees that both $|\delta^{(N)}, \alpha\rangle$ and $|\delta^{(N)} + N, \alpha\rangle$ are physical states. Next a null state is, by definition, a physical state which can be obtained by applying a polynomial of the ℓ 's to another physical state. A null state is thus orthogonal to all physical states including itself. From the above discussion, it follows that $|\delta^{(N)} + N, \alpha\rangle$ is a null state.

Consider the matrix of inner products of all states generated by powers of $\mathcal{L}_{-r}$ applied to a physical state with eigenvalue ε for $\mathcal{L}_0$. Since the inner products are solely derived by using the group structure, the determinant of this matrix is given by Kac's formula with the relevant ε and $\mathcal{C}$. We can therefore conclude that the only possible values of $\delta^{(N)}$ and N must be such that $\delta^{(N)}$ is one of Kac's zeros [Eq. (3.7)] with the null state appearing at the N th level. For a given $\delta^{(N)} = \varepsilon_{m,n}$ the shift operator of smallest degree is such that $N = mn$.

A loophole in the above argument is that the state $|\delta^{(N)}, \alpha\rangle$ could be equal to zero. This does happen as we shall see later on. We only need that $|\delta^{(N)}, \alpha\rangle$ does not vanish for all α . This is always the case in practice.

In the explicit examples considered above one has

$$|\delta^{(2)}_{+2}, \alpha\rangle = \left[\alpha_{-1}^2 - \frac{2}{3} (2\delta^{(2)}_{+1}) \alpha_{-2} \right] |\delta^{(2)}, \alpha\rangle, \quad (3.23)$$

$$|\delta^{(3)}_{+3}, \alpha\rangle = \left[\alpha_{-1}^3 - 2(\delta^{(3)}_{+1}) \alpha_{-1} \alpha_{-2} + (\delta^{(3)}_{+1})(\delta^{(3)}_{+2}) \alpha_{-3} \right] |\delta^{(3)}, \alpha\rangle.$$

As a general example, we now consider the quantum version of the classical shift operators derived in Section 2. In quantum Liouville field theory, the Fourier modes of P and $\tilde{P}$ satisfy²⁾

$$[p_n, p_m] = [\tilde{p}_n, \tilde{p}_m] = n \hbar \delta_{n, -m}. \quad (3.24)$$

Calling $\eta_{\pm}$ the two solutions of the equation

$$2\hbar\eta^2 - \eta + 1 = 0, \quad (3.25)$$

we have introduced^{3), 4)}

$$\begin{aligned} \psi_1^{\pm} &= N \left(e^{i\eta_{\pm}} \sum_n \frac{p_n}{n} z^{-n} \right), \\ \psi_2^{\pm} &= \tilde{N} \left(e^{i\eta_{\pm}} \sum_n \frac{\tilde{p}_n}{n} z^{-n} \right), \end{aligned} \quad (3.26)$$

where $N(\tilde{N})$ means that all p_n ($\tilde{p}_n$) with negative n are located on the right and that zero modes are ordered so that $\phi_{1,2}^{\pm}$ are conformally covariant. We now have general monomials⁴⁾

$$V^{m_+, m_-, m_+, m_-} = (\psi_1^+)^{m_+} (\psi_1^-)^{m_-} (\psi_2^+)^{m_+} (\psi_2^-)^{m_-}, \quad (3.27)$$

where the products can be defined by point splitting techniques. These operators have conformal weights⁴⁾

$$\delta_{(m,n)} = -\frac{1}{2} (m\eta_+ + n\eta_-) - \frac{1}{2} \hbar (m\eta_+ + n\eta_-)^2, \quad (3.28)$$

$$m = m_+ + m_-, \quad n = n_+ + n_-.$$

From (3.25) together with the relation $C = 1 + 3/\hbar$, one can verify that $\delta_{(m,n)}$ is related to Kac's formula (3.7) by

$$\delta_{(m,n)} = \Sigma_{m+1, n+1}. \quad (3.29)$$

Hence the conditions we just established for the existence of shift operators are automatically satisfied. The isobar families are now specified by m and n for generic values of $\hbar$. The shift operator of lowest degree is such that

$$N = (m+1)(n+1). \quad (3.30)$$

Shift operators obviously exist if and only if $n > 0$, $m > 0$ or $n < -2$, $m < -2$.

In the classical limit ($\hbar \rightarrow 0$) $\eta_- \rightarrow 1$ and $\eta_+ \sim 1/2\hbar$. All terms involving η_+ represent non-perturbative contributions in $\hbar$. For the subset of operators such that $m_+ = n_+ = 0$, all formulae go smoothly to the corresponding classical expressions of Section 2.

4. - THE DECOUPLING OF NULL STATES

Consider, in general, a theory which is invariant under a Lie group of transformations generated by a set of operators J_i . The symmetry will be unbroken at the quantum level if all physical states satisfy

$$J_i |phys\rangle = 0. \quad (4.1)$$

For an infinite dimensional symmetry group this condition is usually impossible to satisfy. A well-known example is QED with gauge fixing term $(\partial_\mu A_\mu)^2$. It is well known that $\partial_\mu A_\mu$ is then a free field which can be set equal to zero classically but not quantum mechanically. In this last case, one only imposes

$$\partial_\mu A_\mu^{(-)} |phys\rangle = 0, \quad (4.2)$$

where $\partial_\mu A_\mu^{(-)}$ is the annihilation part. In this Gupta-Bleuler formalism one only ensures that $\langle phys | \partial_\mu A_\mu | phys \rangle$ vanishes. This is sufficient to exclude physical longitudinal photons.

Condition (3.22) is the exact analogue of (4.2) since it ensures that $\langle phys | L_n | phys \rangle$ vanishes for all $n \neq 0$.

Strictly speaking, if one does not want to break conformal invariance, one must look only at correlation functions

$$\langle 0 | \prod_j A_j | 0 \rangle, \quad (4.3)$$

where the intermediate states are only the physical ones. We call them physical Green functions. The vacuum state $|0\rangle$ is obviously physical. However, unless all A_j have conformal weight zero, (4.3) does involve unphysical states. This problem was studied extensively in the old days of dual models since there, as in the example of QED recalled above, only physical states have a chance to possess a positive definite metric⁷⁾.

In general, there exists a set of operators $I^{(\beta)}(z)$ with weight +1. For statistical models we emphasized⁴⁾ that they are the so-called marginal operators whose existence leads to critical manifolds which are not simple points in coupling space.

For dual models, one carefully avoids unphysical states by constructing the S-matrix elements only from marginal operators. For instance, the vertex for emission of the lightest particle has the form

$$W(k, z) = e^{ik \cdot q} \int \frac{d^4 p}{p^2} e^{-\sum_{n \neq 0} \frac{z^{-n}}{n} k_\mu a_\mu^{(n)}} e^{-\sum_{n \neq 0} \frac{z^{-n}}{n} k_\mu a_\mu^{(n)}} \quad (4.4)$$

$$\begin{aligned} [q^\mu, p_\nu] &= i \delta_{\mu\nu}, \\ [a_\mu^{(n)}, a_\nu^{(m)}] &= n \delta_{\mu\nu} \delta_{n,-m}. \end{aligned} \quad (4.5)$$

The conformal generators write

$$L_n = \frac{1}{2} \sum_r a_\mu^{(r)} a_\mu^{(n-r)}. \quad (4.6)$$

W has the weight

$$\delta' = \frac{1}{2} k^2 = -\mu^2, \quad (4.7)$$

where $2\mu^2$ is the mass (we take the slope α' of Regge trajectories to be $1/2$). $W(k, z)$ is marginal for $\mu^2 = -1$ and the lightest particle is a tachyon.

Going back to the general discussion, one easily verifies that, given marginal operators $I^\beta(z)$, the physical Green functions have the form

$$\langle 0 | \prod_e \int \frac{dz_e}{z_e} I^{(\beta_e)}(z_e) | 0 \rangle. \quad (4.8)$$

It is indeed obvious from (1.1) that commuting L_n through a marginal operator gives total derivatives. One chooses the contours of integration such that these total derivatives give zero and it follows that (4.8) involves only physical intermediate states.

The key point is that a null state, even though it is physical, is orthogonal to all physical state. It therefore decouples from all physical Green functions.

Such is not the case, however, for unphysical Green functions in general, even if they involve only covariant operators. This crucial difference between physical and unphysical Green functions can further be seen as follows. A null state is obtained by a polynomial in L applied to a physical state. For physical Green functions we can freely commute this polynomial from one end to the other until it gives zero. Such is not the case for unphysical Green functions in general.

Before closing this section, we give some practical details about (4.8). In 2-D conformally invariant statistical systems, one has two variables

$$z = x + iy, \quad \bar{z} = x - iy.$$

Equation (3.25) is precisely such that

$$\delta_{(-2,0)} = \delta_{(0,-2)} = 1,$$

and there are two isobar families of marginal operators. The physical Green functions are of the form

$$\langle 0 | \prod_e \int \frac{dz_e}{z_e} \frac{d\bar{z}_e}{\bar{z}_e} I^{(\hat{p}_e)}(z_e) I^{(\hat{p}_e)}(\bar{z}_e) | 0 \rangle. \quad (4.12)$$

In string theory, the variables are: σ which distinguishes the points along the string, and τ , the Euclidean time. One lets $z = e^{\tau+i\sigma}$, $\bar{z} = e^{\tau-i\sigma}$. For closed strings, one has periodicity conditions on σ with, say, period 2π . The form is identical to (4.12).

For open strings one can take σ running from zero to π , and all emission operators sitting at $\sigma = 0$, that is, on the real z axis. The variables z_j are the Koba-Nielsen variables⁷⁾. Physical Green functions are obtained by integrating over the real axis with a fixed ordering of the z_j ⁷⁾.

5. - THE VANISHING OF NULL STATES IN THE IRREDUCIBLE CASE

The point of the previous section was to emphasize that, in general, null states only decouple from physical Green functions. On the other hand, Belavin, Polyakov and Zamolodchikov⁶⁾ assumed as a starting point that null states decouple from all Green functions. We now clarify this point by showing that, in our explicit construction suggested by the Liouville quantum solution²⁾⁻⁴⁾, all null states identically vanish. This is expected since null states would span invariant subspaces which cannot exist in an irreducible representation.

The vanishing of null states can be checked using our explicit construction of the L_n 's²⁾⁻⁴⁾ based on the quantum Liouville field theory. One has

$$L_n = \frac{1}{2\pi} \left(N \sum_{\alpha} p_{\alpha} p_{n-\alpha} - i n p_n \right) + \frac{1}{8\pi} \delta_{n,0}. \quad (5.1)$$

According to (3.24), p_0 commutes with all p_n 's. We take it to be diagonal. The ground state vectors, $|\varepsilon\rangle$ (highest weight vectors) are characterized by

$$p_n |\varepsilon\rangle = 0, \quad n > 0, \quad (5.2)$$

$$\varepsilon = \frac{p_0^2}{2\pi} + \frac{1}{8\pi}, \quad (5.3)$$

In general, the null states are of the form

$$\mathcal{P}^{(-)}(L) |\varepsilon_{m,n}\rangle, \quad (5.4)$$

where $\mathcal{P}^{(-)}$ is a polynomial in the L 's with negative indices and where $\varepsilon_{m,n}$ is given by Kac's formula (3.7). Substituting (5.1) in (5.4) and taking (5.2) and (5.3) into account, one can explicitly verify that Kac's formula is precisely such that the null states vanish identically. The simplest example of this vanishing is for the state $L_{-1}|\varepsilon\rangle$ which has zero norm for $\varepsilon = 0$. But for $p_0 = -i/2$, one has precisely both $\varepsilon = 0$ and, according to (5.1), $L_{-1}|0\rangle = 0^*$.

The fundamental difference between irreducible and reducible representations is that, in the latter case, there are many more states than the ones obtained by applying polynomials in L 's to ground states. In particular, there are many additional physical states with eigenvalue $\varepsilon_{m,n}$ for ℓ_0 . Applying well chosen polynomials in L_{-n} to them leads to null states of a completely different nature. This has been studied in detail in string theory⁷⁾.

In order to complete the argument we next rederive the vanishing of null states in the irreducible case from the existence of differential equations. For this we first show how these differential equations are themselves related to the

*) It would seem that $L_{-1}|0\rangle \neq 0$ at $p_0 = -i/2$ if one uses the $\tilde{p}$ representation of L_{-1} . However, at this value of p_0 , the transformation between p and $\tilde{p}$ is no longer canonical. This seems to generalize to the other, more complicated, zero norm states: they are found to vanish identically in the p representation if the imaginary part of p_0 is taken to be negative. In the $\tilde{p}$ representation, this vanishing gives constraint equations between the modes of this representation. The role of p and $\tilde{p}$ is interchanged when the imaginary part of p_0 changes sign.

existence of shift operators. Consider any member V of a given isobar family with weight (3.28). m and n being given, and following our early discussion, we know that $\Delta^{(N)}(V)$ with $N = (n+1)(m+1)$ is a covariant operator with weight $\delta_{(n,m)} + N$. For generic $\hbar, \epsilon$, the set of states

$$\begin{aligned} |\{n\}, \epsilon\rangle &= \prod_i (L_{-i})^{n_i} |\epsilon\rangle, \\ L_{-n} |\epsilon\rangle &= 0, n > 0, (L_0 - \epsilon) |\epsilon\rangle = 0, \end{aligned} \quad (5.5)$$

is complete. The matrix element of any covariant operator $\mathcal{A}$ is thus obtained, purely group theoretically, by applying a suitable differential operator to its matrix element between ground states. Again from covariance, these last matrix elements have the form

$$\langle \epsilon' | \mathcal{A} | \epsilon \rangle = \delta^{\epsilon' - \epsilon} \langle \epsilon' | \mathcal{A}(1) | \epsilon \rangle. \quad (5.6)$$

Since $\Delta^{(N)}$ involves differential operators up to order N one has

$$\langle \epsilon' | \Delta^{(N)}(V) | \epsilon \rangle = P_N(\epsilon' - \epsilon) \delta^{\epsilon' - \epsilon},$$

where P_N is a polynomial of N^{th} degree. The different isobars are distinguished by the possible values of $(\epsilon' - \epsilon)$. This fact which shows up in our explicit construction²⁾⁻⁴⁾, is clear from the present general argument since, by specifying $\epsilon' - \epsilon$, one completely determines all matrix elements.

The polynomial P_N has N zeros which correspond to N members of the family such that $\Delta^{(N)}(V) = 0$. One can check that they are all the isobars with $m_{\pm} > 0$, $n_{\pm} > 0$. This is the quantum extension of the classical discussion of Section 2.

Finally we show, by applying the method of Section 3 that this last result implies the vanishing of the null states. From (3.2) and (3.12), it follows that in general, the only operators which give a non-zero limit are such that they have ground state matrix elements $\langle \epsilon' | \mathcal{A} | \epsilon \rangle$ with $\epsilon' - \epsilon = \delta$. In each isobar family, this requirement selects two operators. This can be seen as follows. From (5.3) it follows that there are two vacuum states ($\epsilon = 0$), $|0, \pm\rangle$, such that $p_0 = \pm 1/2$. Using the explicit expression of V , one verifies that only the $m_{\pm} = m_{\mp} = 0$ ($n_{\pm} = n_{\mp} = 0$) isobar gives a non-zero limit for $z \rightarrow 0$ when applied to $|0, +\rangle$ ($|0, -\rangle$). In both cases the corresponding V has positive indices so that $\Delta^{(N)}(V) = 0$. From (3.16), the associated null state vanishes identically q.e.d.

As before this discussion does not apply to the reducible case since there are many more states. The equivalent of (5.5) is not complete and an operator is not uniquely determined from its ground state matrix elements.

6. - THE EARLIER USES OF THE SHIFT OPERATORS

As an illustration of shift operators applied to a reducible case, we briefly mention our old attempts⁵⁾ to shift the leading intercept of Regge trajectories in the Veneziano model without losing conformal invariance. We thus deal with the corresponding set of operators (4.4)-(4.7) and the central charge is given by (1.3) ($\mathcal{C} = \mathcal{D}$).

In the language of the present paper, Ref. 5) is equivalent to the following. We want to build a marginal operator with $\mu^2 \neq -1$. Consider the ansatz

$$\Delta^{(N)} \mathcal{W}(k, \mathcal{D}). \quad (6.1)$$

From Section 2 we know that it will be covariant if $k^2/2$ is one of Kac's zeros:

$$\frac{1}{2}k^2 = \frac{1}{48} \left[(13-\mathcal{D})(m^2+n^2) - 24mn - 2(1-\mathcal{D}) + (n^2-m^2)\sqrt{(\mathcal{D}-1)(\mathcal{D}-25)} \right]. \quad (6.2)$$

If one uses the $\Delta^{(N)}$ of lowest degree ($N = mn$), (6.1) will give a marginal operator if

$$1 = \frac{1}{48} \left[(13-\mathcal{D})(m^2+n^2) + 24mn - 2(1-\mathcal{D}) + (n^2-m^2)\sqrt{(\mathcal{D}-1)(\mathcal{D}-25)} \right], \quad (6.3)$$

and the mass of the emitted particle is

$$\mu^2 = -\frac{1}{2} k^2 = mn - 1. \quad (6.4)$$

This is related to the intercept α_0 of the leading Regge trajectory by the requirement that this lightest particle lies on it:

$$\alpha_0 = -\mu^2 = 1 - mn. \quad (6.5)$$

Physically, $\mathcal{D}$ must be integer^{*}). For $m \neq n$, (6.3) can be satisfied only if $(\mathcal{D}-1)(\mathcal{D}-25)$ is the square of an integer. This brings back the special values of the central charge^{4),6)}

$$\mathcal{D} = 1 + 6 \frac{(r+s)^2}{rs}, \quad r, s, \text{ relative primes}, \quad (6.6)$$

$$(\mathcal{D}-1)(\mathcal{D}-25) = 36 \frac{(r^2-s^2)^2}{r^2s^2}. \quad (6.7)$$

Equation (6.3) becomes

$$(r-s)^2 = (mr - ns)^2. \quad (6.8)$$

In Ref. 5) we obtained a family of models for all negative odd integer values of α_0 . Conformal invariance was shown to be satisfied for

$$\begin{aligned} \mathcal{D} &= 26, \quad \alpha_0 = -1, \\ \mathcal{D} &= 152/5, \quad \alpha_0 = -3, \\ \mathcal{D} &= 250/7, \quad \alpha_0 = -5. \end{aligned} \quad (6.9)$$

These values of $\mathcal{D}$ are of the form (6.6) with $r = 3, 5, 7$ and $s = 2$. In general, for $r = 2p + 1$, $s = 2$, p integer, Eqs. (6.5) and (6.8) lead to

$$\begin{aligned} m &= 1, \quad n = 2p, \\ \alpha_0 &= 1 - 2p. \end{aligned} \quad (6.10)$$

This agrees with (6.9) for $p = 1, 2, 3$.

^{*}) There are many solutions of (6.6) where $\mathcal{D}$ is an integer. For example $\mathcal{D} = 25, 26, 28, 33, 50, \dots$

Next, we give some details on the connection between the formulae of the present paper and our old dual amplitude in the simplest case $\alpha_0 = -1$, $\mathcal{D} = 26$. From (3.1), one sees that the vertex operator (6.1) writes

$$(\partial_z)^2 w(k, z) + \frac{2}{3} \left\{ \left(\sum_n \Lambda_n z^{-n} \right), w(k, z) \right\}_+ - \frac{2}{3} w(k, z), \quad (6.11)$$

with $k^2/2 = -1$. We can get rid of the differential operator by using

$$\partial_z w = [\Lambda_0, w]. \quad (6.12)$$

Moreover, in the dual amplitude, only physical states with $\Lambda_0 = -1$ contribute. We can replace Λ_0 by -1 on the right and on the left of (6.11). This leads to

$$-\frac{2}{3} \left[2w - w \sum_{n>0} \Lambda_n z^{-n} - \sum_{n<0} \Lambda_n z^{-n} w \right]. \quad (6.13)$$

In the dual amplitude Λ_n with $n < 0$ vanishes when applied to physical bras on the left of (6.13). This allows us, starting from the right-most vertex, to remove one by one all factors $(\sum_{n<0} \Lambda_n z^{-n})w$, thereby exactly reproducing our previous operator construction⁵⁾.

Equation (6.3) has another interesting application. It means that in the set of physical states satisfying

$$(\Lambda_0 - 1) |phys\rangle = \Lambda_n |phys\rangle = 0, \quad n > 0, \quad (6.14)$$

there is a null state at the corresponding values of m, n and $\mathcal{D} = \mathcal{D}_0(m, n)$. Equation (6.14) precisely describes the physical states of the standard Veneziano model with $\alpha_0 = 1$. Consider in this model the vertex $\Delta^{(N)}(w)$ for generic $\mathcal{D}$. The corresponding dual amplitudes describe the emission a state which becomes null at $\mathcal{D} = \mathcal{D}_0$ and hence precisely decouples at this value of $\mathcal{D}$. Dividing the corresponding amplitudes by $\mathcal{D} - \mathcal{D}_0$, one recovers our dual amplitudes with shifted α_0 . This mechanism was put forward by Goddard and Thorn¹⁰⁾.

In general, the existence of shift operators allows us to derive the desired Green functions from the ones which involve only $W(k,z)$ which are trivial. In this connection, we noticed⁵⁾ that for each model (6.9) there appears a null state at the third level. This is now easily verified for all models (6.10) since Eq. (6.3) gives

$$\xi_{3,1} = -2p-2 ,$$

and, according to (6.10),

$$\xi_{3,1} + 3 = 1-2p = \alpha_0 .$$

We made use of this fact to derive differential equations for physical four-point functions⁵⁾. This can be generalized: choosing $m = s - 1$, $n = r - 1$, as an obvious solution of (6.8), one can verify that

$$\xi_{2s-1,1} + 2s-1 = 1-(r-1)(s-1) ,$$

which means that for all r , there is a null state at level $2s - 1$ in the shifted model, which has intercept $\alpha_0 = 1 - (r-1)(s-1)$.

Finally, if $\mathcal{D}$ is not of the form (6.6), one may still build shift operators if one uses Kac's formula (6.2) for $m = n$. This may be related to our general construction of shift operators for generic $\mathcal{D}$ and α_0 ⁵⁾ which we could carry out only level by level. We have not investigated this problem.

Considered as Minkowskian string models, these constructions fall on their face because they all contain negative norm states. In Euclidean space-time to which we now turn, this problem is of course avoided. The theories having no tachyon ($\alpha_0 < 0$), all "external momenta" of vertex operators are pure imaginary; the vertex operators become hermitian, instead of unitarity [see Eq. (4.4)]. There is an infinite number of them, corresponding to all the excited string states. In statistical mechanical language, they are all marginal operators, and duality shows that they form a solution of the conformal bootstrap equations.

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