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V. GLASER'S RESULTS ON THE WAVE FUNCTION

AT THE ORIGIN AND THE SEMI-CLASSICAL LIMIT

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ABSTRACT

We describe V. Glaser's rigorous bounds on the wave function at the origin and the derivatives of the wave function for non-zero angular momentum, for strictly lower bounded potentials. We show also that the method used allows us to obtain the semi-classical limit of the wave function at the origin for large radial excitations and indicate connections with the Thomas-Fermi approximation.

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I. INTRODUCTION

Jurko Glaser was primarily interested in field theory where he produced his masterpiece in collaboration with J. Bros and H. Epstein, the crossing property of scattering amplitudes¹⁾. However, he has always kept an interest for potential models, scattering and bound states. It is in fact, I believe, the way we came in contact in 1959 when I arrived at CERN and described to him my work on analyticity properties of scattering amplitudes produced by superpositions of exponential potentials²⁾. Later on, in collaboration with H. Grosse and W. Thirring, we obtained some non-trivial results on bound states energies in potentials^{3),4)}.

It is probably because Jurko was surrounded by friends who were interested in properties of quarkonium, for instance Reinhold Bertlmann, Harald Grosse or myself, that he tried to do something about the wave function at the origin. Practically all calculations of production and decay processes of quarkonium necessitate the knowledge of the wave function at the origin, in the case of an $\lambda=0$ state, or of the first non-vanishing derivative of the radial wave function in the case of a state with angular momentum different from zero. Semi-classical estimates have been obtained for these quantities, for instance by Quigg and Rosner^{5),6)} or for the case of higher derivatives by Bell and Pasupathy⁷⁾. These semi-classical estimates are supposed to be valid for large quantum numbers. What Jurko tried was to get exact bounds, at the price of an extra assumption, that the potential is non-negative in x space, or, more generally, lower bounded. An essential tool, in this proof, is the use of positivity preserving operators, which he and I learnt, I believe, many years ago, from Derek Robinson. In reconstructing the work of Jurko which he never published in spite of the insistence of many people, I discovered that the method allows us to go further and to give a clear proof of the semi-classical limit results for a relatively general class of potentials (not necessarily lower bounded everywhere) which I describe in the last two sections of that paper.

II. UPPER BOUNDS ON THE WAVE FUNCTION AT THE ORIGIN

For those who do not want to read the Bible⁸⁾, I recall what is a positivity preserving operator (pp operator): It is an operator which, when applied to a positive wave function in x space produces again a positive function. In simpler terms, the matrix elements of this operator A in x space (if they exist) are such that

$$\langle x|A|y\rangle \geq 0, \quad \forall x, \forall y. \quad (1)$$

If property (1) is satisfied, we shall write

$$A \succ 0 \quad (2)$$

Naturally one can give a meaning to

$$A \succ B \quad (3)$$

which is just $A-B \succ 0$.

It is easy to prove the properties:

$$AB \succ 0 \text{ if } A \succ 0 \text{ and } B \succ 0 \quad (4)$$

from

$$\langle x|AB|y\rangle = \int d^4z \langle x|A|z\rangle \langle z|B|y\rangle,$$

$$\left. \begin{array}{l} AC \succ BC \\ CA \succ CB \end{array} \right\} \text{ if } A \succ B \text{ and } C \succ 0 \quad (5)$$

Here we shall deal with the operator

$$e^{-Ht}, \quad (6)$$

$$H = H_0 + V = p^2 + V(x). \quad (7)$$

Let us assume that V is lower bounded. Then we can apply the Trotter formula:

$$e^{-Ht} = \lim_{N \rightarrow \infty} \left[e^{-\frac{H_0 t}{N}} e^{-\frac{Vt}{N}} \right]^N \quad (8)$$

Now $e^{-V(x)t}$, being diagonal in x , is clearly positivity preserving. On the other hand we know that

$$\langle x | e^{-H_0 t} | y \rangle = (4\pi t)^{-\frac{n}{2}} \exp - \frac{(x-y)^2}{4t}, \quad (9)$$

in n space dimensions. Therefore e^{-Ht} is a positivity preserving operator.

Let us prove now that if

$$V(x) \geq 0 \quad \forall x, \text{ since } V > 0, \quad (10)$$

$$e^{-Ht} \leq e^{-H_0 t}. \quad (11)$$

(6) satisfies the integral equation:

$$e^{-Ht} = e^{-H_0 t} - \int_0^t e^{-H\tau} V e^{-H_0(t-\tau)} d\tau \quad (12)$$

The integral over the product of three pp operators is a pp operator and this proves Eq. (11) [it also follows directly from (8)]. If we apply Eq. (11) to the diagonal element, using Eq. (9), we get

$$\langle x | e^{-Ht} | x \rangle < \frac{1}{(4\pi t)^{n/2}}. \quad (13)$$

From this point we assume that V is a confining potential and has a purely discrete spectrum. If we now insert a complete set of eigenstates of H , we get

$$\sum_{k=1}^{\infty} |\psi_k(x)|^2 e^{-E_k t} < \frac{1}{(4\pi t)^{n/2}}, \quad (14)$$

since

$$\psi_k(x) = \langle x | k \rangle. \quad (15)$$

The E_k 's are in increasing order. Hence, from Eq. (14) we get, restricting the sum to the first N terms:

$$\sum_{k=1}^N |\psi_k(x)|^2 < \frac{1}{(4\pi)^{n/2}} \frac{e^{E_N t}}{t^{n/2}} \quad (16)$$

and since this holds for any t we can minimize with respect to t and get

$$\sum_{k=1}^N |\psi_k(x)|^2 < (E_N)^{1/2} \left(\frac{e}{2\pi m}\right)^{1/2} \quad (17)$$

In particular, in three dimensions we get:

$$\sum_{k=1}^N |\psi_k(x)|^2 < E_N^{3/2} \left(\frac{e}{6\pi}\right)^{3/2} \quad (18)$$

One can in particular take $x=0$. It is interesting to compare this bound with the semi-classical estimate^{5),6)}

$$\sum_{k=1}^N |\psi_k(0)|^2 \simeq \frac{1}{6\pi^2} E_N^{3/2} \quad (19)$$

valid for $N \rightarrow \infty$.

It is remarkable that Eqs. (18) and (19) have the same functional form, the coefficient in Eq. (18) being about three times that in Eq. (19). However, Eq. (18) is nevertheless almost optimal. If we take an infinite square well, we find

$$\frac{|\psi_1(0)|^2}{E_1^{3/2}} = \frac{1}{2\pi^2} \quad (20)$$

which is only 10% less than what is given by inequality (18). Let us notice that inequality (18) is valid for any x and does not require spherical symmetry. Naturally, if V is lower bounded:

$$V \geq V_{\min} \quad (21)$$

Eq. (18) becomes for three dimensions

$$\sum_{k=1}^N |\psi_k(x)|^2 < (E_N - V_{\min})^{3/2} \left(\frac{e}{6\pi}\right)^{3/2}$$

III. UPPER BOUNDS ON THE DERIVATIVES OF THE RADIAL WAVE FUNCTIONS FOR THE NON-ZERO ANGULAR MOMENTUM

In the case of spherically symmetric potentials, V. Glaser could obtain an analogue of inequality (18) for the λ th derivative of the radial wave function

and the origin for states of angular momentum ℓ . Again such quantities play a role in quarkonium physics, for instance in the production of the D-state at 3.77 GeV in e^+e^- collisions or in the decay of the P-states.

Here one replaces H by

$$H_\ell = H_{0\ell} + V \quad (22)$$

$$H_{0\ell} = -\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} \quad (23)$$

The operator $e^{-H_{0\ell}t}$ is also positivity-preserving. V. Glaser had a very direct argument to show this, I believe, but I could not reconstruct it. So we shall prove this in a rather pedestrian way:

$$\begin{aligned} \langle r | e^{-H_{0\ell}t} | r' \rangle &= \\ 4\pi \int \frac{d\Omega_{\vec{x}}}{4\pi} Y_{\ell 0}(\vec{x}) \frac{d\Omega_{\vec{x}'}}{4\pi} Y_{\ell 0}(\vec{x}') \langle \vec{x} | e^{-H_0 t} | \vec{x}' \rangle \end{aligned} \quad (24)$$

Here we have taken a normalization such that

$$\begin{aligned} \langle r | r' \rangle &= \frac{\delta(r-r')}{rr'} \\ \text{or } \int r^2 \psi_{\ell n}^2 dr &= 1 \quad \text{or } \int u_{\ell n}^2 dr = 1 \end{aligned} \quad (25)$$

Substituting Eq. (9) (in the case $n=3$), we get

$$\begin{aligned} \langle r | e^{-H_{0\ell}t} | r' \rangle &= \\ \frac{4\pi}{2} \frac{e^{-\frac{r^2+r'^2}{4t}}}{(4\pi t)^{3/2}} \int_{-1}^{+1} P_\ell(\cos\theta) \exp\left(\frac{rr'\cos\theta}{2t}\right) d\cos\theta. \end{aligned} \quad (26)$$

The exponential in Eq. (25) can be expanded in powers of $\cos\theta$. It has positive expansion coefficients, and since

$$\int_{-1}^1 P_\ell(\cos\theta) (\cos\theta)^\ell d\cos\theta \geq 0$$

the positivity of Eq. (24) is established. Hence, copying the argument of the previous section, we see that $\exp(-H_\lambda t)$ is a pp operator. In fact, the integral in Eq. (24) can be calculated explicitly and one finds:

$$\begin{aligned} & \langle r | e^{-H_0 t} | r' \rangle \\ &= \frac{(rr')^{1/2}}{2t} \exp\left(-\frac{r^2+r'^2}{4t}\right) I_{\ell+1/2}\left(\frac{rr'}{2t}\right), \end{aligned} \quad (27)$$

where $I_{\ell+1/2}$ is a modified Bessel function. Furthermore one has

$$\lim_{\substack{r \rightarrow 0 \\ r' \rightarrow 0}} \frac{\langle r | e^{-H_0 t} | r' \rangle}{r^\ell r'^\ell} = \frac{1}{4^{\ell+1} t^{\ell+3/2}} \frac{1}{\Gamma(\ell+3/2)}. \quad (28)$$

If $V(r)$ is positive, we have, as previously

$$\langle r | e^{-H_0 t} | r' \rangle < \langle r | e^{-H_0 t} | r' \rangle \quad (29)$$

and, dividing by $(rr')^\ell$ and following the same line of arguments as in the previous section

$$\sum_{k=1}^N \left(\lim_{r \rightarrow 0} \frac{U_{ke}(r)}{r^{\ell+1}} \right)^2 \leq \frac{\exp(E_{N,e} t)}{t^{\ell+3/2} 4^{\ell+1} \Gamma(\ell+3/2)},$$

or, optimizing in t

$$\sum_{k=1}^N \left(\lim_{r \rightarrow 0} \frac{U_{ke}(r)}{r^{\ell+1}} \right)^2 = \sum_{k=1}^N (a_{ke})^2 \leq \frac{(E_{N,e})^{\ell+3/2}}{4^{\ell+1} \Gamma(\ell+3/2)} \left(\frac{e}{\ell+3/2} \right)^{\ell+3/2} \quad (30)$$

The corresponding semi-classical estimate⁷⁾ is

$$\sum_{k=1}^N (a_{k,l})^2 \approx \frac{(E_{N,l})^{l+3/2}}{\pi (l+3/2) [(2l+1)!!]^2} \quad (31)$$

One can check that the rigorous inequality gives a value above the semi-classical estimate. The ratio between the two increases like $l^{\frac{1}{2}}$. It is around 10 for $l=10$.

IV. SMALL t BEHAVIOUR OF EXP-Ht

Here we present some presumably new results though I must say that somewhat related results can be found in the book of B. Simon on functional integration⁹⁾. Anyway the method, inspired by the one used in the previous section by V. Glaser, is completely original.

We shall prove that

$$\lim_{t \rightarrow 0} \frac{\langle x | e^{-Ht} | y \rangle}{\langle x | e^{-H_0 t} | y \rangle} - 1 = 0. \quad (32)$$

Later on we shall even give more precise estimates of the l.h.s. of Eq. (32) for small t . First we shall consider the simple case of $V(x) \geq 0$, then the case of $V(x) \geq -B$, and later on the case where $V(x)$ is not lower bounded, but where $V(x)$ is submitted to a condition which will be specified later but which amounts to say that the negative part of V is less singular than $r^{-2+\epsilon}$, $\epsilon > 0$ in the case of spherical symmetry.

Before we consider the successive cases, let us make the following general remark: if $V_1(x) \geq V_2(x)$, then:

$$e^{-t(H_0+V_1)} \leq e^{-t(H_0+V_2)}. \quad (33)$$

This follows from the Trotter formula (8), which can be applied for lower bounded potentials. The lower bound can be removed by a limiting procedure as we shall see.

i) Positive potentials

We have

$$\langle x | e^{-Ht} | y \rangle < \langle x | e^{-H_0 t} | y \rangle$$

and from Eq. (12) we can also get a lower bound:

$$\begin{aligned} \langle x | e^{-Ht} | y \rangle &> \langle x | e^{-H_0 t} | y \rangle \\ &- \int_0^t d\tau \int d^3 z \langle x | e^{-H_0 \tau} | z \rangle V(z) \langle z | e^{-H_0(t-\tau)} | y \rangle \end{aligned} \quad (34)$$

We shall assume

$$V(x) < A + B|x|^p \quad (35)$$

and treat the case of three dimensions (an unessential restriction!).

The integral in the right-hand side has to be evaluated. It is

$$\frac{1}{(4\pi)^3} \int_0^t d\tau \int d^3 z \frac{V(z) \exp\left[-\frac{t}{4\tau(t-\tau)} \left(z - \frac{x\tau + y(t-\tau)}{t}\right)^2\right] \exp\frac{(x-y)^2}{4t}}{\tau^{3/2} (t-\tau)^{3/2}} \quad (36)$$

$$= \frac{1}{(4\pi)^3} \int_0^t d\tau \frac{\exp\frac{(x-y)^2}{4t}}{\tau^{3/2} (t-\tau)^{3/2}} I(\tau, z_0(\tau))$$

where

$$z_0(\tau) = \frac{x\tau + y(t-\tau)}{t}$$

$$I(\tau, z_0) = \int d^3 z V(z) \exp\left(-\frac{t}{4\tau(t-\tau)} (z - z_0)^2\right) \quad (37)$$

We can majorize $I(\tau, z_0)$ by splitting the integration region into $|z| < 2|z_0|$, $|z| > 2|z_0|$:

$$\begin{aligned}
 I(\tau, z_0) &< \left(A + 2^\nu B(z_0)^\nu \right) \int d^3 z \exp \left[-\frac{t}{4\tau(t-\tau)} (z-z_0)^2 \right] \\
 &+ \int d^3 z \left(A + B|z|^\nu \right) \exp \left(-\frac{t}{16\tau(t-\tau)} z^2 \right) \\
 &< \left(C_1 + C_2 |z_0|^\nu \right) \left(\frac{\tau(t-\tau)}{t} \right)^{\frac{3}{2}} + C_3 \left(\frac{\tau(t-\tau)}{t} \right)^{\frac{\nu}{2} + \frac{3}{2}} \quad (38)
 \end{aligned}$$

Finally, using $|z_0| < |x| + |y|$, inserting (38) into (36) and integrating over τ , we get:

$$\begin{aligned}
 &\langle x | e^{-Ht} | y \rangle > \\
 &\langle x | e^{-H_0 t} | y \rangle \left[1 - t(D_1 + D_2(|x|^\nu + |y|^\nu)) - D_3 t^{1+\frac{\nu}{2}} \right] \quad (39)
 \end{aligned}$$

Therefore property (32) is proved in this case and more specifically

$$\left| \frac{\langle x | e^{-Ht} | y \rangle}{\langle x | e^{-H_0 t} | y \rangle} - 1 \right| < \text{const} \times t \quad (40)$$

for $t \rightarrow 0$, x, y finite. We shall see that this property enables us to derive semi-classical results on the wave function and the origin in the next section.

ii) Case of V lower bounded

It is completely clear that if $V > -B$ and $V < A + B|x|^\nu$, all we have to do is to replace V by $V+B$ and hence

$$\langle x | e^{-Ht} | y \rangle < \langle x | e^{-H_0 t} | y \rangle e^{Bt}. \quad (41)$$

A lower bound of the same type as (39) will exist and properties (32) and (40) still hold. However, we would like to go further and obtain an inequality independent of the lower bound $-B$, to prepare the next step. From (41) we deduce that there exists

$$r(t) = \sup_{x, y} \frac{\langle x | e^{-Ht} | y \rangle}{\langle x | e^{-H_0 t} | y \rangle} \quad (42)$$

and

$$R(t) = \sup_{0 \leq t' \leq t} r(t') \quad (43)$$

From the semi-group property we have $R(t_1+t_2) \leq R(t_1)R(t_2)$ and hence

$$R(nt) \leq [R(t)]^n. \quad (44)$$

If we call V^- the negative part of the potential, i.e., $V^- = 0$ if $V > 0$, $V^- = -V$ if $V < 0$ we have

$$e^{-Ht} \leq e^{-H_0 t} + R(t) \int_0^t e^{-H_0(t-\tau)} V^- e^{-H_0 \tau} d\tau. \quad (45)$$

To evaluate matrix elements of the right-hand side we have to calculate the analogue of (37) with V replaced by V^- :

$$I(\tau) = \int d^3 z V^-(z) \exp\left(-\frac{t}{4\tau(t-\tau)} \left[z - \frac{x\tau + y(t-\tau)}{t}\right]^2\right)$$

Here we shall majorize $I(\tau)$ by using Hölder's inequality, with $\gamma > 1$

$$\begin{aligned} I(\tau) &< \left[\int d^3 z (V^-(z))^\gamma \right]^{\frac{1}{\gamma}} \\ &\times \left[\int d^3 z \exp\left(-\frac{\gamma}{\gamma-1} \frac{t}{4\tau(t-\tau)} \left(z - \frac{x\tau + y(t-\tau)}{t}\right)^2\right) \right]^{1-\frac{1}{\gamma}} \\ &= \left[\int d^3 z (V^-(z))^\gamma \right]^{\frac{1}{\gamma}} \times \left[\pi \frac{4(\gamma-1)}{\gamma} \frac{\tau(t-\tau)}{t} \right]^{\frac{3}{2} \frac{\gamma-1}{\gamma}} \end{aligned}$$

Then one has to calculate the integral of the upper bound of $I(\tau) [\tau(t-\tau)]^{-3/2}$.

This integral will converge if $\gamma > 3/2$. We therefore impose the condition:

$$\exists \gamma > 3/2 \text{ such that } \int (V^-(x))^\gamma d^3x < \infty \quad (46)$$

This condition will be satisfied for instance by $V = -r^{-2+\eta}$ $\eta > 0$ for $r < R$.
The final result is

$$\begin{aligned} \langle x | e^{-Ht} | y \rangle &< \langle x | e^{-H_0 t} | y \rangle \times \\ &\left[1 + C_\gamma R(t) \left(\int (V^-)^\gamma d^3x \right)^{\frac{1}{\gamma}} t^{1 - \frac{3}{2\gamma}} \right] \end{aligned} \quad (47)$$

There exists t_0 , independent of B , such that

$$C_\gamma t_0^{1 - \frac{3}{2\gamma}} \left[\int (V^-)^\gamma d^3x \right]^{\frac{1}{\gamma}} \leq \frac{1}{2} \quad (48)$$

Now there are two possibilities:

$$a) R(t_0) = r(t_0)$$

Then there exists x and y such that, given ε arbitrarily small,

$$\langle x | e^{-Ht_0} | y \rangle > (1 - \varepsilon) R(t_0) \langle x | e^{-H_0 t_0} | y \rangle,$$

and so, from (47) and (48)

$$R(t_0) \left[1 - \varepsilon - \frac{1}{2} \right] < 1,$$

and hence

$$R(nt_0) < \left(\frac{1}{2} - \varepsilon \right)^{-n}. \quad (49)$$

$$b) R(t_0) < r(t_0)$$

Then there exists $t_M < t_0$ such that $R(t_0) = R(t_M) = r(t_M)$ and we get again Eq. (49).

Therefore we get for $t < t_0$, by inserting (49) in (47),

$$\langle x | e^{-Ht} | y \rangle < \langle x | e^{-H_0 t} | y \rangle \left[1 + K t^{1 - \frac{3}{2\gamma}} \right],$$

where K is independent of the lower bound $-B$ of the potential.

On the other hand, if we have also $V < A+B|x|^\nu$, we can use the monotonicity property (33) to get

$$\langle x | e^{-Ht} | y \rangle > \langle x | e^{-H_0 t} | y \rangle (1 - Ct),$$

for x and y finite, t small enough.

We conclude

$$\left| \frac{\langle x | e^{-Ht} | y \rangle}{\langle x | e^{-H_0 t} | y \rangle} - 1 \right| < \text{const.} \cdot t^{1 - \frac{3}{2\gamma}}, \quad (50)$$

for x, y , finite $t < T$, under conditions (35) and (46).

iii) The case where V is not lower bounded

Then we shall approach V with a monotonous sequence of lower bounded potentials

$$V_B(x) \begin{cases} = V(x) & \text{if } V(x) \geq -B \\ = -B & \text{if } V(x) \leq -B \end{cases}$$

We assume that V satisfies conditions (35) and (46). Then the matrix elements

$$\langle x | e^{-H_B t} | y \rangle$$

form a monotonous increasing sequence when B increases, and for t small enough are bounded by (50). Therefore they converge to a limit which is clearly $\langle x | e^{-Ht} | y \rangle$. This limit still satisfies inequality (50).

In conclusion, under acceptable conditions (which could be slightly refined), we prove property (32).

V. SEMI CLASSICAL RESULTS ON THE WAVE FUNCTIONS

A consequence of (32) is, taking $x=y$

$$\lim_{t \rightarrow 0} \left[(4\pi t)^{3/2} \sum_{k=1}^{\infty} |\psi_k(x)|^2 e^{-E_k t} \right] = 1 \quad (51)$$

There is a Tauberian theorem of Karamata¹⁰⁾ which says that if

$$\lim_{t \rightarrow 0} t^{\gamma} \int e^{-tx} d\mu(x) = D, \quad (52)$$

where μ is a positive measure, and if $\int e^{-tx} d\mu(x)$ converges for all $t > 0$, then

$$\lim_{a \rightarrow \infty} a^{-\gamma} \mu(0, a) = \frac{D}{\Gamma(\gamma+1)} \quad (53)$$

If we take

$$\mu(E) = \sum_{k=1}^{N(E)} |\psi_k(x)|^2 \quad (54)$$

with

$$E_{N(E)} < E < E_{N(E)+1} \quad , \quad N(E) \in \mathbb{N},$$

(51) becomes

$$\lim_{t \rightarrow 0} t^{3/2} \int_0^{\infty} e^{-Et} d\mu(E) = \frac{1}{(4\pi)^{3/2}}$$

Hence we conclude:

$$\lim_{N \rightarrow \infty} \frac{\sum_{k=1}^N |\psi_k(x)|^2}{E_N^{3/2}} = \frac{1}{6\pi^2}. \quad (55)$$

This is precisely the semi-classical limit for the sum of the first N squares of the wave functions at the origin.

Notice that though this statement is less precise than that of Quigg and Rosner⁶⁾

$$|\psi_N(0)|^2 \sim \frac{1}{4\pi^2} E_N^{1/2} \frac{dE_N}{dN} \quad (56)$$

the proof is more rigorous, does not require spherical symmetry and clearly applies to singular potentials, provided condition (46) is satisfied. It applies in particular to the potential $V = \ln r$, for which numerical investigations of Quigg and Rosner⁶⁾ seemed to indicate that $N|\psi_N(0)|^2$ approaches a constant for $N \rightarrow \infty$. In fact, since for such a potential $E_N \sim (\ln N)$ we get from (56)

$$\sum_{k=1}^N |\psi_k(0)|^2 \sim \text{const} \times (\ln N)^{3/2}$$

and if the wave function dependence on N is smooth¹¹⁾

$$|\psi_N(0)|^2 \sim \frac{1}{N} (\ln N)^{1/2} \times \text{const.}$$

One could probably also establish the corresponding results by analogous methods for the l th derivatives of the radial wave function at the origin for angular momentum l . We leave it as an "exercise" to the reader.

In fact, we can get still more precise results by using more accurate estimates on the difference between the matrix element of $\exp -Ht$ and $\exp -H_0 t$. Assume for instance

$$|V(x) - V(y)| < C |x - y|, \quad (57)$$

which is compatible with a linearly confining potential [(57) could in fact be taken locally with C depending on x and y]. Then we can choose $x=0$ and choose the scale of energies such that $V(0)=0$. Then, using the estimates (36), (37), (38) and (39) we find

$$\left| \langle 0 | e^{-Ht} | 0 \rangle - \langle 0 | e^{-H_0 t} | 0 \rangle \right| < C' \quad \text{for } t \rightarrow 0. \quad (58)$$

so that if we define a new measure:

$$\nu(E) = \mu(E) - \frac{E^{3/2}}{6\pi^2} \quad (59)$$

[unfortunately $\nu(E)$ is not necessarily positive!], we have

$$\int_0^\infty e^{-Et} d\nu(E) \text{ finite for } t \rightarrow 0. \quad (60)$$

We cannot go further without assumption. Let us assume that $\nu(E)$ is monotonous beyond a certain value E . Then the quantity (60) has a limit and $\nu(E)$ has a limit for $E \rightarrow \infty$.

This means also that if we write an asymptotic expansion of $\mu(E)$

$$\mu(E) = \frac{E^{3/2}}{6\pi^2} + B E^{1/2}$$

the coefficient B is zero. Returning to a general point and a general value of the potential, we get

$$\lim_{N \rightarrow \infty} \frac{\sum_{k=1}^N |\psi_k(x)|^2 - \frac{[E_N - V(x)]^{3/2}}{6\pi^2}}{E_N^{1/2}} = 0 \quad (61)$$

$\sum_{k=1}^N |\psi_k(x)|^2$ represents the particle density in an independent particle model with fermions. Equation (61) is just telling us that in the large N limit the Thomas-Fermi approximation for the density is valid in an independent particle model in a confining potential satisfying (57). This kind of limit is somewhat different from the one usually considered in the proofs of the validity of the Thomas-Fermi approximation^{12),13)}.

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REFERENCES

- 1) J. Bros, H. Epstein and V. Glaser, Comm. Math. Phys. 1 (1965) 240.
- 2) A. Martin, Nuovo Cimento 14 (1959) 403.
- 3) V. Glaser, H. Grosse, A. Martin and W. Thirring in Studies in Mathematical Physics, E. Lieb, B. Simon and A.S. Wightman editors, Princeton University Press (1976), p. 169.
- 4) V. Glaser, H. Grosse and A. Martin, Comm. Math. Phys. 59 (1978) 197.
- 5) C. Quigg and J.L. Rosner, Phys. Rev. D17 (1978) 2364;
see also:
M. Krammer and P. Céal-Ferreira, Rev. Bras. Fis. 6 (1976) 7.
- 6) C. Quigg and J.L. Rosner, Phys. Reports 56 (1971) 168.
- 7) J.S. Bell and J. Pasupathy, Phys. Lett. 83B (1979) 389.
- 8) M. Reed and B. Simon, Methods of Mathematical Physics, Vol. II, Academic Press, New York (1975), p. 186.
- 9) B. Simon, Functional Integration and Quantum Physics, Academic Press, New York (1979).
- 10) J. Karamata, J. Reine Angew. Math. 164 (1931) 29;
see also: Ref. 9), p. 108.
- 11) J.M. Richard in XIXth Rencontre de Moriond, Vol. II (1984) p. 515, J. Tran Than Van, ed. (Editions Frontières).
- 12) E. Lieb and B. Simon, Phys. Rev. Lett. 31 (1973) 681.
- 13) B. Baumgartner, Comm. Math. Phys. 47 (1976) 215.