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CANONICAL PATH INTEGRAL QUANTIZATION OF GAUGE SYSTEMS

P. Ditsas

Physics Dept., Univ. of Crete

Iraklion, Greece

and

CERN - Geneva

ABSTRACT

It is shown that the (possibly explicitly time-dependent) gauge fixing conditions $z_a(q,p,t) = 0$, invoked for the canonical quantization of gauge theories via the Faddeev path integral formulation, need not satisfy the restrictions $\{z_a, z_b\} = 0$. As a preparation, we re-examine and clarify how the canonical structure on the large (partly unphysical) phase-space induces, through either Poisson brackets or Lagrange brackets or Dirac brackets, the canonical structure on the physical space of observables. The realizability of some particularly convenient coordinate systems on Γ is also proved.

Introduction

The classical canonical formalism for singular lagrangians has been developed fairly thoroughly by Dirac [1 - 3] and Bergmann and collaborators [4 - 6]. Detailed and informative recent treatments can be found in [7 - 9]. We shall concentrate on theories which are singular because of gauge symmetry (gauge theories).

The main motivation for the pioneering work [1 - 6] was to achieve canonical operator quantization of gauge theories, particularly of gravity. Later, advantages of the path-integral quantization procedure were noted, and the canonical (i.e. phase-space) path integral quantization of gauge theories was formulated by Faddeev [10], see also [11].

For simplicity one usually considers a finite dimensional mechanical system exhibiting the essential problems raised by gauge theories. We follow that tradition and so bypass some complications peculiar to field theories, notably constraints and / or gauge-fixing conditions which are not algebraic relations but differential equations (with spatial derivatives) and the possibility of improper linear combinations of constraints [8, 12]. For systems considered (a typical case of practical importance and some terminology are presented in the Appendix A) one can introduce canonically conjugate pairs q_i, p_i ($i = 1, \dots, n$) so that the action functional takes the form (summation over repeated indices in a product is always understood)

$$S = \int [p_i \dot{q}_i - H(p, q) - \lambda_a \Phi_a(q, p)] dt \quad (1)$$

where $H(q, p)$ is the hamiltonian, λ_a ($a = 1, \dots, m < n$) are Lagrange multipliers, and

$$\Phi_a(q, p) = 0 \quad (2)$$

are m nonholonomic functionally independent constraints. Following [1] and [10] we assume that $\Phi_a(q, p)$'s are "irreducible", in the sense that any function which vanishes on points satisfying (2) can be written as $C_a(q, p) \cdot \Phi_a(q, p)$. In gauge theories Φ_a 's and H satisfy

$$\{\Phi_a, \Phi_b\} = C_{abc} \Phi_c, \quad \{H, \Phi_a\} = C_{ab} \Phi_b \quad (3a, b)$$

where $\{f, g\}$ denotes the Poisson bracket (P.B.) of $f(q, p)$ and $g(q, p)$, and C_{abc}, C_{ab} are in general functions of canonical variables. If we call Γ the $2n$ dimensional phase-space parametrized by q 's and p 's, and M its $2n-m$ dimensional subspace specified by constraints (2), then (3a, b) signify that canonical transformations in Γ generated by Φ_a 's and H leave M invariant. Constraints (2) complement the classical equations of motion following from the variational principle for (1) [13]

$$\dot{f} = \{f, H\} + \lambda_a \{f, \Phi_a\} \quad (4)$$

where, under conditions (3a, b), $\lambda_a(t)$'s are completely unrestricted. By (2), an acceptable initial-data (q, p) -point must lie on M , and (3a, b) and (4) then show that the classical motion of such a point remains on M but is not uniquely specified (because of arbitrariness in λ_a 's): the classical initial-value problem is not well-defined. Indeed, one can see that the explicitly time-dependent canonical transformations generated by $\epsilon_a(t) \cdot \Phi_a(q, p)$, with ϵ_a 's arbitrary functions of time, are just the phase-space transcriptions of the gauge symmetry (some finer points in relating the lagrangian and the canonical formalism of gauge theories are reexamined in [14]).

The initial-value problem on M is well-defined for only those functions $f(q, p)$, called observables, which satisfy

$$\{f, \Phi_a\} = d_{ab} \Phi_b \quad (5)$$

for then the last term in (4) vanishes on M. Conditions (5) express gauge-invariance in canonical language: the value of an observable does not change when the argument point moves on M in m independent directions, related to gauge degrees of freedom. So, in order to specify observables on points of M, it is sufficient to know their values on a $2(n-m)$ dimensional submanifold Γ^* of M. Faddeev's approach is based on defining Γ^* by an imposition of m gauge-fixing or supplementary conditions (they supplement set (2))

$$X_a(q, p) = 0 \quad (6)$$

which are demanded (beyond functional independence) to obey

$$\det \{\Phi_a, X_b\} \neq 0, \quad \{X_a, X_b\} = 0 \quad (7a, b)$$

in a neighbourhood of M. He then introduces a canonical coordinate system $p_a, q_a, p_\rho^*, q_\rho^*$ ($\rho=1, \dots, n-m$) such that m momenta p_a coincide with X_a , q_a are their conjugate coordinates, and the remaining canonical pairs q_ρ^*, p_ρ^* are observables: it should be pointed out that the existence of such a coordinate system, with q^* 's and p^* 's having the nontrivial property of being observables, is assumed without proof in [10]. Anyway, using the abovementioned canonical coordinates, Faddeev derived the well-known (16), (17) formulae of [10] for the quantum S-matrix elements, to be called hereafter Faddeev formula (Ff) for brevity.

Most standard references, e.g. [10], [11], [15] - [17], stress the importance of the "commutation conditions" (7b) for proving Ff. Yet, some of them characterize else-

where these conditions as (merely?) convenient, without any further explanation: it seems fair to say that they implicitly raise and leave unanswered the question whether (7b) are necessary for the validity of Ff. The recently circulated List of Errata to [18] recalls the need for (7b), which were initially skipped in the book. A parenthesis on p.114 of [9] states, but with no attempt of justification, that (7b) are not necessary for the derivation of Ff. Kaptanoğlu has shown in [19] that these conditions are not needed if the structure coefficients $C_{abc}(q, p)$ of (3a) have constant values on M: unfortunately, the structure coefficients of some gauge theories (e.g. gravity and supergravity) do not have this property. Finally, Fradkin [20]-[21] sketches a derivation of Ff without invoking (7b), but his treatment is somewhat unsatisfactory in two respects. First, he assumes without a proof, that one can parametrize Γ by a coordinate system $\phi_a, X_a, q_\rho^*, p_\rho^*$, where q_ρ^*, p_ρ^* are observables and such that $p_i \dot{q}_i$ reduces to $p_\rho^* \dot{q}_\rho^*$ on Γ^* . Second, his derivation involves fairly complicated and nontransparent manipulations of functional equations, the intermediate steps (promised in the Appendix of [21]) being never spelt out in some detail.

In the following sections we show that

- (i) the canonical structure of Γ induces very naturally the canonical structure on the set of observables (and thus on Γ^*)
- (ii) the two useful abovementioned coordinate systems on Γ , the one invoked by Faddeev [10] and the one needed by Fradkin [20], are indeed realizable
- (iii) Ff can be proved fairly straightforwardly, without imposing (7b) on X_a 's, and can be extended to explicitly time-dependent X_a 's.

We conclude the paper with a few remarks.

As a first step, we recall in the next section several tools we shall need later, establish some notation and discuss very briefly the status of the phase-space path integrals.

1. Preliminaries and tools

Given a $2n$ dimensional differentiable manifold Γ and a coordinate system ξ_μ ($\mu=1, \dots, 2n$) on it (we restrict ourselves to local considerations, though global treatment is possible by using an atlas), one can introduce the canonical structure on Γ [22] by defining the Poisson bracket as a coordinate-independent mapping from ordered pairs of functions $A(\xi)$, $B(\xi)$ into the set of functions on Γ

$$A(\xi), B(\xi) \xrightarrow{PB} \{A, B\} \equiv w_{\mu\nu} \frac{\partial A}{\partial \xi_\mu} \frac{\partial B}{\partial \xi_\nu} \quad (8)$$

where $w_{\mu\nu}(\xi) \equiv \{\xi_\mu, \xi_\nu\}$ satisfy the axiomatic properties

- (i) $\det w_{\mu\nu} \neq 0$
- (ii) $w_{\mu\nu} = -w_{\nu\mu}$ (9)
- (iii) $\sum_{\text{cyclic}} \{\xi_\mu, \{\xi_\nu, \xi_\lambda\}\} = 0$

this last enforcing the Jacobi identity. A useful implication of (8) is

$$\{A, \xi_\nu\} = \{\xi_\mu, \xi_\nu\} \frac{\partial A}{\partial \xi_\mu} \quad (10)$$

P.B. simplifies particularly in a canonical coordinate system c_μ , singled out by the requirement

$$\{c_\mu, c_\nu\} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \equiv E_{\mu\nu} \quad (11)$$

where I is the $n \times n$ unit matrix.

We shall also use the Lagrange bracket (L.B.) of two functions on Γ , having the "peculiarity" that we must choose (specify) a set of $2n$ independent functions on Γ before we can define the L.B. of any two of them. If $A_1(\xi), \dots, A_{2n}(\xi)$ is a set of such functions, the L.B. among them is defined as

$$(A_k, A_\lambda) \equiv w^{\mu\nu} \frac{\partial \xi_\mu}{\partial A_k} \frac{\partial \xi_\nu}{\partial A_\lambda} \quad (12)$$

where $w^{\mu\nu}(\xi)$ is the inverse of the matrix $w_{\mu\nu}(\xi)$, and each ξ_μ is expressed temporarily in terms of A 's in order to perform the differentiations. We recall two basic properties of L.B.'s (which are easily proved): first, L.B. is the inverse of the P.B. in the sense that (e.g. [7])

$$(A_\mu, A_k) \{A_k, A_\nu\} = \delta_{\mu\nu}, \quad (13)$$

and second, the volume element [22]

$$\left(\prod_\nu d\xi_\nu \right) (\det w^{\mu\nu})^{\frac{1}{2}} = \left(\prod_\nu dA_\nu \right) (\det (A_\mu, A_\nu))^{\frac{1}{2}} \quad (14)$$

is the invariant (under reparametrizations) measure on the phase-space Γ . Notice also that, if instead of ξ_μ we use a canonical coordinate system c_μ , then

$$w^{\mu\nu} \rightarrow E^{\mu\nu} \equiv \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \quad (15)$$

The property of L.B. "dual to (9iii) is

$$\sum_{\text{cyclic}} \frac{\partial}{\partial \xi_\mu} w^{\nu\lambda} = 0 \quad (16)$$

One more remark about the canonical structure. It is well-known that the important sum $p_i \dot{q}_i$, appearing in (1), changes by a total time derivative under a general canonical transformation. It is immediate to see that in arbitrary coordinates ξ_μ it takes the form

$$B^\mu \dot{\xi}_\mu \quad (17a)$$

where $B^\mu(\xi)$ are such that

$$\frac{\partial B^\nu}{\partial \xi_\mu} - \frac{\partial B^\mu}{\partial \xi_\nu} = (\xi_\mu, \xi_\nu) \equiv w^{\mu\nu} \quad (17b)$$

If, in turn, we start from (17a) + (17b) and make a transition from ξ_μ to some canonical coordinates c_μ , then $w^{\mu\nu} \rightarrow E^{\mu\nu}$ and the general solution of (17b) is

$$B^\mu(c) = -\frac{1}{2} E^{\mu\nu} c_\nu + \frac{\partial F}{\partial c_\mu}, \quad F(c) \text{ arbitrary} \quad (18)$$

so that indeed (17a) takes the form $p_i \dot{q}_i$ up to a total time derivative. Notice that a total time derivative gives only path independent end-point contributions to the action integral (1).

We shall also use some notions and properties of function groups [23, §69]: here is a summary of some terminology and main facts.

A set of functions on Γ , having k functionally independent members and the property that the P.B. of any two functions of the set belongs again to the set or is a

constant, is called a function group of rank k . A subset of members of a function group may form a function subgroup: in particular, a function group of rank $k < 2n$ is a subgroup of the "maximal" function group of rank $2n$, the one consisting of all functions on Γ . Two functions are said to be in involution if their P.B. vanishes. Those functions of a group which are in involution with all functions of the group (and thus in particular with each other) are called singular functions of the group and form its null subgroup.

A function group of rank k is completely specified by k independent members $f_1(\xi), \dots, f_k(\xi)$ constituting a basis, any other member of the group being expressible as a function of f 's. If the rank of the antisymmetric matrix $\{f_i, f_j\}$ is $k-l=2r$, then it is possible to find a basis $\psi_1(\xi), \dots, \psi_{2r}(\xi), \phi_1(\xi), \dots, \phi_l(\xi)$ such that ϕ 's are singular functions of the group, while the P.B.'s among the ψ 's are identical to those of a set of $2r$ canonical pairs

$$\{\psi_i, \psi_j\} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the $r \times r$ unit matrix. Such a basis is called canonical. A set of k functions, constituting a canonical basis of a function group of rank $k < 2n$, can be complemented by further $2n-k$ independent functions on Γ , so that the whole set of $2n$ functions forms a canonical basis of the maximal function group.

A given function group of rank k determines another function group of rank $2n-k$, whose functions are in involution with those of the given group. Two such groups are called reciprocal to each other. If two reciprocal function groups have null subgroups, these null subgroups coincide.

We close this section with a comment on phase-space path integrals [24]. They have attracted attention for good

reasons, and have been recently studied extensively in [25] and [26] (where references to earlier work can be found). First, they are directly connected to the canonical operator quantization and allow (when - as usual - momentum integrations are gaussian and performed first) to investigate the relation of that traditional approach to the configuration-space path integral quantization. Second, they have physically transparent (formal) integration measure and are more convenient for establishing unitarity of the S-matrix. Further, it is the classical canonical formalism that elegantly eliminates (explicitly or implicitly) the unphysical degrees of freedom of gauge theories [1 - 10] and is the natural starting point for their quantization. Phase-space path integrals are occasionally blamed as too ambiguous, but this rather shows their virtue of being sensitive to the operator ordering choice and the adopted finite dimensional approximation to the path integral [25, 26], both of which reflect real problems of quantum physics with observable consequences. It is also interesting to notice the role of these integrals in finding some temporarily "missed" vertices in quantum gravity and supergravity [20, 27] (but compare [28]). In the present work we do not pay attention to the subtleties of writing path integrals, these being essentially hidden in the proper choice of the effective hamiltonian to appear in the path integrand exponent.

2. The canonical structure of observables

Because of (2), we are mainly concerned with the behaviour of phase-space functions in a neighbourhood of Γ containing M . A convenient coordinate system in such a neighbourhood is obtained by complementing $\Phi_a(q, p)$'s with $2n-m$ functions $z_s(q, p)$ ($s=1, \dots, 2n-m$), so that all $2n$

functions are functionally independent: then z_s alone provide a parametrization of M .

Given a function $f(\Phi, z)$ on Γ , it is evident from (2) and (3) that it is its restriction (if it does not vanish) on M , defined through

$$\bar{f}(z) \equiv f(0, z) \equiv f(\Phi, z) \Big|_M \quad (19)$$

that contains directly relevant physical information. Further, every transformation $\delta f(\Phi, z)$ induces a transformation of $\bar{f}(z)$ defined through

$$\delta \bar{f}(z) \equiv \delta f(\Phi, z) \Big|_{\Phi=0} \equiv \delta f(\Phi, z) \Big|_M \quad (20)$$

where on the r.h. side one first obtains $\delta f(\Phi, z)$ on Γ and then specializes to points on M . We shall consistently use upper bars to denote restrictions defined by (19) and (20). Many different functions on Γ have the same restriction on M , and the irreducibility of Φ 's means that all such functions differ from their restriction, and among themselves, by terms of the form

$$\Phi_a G_a(\Phi, z) \quad . \quad (21)$$

We need consider only those functions on Γ which have smooth $\Phi \rightarrow 0$ limit, and in (21) only those G_a nonvanishing on M , since otherwise all P.B.'s of $\Phi_a G_a$ vanish on M .

Measurable physical quantities, briefly observables, can be associated with only those f satisfying (5), because the classical time evolution of their restrictions $\bar{f}$ (defined through (4) and (20)) is uniquely determined by the restriction of their initial values. If f and g are observables, then $\{f, g\}$ is again an observable, i.e.

$$\{\{f, g\}, \Phi_a\} = \lambda_{ab} \Phi_b \quad ,$$

as follows immediately from the Jacobi identity

$$\sum_{\text{cyclic}} \{ \{f, g\}, \phi_a \} = 0$$

and the property (5) of f and g . Like any function on Γ , an observable can be written in terms of its restriction $\bar{f}$ on M as follows

$$f(\phi, z) = \bar{f}(z) + \phi_a G_a(\phi, z) \quad (22)$$

The number of physically distinguishable observables, which is also the number of physical degrees of freedom of the system, is just the number of functionally independent restricted observables $\bar{f}(z)$. How many of them are there? (This question has been treated misleadingly in [15]). The characteristic property of restricted observables is that they satisfy (are independent solutions of) a system of m homogeneous linear partial differential equations

$$K_{as} \frac{\partial \bar{f}}{\partial z_s} \equiv D_a \bar{f} = 0 \quad (23)$$

where the $m \times (2n-m)$ matrix K_{as} is

$$K_{as}(z) \equiv \{ \phi_a, z_s \} \Big|_M \quad (24)$$

Eqs (23) are obtained by rewriting (5) (recall (10)) as

$$\{ \phi_a, \phi_b \} \frac{\partial f}{\partial \phi_b} + \{ \phi_a, z_s \} \frac{\partial f}{\partial z_s} = - d_{ab} \phi_b,$$

then restricting on M (all $\phi_a=0$), and finally using

$$\frac{\partial f}{\partial z_s} \Big|_M \equiv \frac{\partial f}{\partial z_s} \Big|_M = \frac{\partial \bar{f}}{\partial z_s} + \phi_a \frac{\partial G_a}{\partial z_s} \Big|_M = \frac{\partial \bar{f}}{\partial z_s} \quad (25)$$

Let us show that the rank of K_{as} is m , so that all equations (23) are independent. For infinitesimal η , the m generators $\eta\phi_a(q, p)$ generate m independent infinitesimal displacements δz_a on M defined as follows, each δz_a having $2n-m$ components distinguished by s

$$(\delta z_a)_s \equiv \eta \{z_s, \phi_a\} \Big|_M = -\eta K_{as} \quad . \quad (26)$$

Each δz_a has at least one nonzero component. So the rows of the matrix K_{as} can be considered as m nonzero and independent $(2n-m)$ -dimensional vectors, which implies that the rank of the matrix is m .

Eqs (23) are not only independent but, as a consequence of (3a), they also form a complete system, in the sense that for any (nonsolution) $\bar{h}(z)$

$$(D_a D_b - D_b D_a) \bar{h} = \gamma_{abc} D_c \bar{h} \quad . \quad (27)$$

To show this, notice first that

$$D_a \bar{h} = \{\phi_a, h\} \Big|_M \quad (28a)$$

$$D_a D_b \bar{h} = \{\phi_a, \{\phi_b, h\}\} \Big|_M \quad . \quad (28b)$$

Next, applying (28b), the Jacobi identity and (3a) we get

$$\begin{aligned} (D_a D_b - D_b D_a) \bar{h} &= (\{\phi_a, \{\phi_b, h\}\} - \{\phi_b, \{\phi_a, h\}\}) \Big|_M = \\ &= \{\{\phi_a, \phi_b\}, h\} \Big|_M = \{C_{abc} \phi_c, h\} \Big|_M = \bar{C}_{abc} D_c \bar{h} \end{aligned}$$

so that (27) is true with

$$\gamma_{abc}(z) = \bar{C}_{abc}(z) \equiv C_{abc}(\phi, z) \Big|_M \quad .$$

We can now appeal to a classical theorem, see e.g. [23, Ch.1], that a complete system of r homogeneous linear partial differential equations of the first order in $N > r$ variables admits exactly $N-r$ independent solutions. In our case $N=2n-m$ and $r=m$, and we conclude that there are exactly $2(n-m)$ independent restricted observables $\bar{f}(z)$, i.e. independent physical degrees of freedom.

According to our last result, we can choose z_s so that $2(n-m)$ among them, denoted z_h^* ($h=1, \dots, 2n-2m$) are observables

$$K_{ah} \equiv \{\phi_a, z_h^*\} \Big|_M = 0 \quad (29a)$$

while the rest m , denoted z_b ($b=1, \dots, m$), will be called gauge variables and have a characteristic property that for each of them there exists at least one ϕ_a such that

$$K_{ab} \equiv \{\phi_a, z_b\} \Big|_M \neq 0 \quad (29b)$$

In fact, after such a choice of z_s

$$K_{as} \equiv (K_{ah} \ K_{ab}) = (0 \ K_{ab}) \quad (30)$$

and since the rank of K_{as} is m , (30) implies a characteristic property of gauge variables which is stronger than (i.e. contains) (29b)

$$\det K_{ab} \neq 0 \quad (31)$$

Eqs (23) now reduce to

$$K_{ab} \frac{\partial \bar{f}}{\partial z_b} = 0$$

and multiplying on the left by K_{ab}^{-1} we get

$$\frac{\partial \bar{f}}{\partial z_a} = 0 \quad (32)$$

In conjunction with (22) and (25), this leads to the important information, which seems to have passed unnoticed or at least unstressed in the literature, that any observable $f(\Phi, z^*, z)$ is of the form

$$f(\Phi, z^*, z) = \bar{f}(z^*) + \Phi_a G_a(\Phi, z^*, z) \quad (33)$$

and has the property

$$\frac{\partial f}{\partial z_a} = 0 \quad (34)$$

"Gauge" transformations generated by Φ_a 's change only the form of G_a in (33).

In order to obtain the canonical structure of observables, we need two more bits of preparatory information. The first, whose proof is too easy to dwell on, is that if $f(\Phi, z^*, z)$ and $g(\Phi, z^*, z)$ are observables then

$$\{f, g\}|_M = \{\bar{f}, g\}|_M = \{f, \bar{g}\}|_M = \{\bar{f}, \bar{g}\}|_M \quad (35)$$

Second, we want to show that the $(2n-2m) \times (2n-2m)$ matrix

$$W_{h\ell} \equiv \{z_h^*, z_\ell^*\} |_M \quad (36)$$

is nonsingular. To this end, let denote the (Φ_a, z_a, z_h^*) coordinate system by ξ_λ ($\lambda=1, \dots, 2n$) and express ξ 's in terms of a canonical coordinate system c_μ ($\mu=1, \dots, 2n$): then the functional independence of $\xi_\lambda(c)$'s implies

$$\det \left(\frac{\partial \xi_\lambda}{\partial c_\mu} \right) \neq 0 \quad .$$

Thus, from

$$\{\xi_k, \xi_\lambda\} = E_{\mu\nu} \frac{\partial \xi_k}{\partial c_\mu} \frac{\partial \xi_\lambda}{\partial c_\nu}$$

we get

$$\det\{\xi_k, \xi_\lambda\} = \det^2 \left(\frac{\partial \xi_\lambda}{\partial c_\mu} \right) \neq 0. \quad (37)$$

In particular this is true on M, where a straightforward calculation gives

$$\det\{\xi_k, \xi_\lambda\} \Big|_M = \det^2(K_{ab}) \det w_{h\ell} \quad (38)$$

Comparison of (37) and (38) shows that $w_{h\ell}$ is nonsingular. Notice that the elements $w_{h\ell}$ are observables restricted on M and are thus functions of z^* 's only.

We are now ready to derive the important fact that the bilinear "bracket" function $\{ , \}^*$ defined on restricted observables by

$$\{\bar{f}, \bar{g}\}^* \equiv \{\bar{f}, \bar{g}\} \Big|_M \quad (39)$$

is just the Poisson bracket on the set of restricted observables and induces the canonical structure on them. This emerges by adjusting to the present case the statements associated with (8) and (9). Restricted observables are defined on a $2(n-m)$ dimensional differentiable manifold Γ^* parametrized by a coordinate system consisting of $2(n-m)$ independent observables z_h^* , and $\{ , \}^*$ provides a coordinate-independent mapping from ordered pairs of functions $\bar{f}(z^*)$, $\bar{g}(z^*)$ into the set of functions on Γ^*

$$\bar{f}(z^*), \bar{g}(z^*) \rightarrow \{\bar{f}, \bar{g}\}^* \equiv w_{h\ell} \frac{\partial \bar{f}}{\partial z_h^*} \frac{\partial \bar{g}}{\partial z_\ell^*} \quad (40)$$

where $w_{h\ell}(z^*) \equiv \{z_h^*, z_\ell^*\} \big|_M$ has the properties required by canonical structure

$$(i) \quad \det w_{h\ell} \neq 0$$

$$(ii) \quad w_{h\ell} = -w_{\ell h}$$

$$(iii) \quad \sum_{\text{cyclic}} \{z_h^*, \{z_\ell^*, z_m^*\}^*\}^* = 0$$

Among these, (i) has already been proved, while (ii) is evident. (iii) is obtained from the Jacobi identity for z_h^* , z_ℓ^* and z_m^* on Γ , by first restricting that identity on M and then invoking (35)

$$\begin{aligned} \sum_{\text{cyclic}} \{z_h^*, \{z_\ell^*, z_m^*\}\} &= 0 = \{z_h^*, \{z_\ell^*, z_m^*\}\} \big|_M \\ &= \{z_h^*, \{z_\ell^*, z_m^*\} \big|_M\} \big|_M \end{aligned}$$

The Lagrange bracket (L.B.) of z_h^* 's on Γ^* is

$$(z_h^*, z_\ell^*)^* \equiv w^{h\ell} \tag{41}$$

where $w^{h\ell}$ is the inverse of $w_{h\ell}$, and the L.B. of two restricted observables $\bar{f}$ and $\bar{g}$ (as members of a fixed set of $2(n-m)$ functions on Γ^*) is

$$(\bar{f}, \bar{g})^* = w^{h\ell} \frac{\partial z_h^*}{\partial \bar{f}} \frac{\partial z_\ell^*}{\partial \bar{g}} \tag{42}$$

In order to settle if

$$w^{h\ell} = (z_h^*, z_\ell^*) \big|_M \tag{43}$$

is true, and also to gain some further useful information,

let again denote the set (Φ_a, z_a, z_h^*) by ξ_λ ($\lambda=1, \dots, 2n$) and examine the relations (recall (13))

$$\{\xi_k, \xi_\lambda\} (\xi_\lambda, \xi_\mu) = \delta_{k\mu}$$

Restricting these relations to M and taking account of the properties (3a), (29a) and (31) of our coordinate system, we find immediately (43) and

$$(z_a, z_h^*) \big|_M = (z_a, z_b) \big|_M = 0 \quad (44)$$

$$(z_a, \Phi_b) \big|_M = K_{ab}^{-1} \quad (45)$$

$$(z_a, z_h^*) \big|_M = K_{ab} (\Phi_b, z_\ell^*) \big|_M w_{\ell h} \quad (46)$$

plus a more complicated relation which shows that (Φ_a, Φ_b) vanish on M only if both $\{z_a, z_b\}$ and (Φ_a, z_ℓ^*) vanish on M . In this context it is worth noticing that we can, with no loss of generality, impose on our observable coordinates z_h^* one more requirement, namely

$$(z_h^*, z_a) \big|_M = 0 \quad (47)$$

Indeed, if these are not satisfied by z_h^* , then they are by the new observable coordinates

$$z_h^{*'} = z_h^* - \{z_h^*, z_a\} K_{ab}^{-1} \Phi_b.$$

With such a choice one also gets

$$(\Phi_a, z_h^*) \big|_M = 0 \quad (48)$$

according to (46). Another useful observation is that as long as z_a 's continue to satisfy (31) for all times, they

can be chosen explicitly time-dependent: then K_{ab} is also explicitly time-dependent. All relations up to including (46) remain valid with a time-independent choice of z_h^* 's, and only if we demand in addition (47) we are forced (in general) to select time-dependent z_h^* 's.

Finally, let us show that if $\bar{f}_h(z^*)$ ($h=1, \dots, 2n-2m$) are independent functions on Γ^* then

$$(\bar{f}_h, \bar{f}_\ell)^* = (\bar{f}_h, \bar{f}_\ell) \Big|_M \quad (49)$$

which is "dual" to (39) and can be used to introduce canonical structure on Γ^* [22]. According to (13)

$$\{\bar{f}_h, \bar{f}_i\}(\bar{f}_i, \bar{f}_\ell) + \{\bar{f}_h, \phi_a\}(\phi_a, \bar{f}_\ell) + \{\bar{f}_h, z_a\}(z_a, \bar{f}_\ell) = \delta_{h\ell}$$

where

$$(z_a, \bar{f}_\ell) = (z_a, z_h^*) \frac{\partial z_h^*}{\partial \bar{f}_\ell}, \quad \{\bar{f}_h, \phi_a\} = \frac{\partial \bar{f}_h}{\partial z_\ell^*} \{z_\ell^*, \phi_a\}.$$

Restricting to M and recalling (29a) and (44) we find

$$\{\bar{f}_h, \bar{f}_i\}^* (\bar{f}_i, \bar{f}_\ell) \Big|_M = \delta_{h\ell}$$

which proves the assertion. We can combine (49) with the "dual" to (35)

$$(f_k, f_\ell) \Big|_M = (\bar{f}_k, f_\ell) \Big|_M = (f_k, \bar{f}_\ell) \Big|_M = (\bar{f}_k, \bar{f}_\ell) \Big|_M \quad (50)$$

which nevertheless are true only if the $2(n-m)$ independent nonrestricted observables f_h are chosen to satisfy

$$\{f_h, z_a\} \Big|_M = 0 \quad (51)$$

which also implies

$$(f_h, \phi_a) \Big|_M = 0, \quad (52)$$

as seen from

$$\{f_h, z_b\}(z_b, \phi_a) + \{f_h, \phi_b\}(\phi_b, \phi_a) + \{f_h, f_\ell\}(f_\ell, \phi_a) = 0$$

Imposing (51) is always possible, since if some f_h does not satisfy them, then the observable with the same restriction on M

$$f'_h = f_h - \{f_h, z_a\} K_{ab}^{-1} \phi_b \quad (53)$$

does satisfy them.

To recapitulate, all observables having the same restriction on M form a physically indistinguishable class of functions on Γ . The canonical structure on the physical phase-space Γ^* is introduced by the physical P.B. among observables, which is simply their P.B. on Γ subsequently restricted to M . Alternatively we can obtain the canonical structure on Γ^* by introducing the physical L.B. among observables: this is just their L.B. on Γ subsequently restricted to M , under the condition that the $2(n-m)$ independent observables used to form the L.B. on Γ are chosen to satisfy (51) (otherwise one must work with restricted observables as in (49)).

Since $\{ , \}^*$ is the P.B. on Γ^* , functions on Γ^* form a function group of rank $2(n-m)$, with no singular functions and having z_h^* 's as a basis. As recalled in the previous section, for such a function group one can always introduce a canonical basis q_ρ^*, p_ρ^* ($\rho=1, \dots, n-m$)

$$\{q_\rho^*, q_\sigma^*\}^* = \{p_\rho^*, p_\sigma^*\}^* = 0, \quad \{q_\rho^*, p_\sigma^*\}^* = \delta_{\rho\sigma} \quad (54)$$

In general, q_ρ^* and p_ρ^* are not a part of a canonical coordinate system on Γ : in that case (54) would evidently be true with $\{ , \}$ in place of $\{ , \}^*$.

To close this section, let us establish connection with the Dirac brackets (D.B.) [1 - 3]. If ϕ_a 's and z_a 's are denoted collectively Ψ_A ($A=1, \dots, 2m$), then the D.B. of two functions f and g on Γ is defined as

$$\{f, g\}_D \equiv \{f, g\} - \{f, \Psi_A\} C_{AB}^{-1} \{\Psi_B, g\} \quad (55)$$

where

$$C_{AB} \equiv \{\Psi_A, \Psi_B\}$$

is nonsingular [1]: on M , $\det C_{AB} = \det^2 K_{ab}$. It is easy to check that D.B. vanishes by construction if either f and/or g belong to the set of Ψ_A 's

$$\{f, \Psi_A\}_D = \{\Psi_A, \Psi_B\}_D = 0 \quad (56)$$

while for general f and g (e.g. observables) it reduces, after a straightforward calculation, to

$$\{f, g\}_D = \{z_h^*, z_\ell^*\}_D \frac{\partial f}{\partial z_h^*} \frac{\partial g}{\partial z_\ell^*} \quad (57)$$

Let us choose our z_h^* coordinates so that (47) is satisfied: as already noticed, this is no real restriction. Then on M (57) becomes

$$\{f, g\}_D \Big|_M = w_{h\ell} \frac{\partial f}{\partial z_h^*} \frac{\partial g}{\partial z_\ell^*} \quad (58)$$

Thus, acting on observables, D.B. reduces on M to the physical P.B. $\{ , \}^*$.

A characteristic property of $\{z_h^*, z_\ell^*\}_D$ is

$$\{z_h^*, z_\ell^*\}_D (z_\ell^*, z_m^*) = \delta_{hm} \quad (59)$$

i.e. it is the inverse of (z_ℓ^*, z_m^*) . Indeed, by (56) the l.h. side of (59) can be written

$$\begin{aligned} & \{z_h^*, z_\ell^*\}_D (z_\ell^*, z_m^*) + \{z_h^*, \Psi_A\}_D (\Psi_A, z_m^*) = \\ & = \{z_h^*, \Xi_\lambda\}(\Xi_\lambda, z_m^*) - \{z_h^*, \Psi_A\} C_{AB}^{-1} \{\Psi_B, \Xi_\lambda\}(\Xi_\lambda, z_m^*) = \delta_{hm} \end{aligned}$$

where Ξ_λ ($\lambda=1, \dots, 2n$) denote all the coordinates z_h^* and Ψ_A , and (13) was invoked. Starting from (59) and interpreting (z_h^*, z_ℓ^*) as the physical L.B. of observable coordinates (which according to (41) + (43) is not true), one is often misled to consider the D.B. (57) as the physical P.B. even before restricting to M.

What is the content of the D.B. (57) when at least one among f and g is a nonobservable? To this end we must first realise that with any function $f(z^*, \Psi)$ on Γ , having a smooth $\Psi_A \rightarrow 0$ limit (such functions suffice for physics), we can associate a restricted observable $f^*(z^*)$ by

$$f(z^*, \Psi) \equiv f^*(z^*) + \Psi_A G_A(z^*, \Psi) \quad (60)$$

$$\text{or} \quad f^* \equiv f|_{\Gamma^*} \quad (61)$$

where G_A are such that $\Psi_A G_A$ vanish on Γ^* . For an observable, in particular, $f|_{\Gamma^*} = f|_M \equiv \bar{f}$. It is now easy to show that

$$\{f, g\}_D|_{\Gamma^*} = w_{h\ell} \frac{\partial f^*}{\partial z_h^*} \frac{\partial g^*}{\partial z_\ell^*} = \{f^*, g^*\}|_M \quad (62)$$

Previously, we have introduced the physical phase-space by considering only observables and their ordinary P.B.'s on Γ , and subsequently restricting to M: observables on Γ having the same restriction on M were physically identical.

According to (60) + (62) we can obtain the same goal by a different procedure, namely consider all functions and their D.B.'s on Γ , and subsequently restrict to Γ^* : functions on Γ having the same restriction on Γ^* are physically indistinguishable. An advantage of the second procedure is that we need not separate observables from nonobservables on Γ , a disadvantage is that the D.B. is more complicated than the P.B. (in particular one must first obtain the matrix C_{AB}^{-1}). The two procedures are completely equivalent.

From the preceding results, it is clear that infinitesimal canonical transformations on Γ^* can be expressed either as

$$\delta_{\varepsilon g} \bar{f} = \varepsilon \{f, g\} \Big|_M \quad (63)$$

where f and g are observables on Γ , or as

$$\delta_{\varepsilon g^*} f^* = \varepsilon \{f, g\} \Big|_{D \Big| \Gamma^*} \quad (64)$$

where f and g are any functions on Γ . The variety of transformations on Γ^* obtained either way is identical.

3. Realizability of useful coordinates

As mentioned in the Introduction, both Fradkin [20] and Faddeev [10] used convenient coordinate systems on Γ without discussing their realizability: it will be proved in this section (which is thus somewhat "pedantic" in comparison to the other sections).

In [20] a coordinate system q_ρ^*, p_ρ^* ($\rho=1, \dots, n-m$), ϕ_a, z_a ($a=1, \dots, m$) is invoked, where ϕ_a and z_a are just those of the previous section, while q_ρ^*, p_ρ^* are observables and canonical in the sense that ($i=1, \dots, n$)

$$p_i \dot{q}_i \Big|_{\Gamma^*} = p_\rho^* \dot{q}_\rho^* \quad (65)$$

up to a total time derivative depending only on q^* 's and p^* 's. We shall show that (65) is a basic feature of just the coordinate system whose existence was proved in the previous section (the part preceding eqs (54)). For convenience, we shall denote these coordinates either by ξ_λ ($\lambda=1, \dots, 2n$) or c_h^*, ψ_A or c_h^*, ϕ_a, z_a ($h=1, \dots, 2n-2m$), and we summarize their basic properties found in section 2

$$w_{h\ell} \equiv \{c_h^*, c_\ell^*\} \Big|_M = E_{h\ell} \quad , \quad w^{h\ell} \equiv (c_h^*, c_\ell^*) \Big|_M = E^{h\ell} \quad (66a)$$

$$\{c_h^*, \phi_a\} \Big|_M = (z_a, c_h^*) \Big|_M = (z_a, z_b) \Big|_M = 0 \quad (66b)$$

$$\det K_{ab} \neq 0, \quad K_{ab} \equiv \{\phi_a, z_b\} \Big|_M, \quad (z_a, \phi_b) \Big|_M = K_{ab}^{-1} \quad (66c)$$

where $E_{h\ell}$ and its inverse $E^{h\ell}$ are defined in analogy with (11) and (15). Returning to (65), we have according to (17a) and (17b)

$$p_i \dot{q}_i = B^\mu \dot{\xi}_\mu = B^A \dot{\psi}^A + B^h \dot{c}_h^*$$

$$\frac{\partial B^\nu}{\partial \xi_\mu} - \frac{\partial B^\mu}{\partial \xi_\nu} = (\xi_\mu, \xi_\nu) \quad .$$

Restricting the last two relations to Γ^* we get

$$p_i \dot{q}_i \Big|_{\Gamma^*} = b^h \dot{c}_h^* \quad , \quad \frac{\partial b^\ell}{\partial c_h^*} - \frac{\partial b^h}{\partial c_\ell^*} = (c_h^*, c_\ell^*) \Big|_M = E^{h\ell} \quad (67a, b)$$

where

$$B^\mu(\xi) = b^\mu(c^*) + \Psi_A G_A^\mu(c^*, \Psi) \quad .$$

In analogy with (18), the solution of (67b) is

$$b^h(c^*) = -\frac{1}{2} E^{h\ell} c_\ell^* + \frac{\partial F(c^*)}{\partial c_h^*}$$

and substituting this in the r.h. side of (67a) we find (65) up to a total time derivative depending on c^* 's. Relation (65) will be used in the next section.

In [10], gauge variables z_a (denoted there X_a) are first chosen so that

$$\{z_a, z_b\} = 0 \quad (68)$$

and then a canonical coordinate system q_a, p_a ($a=1, \dots, m$), q_ρ^*, p_ρ^* ($\rho=m+1, \dots, n$) on Γ is invoked, where $p_a = z_a$ and q_ρ^*, p_ρ^* are observables. To prove its realizability, let us start with a general coordinate system z_a, ϕ_a, z_h^* , where z_h^* ($h=1, \dots, 2n-2m$) are observables, and notice that under conditions (68) one can replace z_h^* 's by new observable coordinates g_h^* which satisfy

$$\{g_h^*, z_a\} \equiv T_a g_h^* = 0 \quad . \quad (69)$$

These are m independent homogeneous linear partial diffe-

rential equations of the first order in $2n-m$ variables, if we look for solutions of the form $g_h^*(\Phi, z^*)$ which ensures them to be observables. The system (69) is complete because of (68) and the Jacobi identity

$$\{g_h^*, z_a\}, z_b\} - \{g_h^*, z_b\}, z_a\} = \{g_h^*, \{z_a, z_b\}\} = 0,$$

and thus has $(2n-m)-m = 2(n-m)$ independent observables as solutions.

Now, according to (68), z_a 's form a basis of a completely commutative function group Y of rank m . Its reciprocal function group Y' is of rank $2n-m$ and has z_a, g_h^* as a basis. The $\{g_h^*, g_\ell^*\}$ are observables, i.e. must be of the form

$$f(g^*) + \Phi_a G_a(\Phi, z, g^*)$$

while simultaneously - as members of Y' - they must be independent of Φ 's, so they are functions only of g^* 's. Thus g^* 's form a function subgroup of rank $2(n-m)$ (with respect to $\{ , \}$ on Γ), and $q_\rho^*(g^*), p_\rho^*(g^*)$ can be chosen as a canonical basis of that subgroup. Complementing with z_a 's, we obtain a canonical basis of Y' (notice that the $(2n-m) \times (2n-m)$ matrix of P.B's among the basis functions z_a, g_h^* of Y' has rank $2(n-m)$). Finally, as recalled in section 1, this can always be extended to a canonical basis of the maximal function group of all functions on Γ , realizing the coordinates of [10].

We now turn to the quantization problem.

4. The Faddeev formula

Let us parametrize Γ by a coordinate system q_ρ^*, p_ρ^* ($\rho=1, \dots, n-m$), Φ_a, z_a ($a=1, \dots, m$), the same one that we

used in the two previous sections and whose basic properties are summarized by (66) and (65): as previously, for reasons of convenience these coordinates will also be denoted by ξ_λ or q_ρ^* , p_ρ^* , ψ_A . Our z_a correspond to the x_a of [10] (when expressed in terms of q 's and p 's of (1) they are just the l.h. sides of the gauge-fixing conditions) and we shall allow them to be explicitly time-dependent, as needed e.g. in "generally covariant" models where space-time coordinates change in gauge transformations. Then K_{ab} become explicitly time-dependent, but (as already noticed in section 2) all (66) remain valid with a time-independent choice of q^* 's and p^* 's, to which we stick.

Quantization consists in summing, with a proper quantum amplitude, over all those trajectories in the physical phase-space Γ^* that satisfy the end-point or asymptotic conditions corresponding to the initial and final quantum states. E.g. the $|q^{*'}\rangle \rightarrow |q^{*''}\rangle$ matrix element of the evolution operator is (we set $\hbar=1$ and ignore $1/2\pi$ factors to simplify the appearance of formulae)

$$\langle q^{*''} | e^{-i\bar{H}(t'' - t')} | q^{*'} \rangle = \int \Delta \mu \exp \left[i \int_{t'}^{t''} [p_\rho^* \dot{q}_\rho^* - \bar{H}(q^*, p^*)] dt \right] \quad (70a)$$

$$\Delta \mu \equiv \prod_t d\mu(t) \equiv \prod_t \prod_\rho dq_\rho^*(t) dp_\rho^*(t) \quad (70b)$$

where $\bar{H}(q^*, p^*)$ is the restriction on M of the observable $H(q, p)$ appearing in (1), and we integrate over all trajectories $q^*(t)$, $p^*(t)$, $t' \leq t \leq t''$, joining the surfaces $q^* = q^{*'}$ and $q^* = q^{*''}$ in Γ^* . These $(n-m)$ -dimensional surfaces, to be called Σ' and Σ'' , are parametrized by varying p_ρ^* 's.

We must comment here on the use of $\bar{H}(q^*, p^*)$ in (70). As long as we employ any time-independent coordinates on Γ , the role of the hamiltonian in the action functional on Γ is played by $H(\xi) \equiv H(q(\xi), p(\xi))$. In the present case

the z_a subset of coordinates is explicitly time-dependent and the role of the hamiltonian on Γ is played by a different function $H'(\xi, t) \equiv H(\xi) + R(\xi, t)$, see e.g. [13]. But we expect two descriptions on Γ , differing only in their gauge variables, to reduce to the same description on the physical phase space Γ^* , i.e. we expect $R|_M = 0$. This can be seen rigorously. As shown in Appendix B, $R(\xi, t)$ is determined (up to an irrelevant arbitrary function of time) by

$$(\xi_\mu, \xi_\nu) \frac{\partial \xi_\nu}{\partial t} = \frac{\partial R}{\partial \xi_\mu} \quad (71)$$

which in our case becomes

$$(\xi_\mu, z_a) \frac{\partial z_a}{\partial t} = \frac{\partial R}{\partial \xi_\mu} \quad (72)$$

Restricting on M we find (recall (66b))

$$\left. \frac{\partial R}{\partial q_\rho^*} \right|_M = \left. \frac{\partial R}{\partial p_\rho^*} \right|_M = \left. \frac{\partial R}{\partial z_a} \right|_M = 0$$

which implies that $R(\xi, t)$ must be of the form $\phi_a G_a(\xi, t)$ and vanishes on M . We are thus justified in using $\bar{H}(q^*, p^*)$.

We want now to express (70) as a path integral over Γ . First, the "instantaneous" functional integration measure $d\mu(t)$ can be rewritten

$$d\mu(t) = \prod_\mu d\xi_\mu \prod_A \delta(\Psi_A) \quad (73)$$

According to expression (14) for the invariant volume element on Γ , we have

$$\prod_\mu d\xi_\mu(t) (\det(\xi, \xi))^{\frac{1}{2}} = \prod_i dq_i(t) dp_i(t) \quad (74)$$

where

$$(\xi_\mu, \xi_\nu) = \frac{\partial q_i}{\partial \xi_\mu} \frac{\partial p_i}{\partial \xi_\nu} - \frac{\partial q_i}{\partial \xi_\nu} \frac{\partial p_i}{\partial \xi_\mu}$$

are the L.B.'s of ξ 's. Multiplying both sides of (74) by $(\det\{\xi, \xi\})^{\frac{1}{2}}$, using (13) and substituting in (73) we get

$$d\mu(t) = \prod_i dq_i(t) dp_i(t) \prod_A \delta(\Psi_A) (\det\{\xi, \xi\})^{\frac{1}{2}}$$

A straightforward calculation of $\det\{\xi, \xi\}$ on Γ^* , by invoking (66) (true in M which contains Γ^*), gives our final expression for the integration measure

$$\Delta\mu = \prod_t \left[\prod_i dq_i(t) dp_i(t) \prod_a \delta(\phi_a) \delta(z_a) \det\{\phi, z\} \right] \quad (75)$$

over all trajectories $q(t), p(t), t' \leq t \leq t''$ in Γ , joining the surfaces Σ' and Σ'' and subject to the constraints

$$\phi_a(q, p) = z_a(q, p) = 0 \quad . \quad (76)$$

Σ' and Σ'' are parametrized by those q 's and p 's which, in addition to the above constraints, satisfy

$$q_p^*(q, p) = q_p^{*'} \quad \text{for } \Sigma'$$

$$q_p^*(q, p) = q_p^{*''} \quad \text{for } \Sigma'' \quad .$$

We have now only to recall (65) and $\bar{H} \equiv H|_M$ and to gather all pieces together to get finally

$$\langle q^{*''} | e^{-i\bar{H}(t'' - t')} | q^{*'} \rangle = \int \Delta\mu \exp \left[i \int_{t'}^{t''} [p_i \dot{q}_i - H(q, p)] dt \right] \quad (77)$$

where $\Delta\mu$ is given by (75). This is just the Faddeev formula.

Concluding remarks

In the operator method of quantization, based on using Dirac brackets as a guide for quantum commutation relations of physical quantities [1, 8, 9], there are no restrictions (7b). It is thus both conceptually satisfactory and practically important that they are also not needed in the Faddeev method, which is essentially a realization of the Dirac's programme in the path integral approach.

A basic advantage of the Faddeev method is that, after integration over momenta (these are gaussian integrations in cases of interest), Ff turns into a useful tool for deriving Feynman rules in nonconvariant configuration- space gauges. As a next step, whenever a "method of integration over the gauge group" [29, 10, 11, 16, 17, 20] works, one can transform to configuration-space path integral formulae in covariant gauges.

Preparing the ground for Ff, we had to identify the canonical coordinates of the physical phase-space Γ^* in terms of functions on Γ , and were thus led to reexamine and clarify how the canonical structure on Γ induces the canonical structure on Γ^* . This was seen to be describable in three equivalent ways, using either P.B.'s or L.B.'s or D.B.'s, and in the whole procedure the process of restricting functions and brackets (i.e. transformations) on Γ to either M or Γ^* played an important role. One should remark here that using L.B.'s may not be too impractical: from the equations of motion on Γ (see Appendix B)

$$(\xi_\mu, \xi_\nu) \frac{d\xi_\nu}{dt} = \frac{\partial}{\partial \xi_\mu} (H + \lambda_a \phi_a)$$

one gets immediately (recall (66b))

$$(z_h^*, z_\ell^*) \left|_M \frac{dz_\ell^*}{dt} = \frac{\partial H}{\partial z_h^*} \right|_M \quad (78)$$

so that if one manages to express $\left. \frac{\partial H}{\partial z_n^*} \right|_M$ as a linear form of $\dot{z}^*$'s, the coefficients provide the physical L.B.'s.

We also proved the realizability of two coordinate systems on Γ having particularly useful properties.

An observation worth mentioning here, because it is possibly of practical importance, is that apart from the basic conditions

$$\{f, \phi_a\} \Big|_M = 0$$

on observables f , these can always be chosen, with no loss of generality, to satisfy also

$$\{f, z_a\} \Big|_M = 0 \quad .$$

This is ensured simply by replacing each observable f with the observable (recall (53))

$$f - \{f, z_a\} K_{ab}^{-1} \phi_b \quad . \quad (79)$$

Finally, we should comment on the fact that when applying the Faddeev path integral approach or the Dirac bracket approach one can start from two different choices of the large phase-space Γ , according to whether one treats the λ_a 's in (1) as (i) Lagrange multipliers or as (ii) phase-space variables whose canonically conjugate momenta Π_a should be introduced. Ff is usually applied with the first choice (as in present work), in which case $\phi_a = 0$ are the so-called lagrangian or secondary hamiltonian constraints [1-6, or Appendix A]. This is at variance with the usual practice with the D.B. method [8, 9], but in fact conforms to the spirit of Dirac's own work [2, 3, 30] and the results of Appendix A. Given the equivalence of the two approaches, the message of these remarks is that one

introduces an unnecessary complication in the D.B. method by treating λ 's as canonical variables.

What complications would such a treatment of λ_a 's cause in the path integral method? It is completely straightforward to show along the lines of the present work that, if one was to employ a $2(n+m)$ dimensional phase-space Γ' parametrized by $q_i, \lambda_a, p_i, \Pi_a$, the r.h. side of the Ff (77) would become

$$\int_{\Delta\tau} \exp \left[\int_t^{t''} [p_i \dot{q}_i + \Pi_a \dot{\lambda}_a - H(q, p)] dt \right], \quad (80)$$

$$\Delta\tau \equiv \prod_t \left[\prod_i dq_i(t) dp_i(t) \prod_a d\lambda_a(t) d\Pi_a(t) \prod_A \delta(\Phi_A) \delta(X_A) \det\{\Phi, X\} \right]$$

where $\Phi_A = 0$ ($A=1, \dots, 2m$) denote collectively the constraints $\Phi_a = \Pi_a = 0$, and $2m$ gauge-fixing conditions have been introduced

$$X_A(q, \lambda, p, \Pi) = 0$$

with $\det \{\Phi, X\} \neq 0$ in a neighbourhood of the $2n$ dimensional subspace M' specified by $\Phi_A = 0$. Such form of path integral is used e.g. in [9]. It is clear that, after integrating (80) over Π 's and λ 's, we must arrive at a path integral which is identical (up to a constant factor) to (75) + (77) with properly chosen gauge-fixing conditions $z_a(q, p) = 0$.

As a closing remark, we should mention that very recently Barbashov et al. [31] have also extended the proof of Ff to the case of explicitly time-dependent gauge-fixing conditions, but their proof still requires (and in fact relies heavily on) the conditions $\{z_a, z_b\} = 0$.

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Appendix A

We examine here a mechanical system which is sufficiently general to serve as a finite-dimensional analogue of gauge field theories of current interest and how its action is naturally brought to form (1) of the main text. The presentation is motivated from [2]. A somewhat parallel approach, using the so-called first order formalism, is outlined in [17, Ch. IV].

Consider a system described in terms of coordinates q_i ($i=1, \dots, n$), Q_λ ($\lambda=1, \dots, s$) and x_ρ ($\rho=1, \dots, r$) whose lagrangian is of the form

$$L(q, Q, x, \dot{q}, \dot{Q}) = \frac{1}{2} \dot{q}_i A_{ij} \dot{q}_j + B_i \dot{q}_i + C_\lambda \dot{Q}_\lambda + D \quad (1)$$

where A_{ij} , B_i , C_λ , D are functions of $z \equiv (q, Q, x)$ and the symmetric matrix A_{ij} is assumed invertible. Since

$$\det \left(\frac{\partial^2 L}{\partial \dot{z}_A \partial \dot{z}_B} \right) = 0 \quad (A, B = 1, \dots, n+s+r) \quad (2)$$

the lagrangian is singular and leads to constraints. The primary constraints are those arising because the r.h. sides of the equations defining momenta

$$\Pi_A = \frac{\partial L}{\partial \dot{z}_A} \quad (3)$$

are not functionally independent in $\dot{z}_A$'s (recall (2)): the secondary constraints, also called lagrangian constraints, are obtained when, in addition, the equations of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{z}_A} \right) - \frac{\partial L}{\partial z_A} = 0 \quad (4)$$

are taken into account.

Our primary constraints are

$$\Pi_\lambda - C_\lambda = 0 \quad , \quad \Pi_\rho = 0,$$

and let us assume, as often met in practice, that the P.B.'s among primary constraints vanish

$$\{\Pi_\rho, \Pi_\lambda - C_\lambda\} = 0 \quad , \quad \{\Pi_\lambda - C_\lambda, \Pi_\mu - C_\mu\} = 0,$$

which imply

$$\frac{\partial C_\lambda}{\partial x_\rho} = 0 \quad , \quad \frac{\partial C_\lambda}{\partial Q_\mu} - \frac{\partial C_\mu}{\partial Q_\lambda} = 0$$

respectively. Then C_λ 's must be of the form

$$C_\lambda = \frac{\partial F(q, Q)}{\partial Q_\lambda}$$

and

$$C_\lambda \dot{Q}_\lambda = \frac{dF}{dt} - \frac{\partial F}{\partial q_i} \dot{q}_i \quad .$$

So, instead of (1), we can adopt $L - \frac{dF}{dt}$ as a physically equivalent lagrangian (denoted for simplicity by the same symbol) having no $\dot{Q}_\lambda$ velocities

$$L(q, Q, \dot{q}) = \frac{1}{2} \dot{q}_i A_{ij} \dot{q}_j + A_i \dot{q}_i + D \quad , \quad A_i \equiv B_i - \frac{\partial F}{\partial q_i} \quad (5)$$

Since Q_λ and x_ρ appear in (5) on equal footing, we write them collectively Q_a ($a = 1, \dots, s+r \equiv m$).

We thus restart with the new L (5), whose primary constraints are

$$\Pi_a = 0 \quad (6)$$

while the remaining relations (3) give now

$$p_i = A_{ij} \dot{q}_j + A_i, \quad \dot{q}_i = A_{ij}^{-1} (p_j - A_j) \quad (7a,b)$$

Let us introduce a preliminary hamiltonian H' by performing a Legendre transform (L.t.) of L on the submanifold restricted by (6)

$$H'(q, p, Q) = p_i \dot{q}_i - L = \frac{1}{2} (p_i - A_i) A_{ij}^{-1} (p_j - A_j) - D \quad (8)$$

and recall the basic property of the L.t.

$$\frac{\partial H'}{\partial z_A} = - \frac{\partial L}{\partial z_A} \bigg|_{\dot{q}(q,p,Q)}$$

where velocities on the r.h. side are substituted from (7b). The secondary constraints, arising either from (4) or from demanding $\{\Pi_a, H'\}$ to vanish, are

$$\frac{\partial H'}{\partial Q_a} = - \frac{\partial L}{\partial Q_a} \bigg|_{\dot{q}(q,p,Q)} \equiv X_a(q, p, Q) = 0 \quad (9)$$

In gauge theories H' , Π_a 's and X_a 's are first class quantities in Dirac terminology, i.e. their P.B. with any constraint either vanishes or is a linear combination of constraints (linear combination of only secondary constraints in our model, since H' and X_a 's are Π_a -independent). This implies in particular

$$\{X_a, \Pi_b\} \equiv \frac{\partial X_a}{\partial Q_b} = \gamma_{abc} X_c$$

If X_a 's involve Q 's at all (i.e. if $\gamma_{abc} \neq 0$), they can satisfy the above equations only by being of the form

$$X_a(q, p, Q) = \theta_{ab}(q, p, Q) \phi_b(q, p) \quad (10)$$

where θ_{ab} is a nonsingular matrix. But then (9) imply the equivalent set of constraints

$$\Phi_a(q, p) \equiv \Theta_{ab}^{-1}(q, p, Q) X_b(q, p, Q) = 0 \quad . \quad (11)$$

The first-class property of X_a 's demands further

$$\{X_a, X_b\} = \delta_{abc} X_c$$

and substituting here X_a 's as in (10) shows immediately that $\Phi_a(q, p)$ must satisfy involution relations of the form (3a) of the main text. Finally, examining

$$\{H', X_a\} = \gamma_{ab} X_b$$

or simply integrating (recall (9))

$$\frac{\partial H'}{\partial Q_a} = \Theta_{ab} \Phi_b$$

we find that H' must be of the form

$$H'(q, p, Q) = H(q, p) + \lambda_a(q, p, Q) \Phi_a(q, p) \quad . \quad (12)$$

Since Q 's are now completely unrestricted and appear only in λ 's, it is natural to adopt instead λ_a as new variables and to interpret them as Lagrange multipliers.

A combination of (8) and (12) leads to an action integral having same form with (1) of the main text.

Appendix B

We want to prove (71) of the main text. It is easy to see, e.g. starting from

$$S = \int [p_i \dot{q}_i - H(q, p)] dt, \quad (i = 1, \dots, n)$$

that the action functional for the variational principle

takes in general coordinates u_μ ($\mu=1, \dots, 2n$) the form

$$S = \int [B^\mu(u) \dot{u}_\mu - H(u)] dt$$

where

$$\frac{\partial B^\nu}{\partial u_\mu} - \frac{\partial B^\mu}{\partial u_\nu} = (u_\mu, u_\nu) \equiv u^{\mu\nu}.$$

An infinitesimal path variation δu_μ in phase-space gives, up to end-point contributions,

$$\delta S = \int [u^{\mu\nu} \dot{u}_\nu - \frac{\partial H}{\partial u_\mu}] \delta u_\mu dt$$

leading (e.g. [7]) to the equations of motion

$$u^{\mu\nu} \dot{u}_\nu = \frac{\partial H}{\partial u_\mu}. \quad (13)$$

If we change from u_μ 's to other explicitly time-dependent coordinates $\xi_\mu = \xi_\mu(u, t)$ we must put in the action a new hamiltonian $H' \equiv H + R$ in order to get the same physics. The new equations of motion are

$$\xi^{\mu\nu} \dot{\xi}_\nu = \frac{\partial H'}{\partial \xi_\mu}, \quad \xi^{\mu\nu} \equiv (\xi_\mu, \xi_\nu), \quad (14)$$

and we can find the equations determining R by comparing (13) and (14). To this end we shall need

$$\dot{\xi}_\mu = \frac{\partial \xi_\mu}{\partial u_\nu} \dot{u}_\nu + \frac{\partial \xi_\mu}{\partial t}, \quad \frac{\partial H'}{\partial \xi_\mu} = \frac{\partial H}{\partial u_\nu} \frac{\partial u_\nu}{\partial \xi_\mu} + \frac{\partial R}{\partial \xi_\mu}$$

$$\xi^{\mu\nu} = u^{\lambda\rho} \frac{\partial u_\lambda}{\partial \xi_\mu} \frac{\partial u_\rho}{\partial \xi_\nu}, \quad \frac{\partial u_\mu}{\partial \xi_\nu} \frac{\partial \xi_\nu}{\partial u_\lambda} = \delta_\mu^\lambda.$$

Using these, (14) can be rewritten

$$\begin{aligned} & (u^{\lambda\rho} \frac{\partial u_\lambda}{\partial \xi_\mu} \frac{\partial u_\rho}{\partial \xi_\nu}) \frac{\partial \xi_\nu}{\partial u_\sigma} \dot{u}_\sigma + \xi^{\mu\nu} \frac{\partial \xi_\nu}{\partial t} = \\ & = \frac{\partial H}{\partial u_\lambda} \frac{\partial u_\lambda}{\partial \xi_\mu} + \frac{\partial R}{\partial \xi_\mu} \end{aligned}$$

or

$$u^{\lambda\rho} \dot{u}_\rho \frac{\partial u_\lambda}{\partial \xi_\mu} + \xi^{\mu\nu} \frac{\partial \xi_\nu}{\partial t} = \frac{\partial H}{\partial u_\lambda} \frac{\partial u_\lambda}{\partial \xi_\mu} + \frac{\partial R}{\partial \xi_\mu}$$

By (13), the first terms on each side cancel each other and we end with (71) of the main text.

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