CONFINED QUANTUM ELECTRODYNAMICS IN 1+1 DIMENSIONS :
A PERTURBATIVE ANALYSIS

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A B S T R A C T

The spectrum of confined QED in 1+1 dimensions is analysed using perturbation theory. The mass spectra of systems made up of massless fermions are calculated to $O(e^2)$ and compared to the mass spectra obtained using non-perturbative methods. Systems containing heavy fermions are also studied and an analogy with the 3+1 dimensional Bag model is pointed out.

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1 INTRODUCTION

This paper contains a model study of the properties of hadrons consisting of both heavy and light quarks within the context of bag models. In this setting there are two ways of describing quarks, depending on whether they are heavy or light.

For light quark spectroscopy, the only feasible bag model approach is the "static cavity approximation"^[1], where the "constituent" quarks are described by Dirac wave functions in a fixed (usually spherical) cavity with MIT-type boundary conditions imposed on the surface. This "shell model" description has obvious problems with momentum conservation^[2], but, phenomenologically, it has been quite successful.

The bag picture of hadrons containing heavy quarks is quite different. Since a heavy quark by definition moves much slower ($v_Q \approx 1/m_Q$) than the bag surface ($v_{\text{surf}} \approx 1/\Lambda_{\text{QCD}}$) or the massless quarks and gluons, it is reasonable to use a Born-Oppenheimer approach^[3] to separate the heavy quark dynamics from that of the rest of the system. For hadrons containing only one heavy quark, e.g. $Q\bar{q}$ or Qqq systems^[4,5,6], this again results in a description based on a fixed cavity, but this time without any serious problems due to a large C.M. motion. For systems containing several heavy quarks, such as $Q\bar{Q}$ ^[7,8,9] or $Q\bar{Q}g$ ^[10], the Born-Oppenheimer approximation results in a potential model description of the heavy quark motion.

The immediate question here is, of course, when can a quark be considered heavy? The Born-Oppenheimer approximation is clearly justified in the $m_Q \rightarrow \infty$ limit, but it is not clear whether, e.g., $m_s/\Lambda_{\text{QCD}} \approx 3$ or $m_c/\Lambda_{\text{QCD}} \approx 10$ can be considered large. Traditionally one has regarded m_s as small and m_c as large, although the static cavity approximation has been applied to charmonium states^[11,12], and the Born-Oppenheimer approximation to all strange states^[8].

In the second part of this article we will study the transition from the $m=0$, "spread-out cavity mode" description to the $m = \infty$, "localized heavy" particle description. We will present some results about confined 2-dimensional QED (i.e. the massive^[13] Schwinger model^[14]) that we believe are of relevance for understanding the localization of heavy quark wave functions in the context of a bag model. To avoid misunderstanding, we stress already at this point that the massive Schwinger model does not provide a realistic model for hadrons. We shall, therefore, not be able to give any quantitative results concerning the applicability of various approximation schemes. We do, however,

believe that when it comes to the particular aspect studied here, i.e. localization of heavy quark wavefunctions, the analogy between QED_2 and QCD_4 is a useful one.

In the course of this work we discovered several interesting properties of massless confined QED_2 . Just like ordinary massless QED_2 ^[14], this model is exactly solvable, and the spectrum is that of a free massive Bose field with Dirichlet boundary conditions^[15,16]. In spite of this bosonization, states with non-zero quark number are well defined, and correspond to topologically non-trivial background fields in the bose description^[17]. These results have been obtained by purely formal arguments involving the use of a canonical formulation with boundary-dependent commutation relations^[15] or manipulations of path integrals^[16,17]. In view of this, it is quite appealing that they can be checked in perturbation theory. Note that confined QED_2 , as opposed to the usual Schwinger model, possesses a perturbation expansion and has a spectrum containing quark states. This comes about because the length of the confining box ($=2a$) acts as an infrared cutoff, so a perturbation series can be defined in the dimensionless parameter $g=ea$ where e is the gauge theory coupling constant. Obviously, in the weak coupling limit, $g \rightarrow 0$, one recovers the free quark theory. The first section of our paper is devoted to a perturbative analysis of the spectrum of confined QED_2 .

Thus, the unifying theme of this article is perturbation theory for confined QED_2 , but the emphasis in the two parts is quite different. The first part is mainly of formal interest in that it confirms non-perturbatively obtained results in the context of perturbation theory. The second part, hopefully, contributes to the understanding of heavy quarks in the bag model.

The paper is organized as follows. In Section 2 we give the results for the mass spectrum of systems of arbitrary integer charge, containing only massless quarks, to first order in g^2 . We also verify to this order the correspondence between the charged quark system and a system of massive bosons quantized on top of a charged, topologically non-trivial background. In Section 3 we study how the transition from "delocalized light" quark to "localized heavy" quark occurs as a function of the quark mass for systems containing only heavy quarks and systems containing both heavy and massless quarks. In Sections 2 through 4 we will use the term 'fermion' rather than 'quark' for the fundamental particles of the theory, which is more appropriate within the context of QED_2 .

2 CONFINED QED₂ WITH MASSLESS FERMIONS

The action for the confined, massive Schwinger model is given by :

$$S = \int d^2x \{ -1/4 F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi \} \quad (2.1)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The fermion mass m is put to zero in this section. The space integration will always be restricted to the interval $[-a, a]$, and we use the notations $\int d^2x = \int_{-\infty}^{\infty} dt \int dx$, and $\int dx = \int_{-a}^a dx$. The fermion fields are confined by the following gauge-invariant (MIT bag^[1]) boundary condition :

$$(1 \pm i\gamma^1)\psi(\pm a) = 0 \quad (2.2)$$

By choosing the Coulomb gauge ($\partial_1 A^1 = 0$) we are left with only the instantaneous Coulomb interaction. The Hamiltonian then is given entirely in terms of the fermion fields :

$$H = \int dx [1/4 F^{\mu\nu} F_{\mu\nu} - i\bar{\psi}\gamma^1 \partial_x \psi + e\rho A^0 + m\bar{\psi}\psi] \quad (2.3)$$

$$= H_F + 1/2 e^2 \int dx \rho(x) \int dx' D(x, x') \rho(x') + 1/2 A^0 E|_{-a}^a \quad (2.4)$$

where we introduced the free fermion Hamiltonian $H_F = \int dx \bar{\psi}(-i\gamma^1 \partial_x + m)\psi$, the charge density operator $\rho(x) = \psi^\dagger(x)\psi(x)$, and the electric field E . The static photon propagator is given by:

$$D(x, x') = -1/2 |x - x'| + \lambda a. \quad (2.5)$$

where λ is a gauge-dependent constant. We can rewrite Eq.(2.4) in the form:

$$H = H_F + 1/2 E^2 + A^0 E|_{-a}^a \quad (2.6)$$

where all gauge-dependence is contained in the boundary term.

In the $e=0$ limit the fermions satisfy the free Dirac equation:

$$(i\not{\partial} - m)\psi = 0 \quad (2.7)$$

which has the following solutions:

$$u_{n\epsilon}(r, t) = \begin{bmatrix} \cos(k_n r - \alpha) \\ \frac{ik_n}{\omega_n + m} \sin(k_n r - \alpha) \end{bmatrix} N e^{-i\omega_n t} \quad (2.8)$$

and

$$v_{m\eta}(r,t) = \begin{bmatrix} -k_m \cos(k_m r + \alpha) \\ \omega_m + m \\ i \sin(k_m r + \alpha) \end{bmatrix} N e^{i\omega_m t} \quad (2.9)$$

where $\alpha=0$ for positive parity ($\epsilon, \eta=+1$) solutions, and $\alpha=\pi/2$ for the negative parity ($\epsilon, \eta=-1$) solutions. $\omega = (k^2 + m^2)^{1/2}$ and the momentum eigenvalues $k_{n,m}$ are determined from Eq.(2.2). $N = \{\omega a(\omega + ma)/[2(\omega a)^2 + ma]\}^{1/2}/\sqrt{a}$ is the normalization constant. The two solutions are related by a C-parity transformation :

$$v_{m\eta}(r,t) = i\gamma_5 u_{m-\eta}^*(r,t). \quad (2.10)$$

2.1 The $Q = 0$ Sector

The $e=0$ eigenfrequencies corresponding to Eq.(2.2) are $\omega_f^n = (2n-1)\pi/4a$, where n is any positive integer. The spectrum of the $Q=0$ fermion system is identical^[15,16] to that of a free, massless scalar field ϕ satisfying Dirichlet boundary conditions :

$$\phi(\pm a) = 0. \quad (2.11)$$

For a massless field, Eq.(2.11) gives bose eigenfrequencies $\omega_b^n = n\pi/2a$. The equivalence between the fermion and boson spectrum is illustrated in Table 1 where we list the first few levels, in both the fermion and boson representations. The energy levels are trivially the same, and by evaluating the partition functions one can also show that the degeneracies of the levels of the two spectra are the same^[16].

The interaction between the fermions will lift the degeneracy of the levels seen in Table 1. The corresponding boson theory will remain non-interacting, but the boson will acquire a mass $m^2 = e^2/\pi$ ^[13-15]. In the fermion representation, the level shifts can be straightforwardly computed using perturbation theory. There are, to lowest order in ea , three interaction terms which contribute for systems containing just a fermion-antifermion pair. These terms are the scattering term S (Fig.1a), the annihilation term A (Fig.1b), and the self-energy terms Σ (Figs. 1c and 1d). At higher energy levels, we will also encounter states containing more than one $f\bar{f}$ pair. For these levels, operators T ($T^\dagger$), which create (annihilate) $f\bar{f}$ pairs as depicted in Figs. 1e and 1f,

will also contribute to the level shifts. In these states also f-f and $\bar{f}$ - $\bar{f}$ scattering contributions to the energy will be present.

We will label the interaction terms by the energies and parities of the fermion modes involved. For example, the diagram describing the scattering of a fermion in mode (k, ϵ) , with energy $\omega_k = (2k-1)\pi/4a$ and parity ϵ , and an antifermion in mode (q, η) into a fermion mode (k', ϵ') and an antifermion mode (q', η') will be denoted by $S(k, \epsilon, q, \eta; k', \epsilon', q', \eta')$. For massless fermions we find the following results :

$$aS(k, \epsilon, q, \eta; k', \epsilon', q', \eta') = \begin{cases} \alpha a^2 (-1/3 + \lambda) & \text{if } (k, \epsilon) = (k', \epsilon') \\ & \text{and } (q, \eta) = (q', \eta') \\ \frac{\alpha z z'}{4(\omega_k - \omega_{k'})^2} & \text{if } |k - k'| = |q - q'| \neq 0 \\ & \text{and } \epsilon\eta = \epsilon'\eta' \\ \frac{-\alpha \cos((\omega_k - \omega_{k'})a)}{2(\omega_k - \omega_{k'})^2} & k \neq k', q = q' \text{ and} \\ & \epsilon\eta = \epsilon'\eta' \\ & \text{and } k \rightarrow q, k' \rightarrow q' \\ 0 & \text{otherwise} \end{cases} \quad (2.12)$$

with z and z' equal to $\text{sign}(\omega_{\text{odd}} - \omega_{\text{even}})$ for transitions between a mode with even parity and one with odd parity, and $+1$ for all other transitions. We define $\alpha = 1/2e^2$. Eq.(2.12) is also valid for f-f and $\bar{f}$ - $\bar{f}$ scattering, provided one multiplies the result with a factor (-1) for every positive parity antifermion mode $(q\eta)$ in the diagram.

$$aA(k, \epsilon, q, \eta; k', \epsilon', q', \eta') = \begin{cases} \frac{\alpha w}{4(\omega_k + \omega_{k'})^2} & \text{if } k+q = k'+q' \text{ and } \epsilon\eta = \epsilon'\eta' \\ 0 & \text{otherwise} \end{cases} \quad (2.13)$$

with $w = -1$ for diagrams with negative parity initial and final states, where the (anti)fermions in the initial and the final state have opposite parity, and $+1$ otherwise. For the self-energy diagram of the fermion one has :

$$\begin{aligned} \Sigma(k, \epsilon, k', \epsilon) = & \sum_{k'', \epsilon''} S(k, \epsilon, k'', \epsilon''; k'', \epsilon'', k', \epsilon) + \\ & - \sum_{q, \eta} A(k, \epsilon, q, \eta; k', \epsilon, q, \eta) \end{aligned} \quad (2.14)$$

Here the first sum is over fermion modes, and the second one over antifermion modes. For massless fermions, Σ is only non-zero for transitions between identical levels. The second term in Eq.(2.14) then cancels most contributions from the first one and Eq.(2.14) simplifies to :

$$a\Sigma(k) = -1/3 + \lambda + 2/\pi^2 \sum_{n=1}^{k-1} 1/n^2 \quad (2.15)$$

Corresponding results hold for the antifermion self-energy diagram.

The matrix element of the Hamiltonian H of Eq.(2.4) for the transition depicted in Fig. 1e is :

$$aT(k, \epsilon; k', \epsilon', k'', \epsilon'', q, \eta) = \begin{cases} \frac{\alpha\beta zw}{4(\omega_k - \omega_{k'})^2} & \text{if } |k - k'| = q + k'' \\ & \text{and } \epsilon = \epsilon' \epsilon'' \eta \\ 0 & \text{otherwise} \end{cases} \quad (2.16)$$

In Eq.(2.16) z and w are as defined below Eq.(2.12) and (2.13), respectively. $\beta = +1$ when both antifermions have the same parity and -1 when they have opposite parity. Analogous expressions can be given for the creation of an $f\text{-}\bar{f}$ pair from an antifermion line, and for the annihilation of $f\text{-}\bar{f}$ pairs.

The splitting of the fermion levels listed in Table 1 is given, to $O(g^2)$, in Table 2. It is easy to verify that the fermion levels in Table 2 agree with the boson levels given in Table 1, provided we substitute the energies of the massive bosons, i.e., $a\omega_b^n = ((n\pi/2)^2 + g^2/\pi)^{1/2}$, for each of the eigenenergies, and then Taylor expand the result to $O(g^2)$.

Notice that the Taylor expansions for the ω_b^n 's all have a finite radius of convergence. This is very different from, e.g., QED_4 , where the perturbation expansion is an asymptotic series.

Table 2 possesses several interesting features. We see that all levels are raised in energy, when we let the dimensionless coupling constant ea become finite. This phenomenon is, of course, a direct reflection of the fact that the boson acquires a mass. Then, if we label the levels in Table 2 with $N \in \{1, 2, \dots\}$, the smallest splitting in each level has $\Delta = 1/N \cdot g^2/\pi^2 a$, corresponding to a single excitation of the boson mode with energy $N\pi/2a$. All other states in the level correspond to multi-boson states. Furthermore, the fermion wave function for the lowest sublevel shows that this level is built up from only single $f\text{-}\bar{f}$ pairs, which all contribute, up to a sign, with an equal weight $1/\sqrt{N}$. The contributions to the energy of this state due to the sum of the Σ and S terms, and of the T terms in the Hamiltonian are zero. The shift, therefore, is entirely due to the annihilation term. We note that in QCD_2 , where the annihilation diagram is absent for modes containing only one singlet $f\text{-}\bar{f}$ pair, the lowest level in each multiplet does not shift, when we let ga (where g is now the QCD coupling constant) become finite.

To complete the discussion of the energy spectrum to $O(g^2)$, we note that there is also a zeropoint energy term H_0 (see Fig. 1g), which can be computed exactly. We find :

$$\begin{aligned} H_0 &= \sum_{k, \epsilon, q, \eta} A(k, \epsilon, q, \eta; k, \epsilon, q, \eta) \\ &= g^2/2a\pi^2 [1 + \gamma_E + \lim_{N \rightarrow \infty} (\pi^2 N/6 + \log(N) + \dots)] \end{aligned} \quad (2.17)$$

where γ_E is Euler's constant. This energy is, of course, independent of the state in which it is evaluated. Therefore, we will omit it in the considerations below.

2.2 The $Q \neq 0$ Sectors

Knowing the unperturbed $Q=0$ spectrum, one can easily construct the unperturbed $Q=q$ levels and the corresponding fermion states, where q is any integer. The construction simply consists of filling up the q lowest energy levels to obtain the ground state in the charge $Q=q$ sector. Since this is equivalent to a redefinition of the zero energy level (or the surface of the Dirac sea), the excitation spectrum obviously remains the same as for $Q=0$ and the wave functions for $e=0$ can be trivially read off from Table 1. For example, in the notation of

Table 1, the three $n=3$ wave functions in the $Q=2$ sector are $(1+9)$, $(3+7)$ and $(1+3+5+\overline{1})$. This construction also follows from the calculation of the thermodynamical partition function in Ref.16, where we found an extra energy $E_0=\pi q^2/4a$ in the charge q sector and noticed $E_0=\sum \omega_f^n$, where the sum is over the first q levels^[17].

The energy spectrum of the $Q=q$ system for $g=0$ is identical to that of a massless boson field satisfying Eq.(2.11), provided we subtract an energy $\omega_0=\pi q^2/4a$, i.e., the expectation value of the Hamiltonian for the $Q=q$ vacuum state^[17]. In the boson formulation ω_0 can be interpreted^[17] as the energy of a background field ϕ_0 . We can take ϕ_0 to be a time-independent solution of $\partial^2\phi = 0$. This gives $\phi_0 = q\sqrt{\pi/2a}(x+\eta)$, with η a constant. The field ϕ_0 corresponds to a boson current^[18] $j^\mu(x)=-\epsilon^{\mu\nu}\partial_\nu\phi_0(x)/\sqrt{\pi}$, which carries a topological charge $Q = \int dx j^0(x) = q$. In Ref.17 we identified this topological charge with the fermion number.

When ea becomes finite and the fermions interact, we find that the energy spectrum for the interacting $Q=q$ system is identical to that of the interacting $Q=0$ system, up to a charge-dependent term E_q . This very non-trivial property of the charged states is intimately related to the simple structure of the $m=0$ vacuum. The equal spacing of the fermion energy levels has as a consequence that the exchange scattering diagrams cancel against the self-energy graphs. Left over are the diagonal, or classical, interaction terms, where the fermions always stay in their initial modes. This absence of quantum correlations^[19], together with the constant charge densities computed from Eqs.(2.8) and (2.9), implies that to $O[(ea)^2]$ the background energy is just that of a constant classical charge distribution q :

$$b_q = \pi q^2/4a [1 + 2g^2/\pi(-5/6 + 2\lambda)] \quad (2.18)$$

This result again coincides with the exact bosonic solution in Ref.17 where the spectrum was shown to be that of a free massive ($m^2=e^2/\pi$) boson quantized around a non-trivial background ϕ_0 , satisfying $(\square - m^2)\phi_0=0$. The value

$$E_q = c^2 a^{-1} \sinh(ma) \cosh(ma) / ma \rightarrow \pi q^2/4a [1 + 2g^2/\pi(1/3)] \quad (2.19)$$

$$g^2 \rightarrow 0$$

where c was derived in Ref.17 using a covariant gauge, cannot be compared directly with the Coulomb gauge result Eq.(2.18). Note that

the energy of the charged state does depend on the gauge parameter λ , corresponding to a constant electric potential, which must be fixed by a boundary condition^[17]. Using covariant perturbation theory and the boundary conditions of Ref. 17, we have checked Eq.(2.19) to $O[(ea)^2]$. [Eqs.(2.18) and (2.19) coincide if we fix the arbitrary gauge constant λ to $\lambda=7/12$.]

Since the Hamiltonian H and the charge operator Q commute, the eigenstates of H for $q \neq 0$ are given by the same linear combinations of the basis states as those listed in Table 2 for the $Q=0$ system. The basis states themselves are, of course, the appropriate $Q=q$ states, as constructed by applying the prescription given above.

3 CONFINED QED₂ WITH MASSIVE FERMIONS

When the fermions are given a mass, the Schwinger model is no longer exactly solvable^[13]. However, the physical spectrum of the model in the charge zero sector still contains only bosons, i.e., the fermions are still confined. We will study in this section the energies and densities of the fermions as a function of their mass. We will compare our results on the heavy quark wave functions with those obtained by Barnes in a 3+1 dimensional model^[12].

The results of Eqs.(2.12)-(2.16) are easily modified to incorporate a mass m [we distinguish heavy fermions (F) from light fermions (f)]. We will consider mainly systems containing one fermion-antifermion pair.

We will find below that already fermions with a mass $ma=10.0$ interact (in their lowest mode) very much as if they are infinitely heavy. However, we do not expect these fermions to have a narrow peak in the density. The fermion-density for the ground state is given by:

$$\rho_F^0(r) = \int dr' \psi_0^\dagger(r, r') \psi_0(r, r') \quad (3.1)$$

where

$$\psi_0(r, r') = \sum_i w_i u_{n_i \epsilon_i}(r) \otimes v_{m_i \eta_i}^*(r') \quad (3.2)$$

is the ground state product wave function in terms of the fermion basis states (see Table 1), which contribute each with a weight w_i . This density mainly reflects the wave function of the fermion-antifermion

bound state. When one compares it to the density of a fermion in its lowest cavity mode, one obtains a measure of the strength of the interactions between the fermions. We find that, for fermions with a mass $ma=10.0$, the "static cavity" approximation is still quite good. This is a signal of the weakness of the QED₂ interactions. One has to go to higher masses to find substantial deviations from the "static cavity" description. It is only for masses $ma \geq 100.0$, that one sees a clear localization of the fermion density near the origin of the one-dimensional bag.

3.1 The $F-\bar{F}$ system

In Fig.2 we show the dependence of the level splittings on the fermion mass m for $g^2=1.0$. For $g \equiv ea=0$, the splitting between different energy levels n and n' is due to the kinetic energy of the particles. For large m (i.e. $ma \gg n\pi$), the kinetic energy will go to zero, and all the energy levels become degenerate. Fig.2 shows that the initial decrease of the energy of a mode in the n 'th level slows down around $(ma)^{1/2} \approx n$, to approach a constant value. It is at this point that $O(g^2)$ effects start to dominate the energy. Mixing between this level and the lower-lying levels becomes noticeable, and one starts to see the effects of this mixing on the fermion densities. (The mixing between states in different levels is not included in Fig. 2.)

The mixing is dominated by the scattering diagram (Fig.1a), since the contribution of the annihilation diagram (Fig.1b) vanishes for $m \rightarrow \infty$ (it is proportional to the overlaps of large and small spinor components), and the self-energy terms (Figs.1c and 1d) reduce to: $\Sigma_{\infty}(k, \epsilon; k', \epsilon') = \lambda \delta_{kk'} \delta_{\epsilon\epsilon'}$. These cancel the gauge-dependent part of the scattering diagram. Then only the gauge-independent part of the scattering diagram remains. For finite mass this situation is already realized to 1.0 % accuracy for $ma = 10$. One should, therefore, regard a fermion with this mass as heavy.

In Fig.3 we show the density $(\psi^\dagger \psi)$ of one of the fermions in the ground state for selected masses and coupling constants. The mixing includes the contributions from the first five positive parity levels. Fig.2 shows that all positive parity levels are an energy $a\Delta=O(1)$ or more away from the ground state. For comparison, note that the "off-diagonal" matrix elements are $a\Delta \approx 1.25g^2/\pi^2$ or less, so mixing is small, especially with higher levels (omitting the fifth level hardly affects the results in Fig.3).

Fig. 3 demonstrates that there is an increased peaking of the density when the mass of the fermion is increased. However, most of this effect is already there for $g^2=0$ [the uniform density changes into a $\cos^2(\pi r/2a)$ one]. As seen from comparing the results for $g^2=1.0$ and 2.0 with those for $g^2=0.0$ in Fig.3, the additional localization, which comes from the constructive interference of several modes, is not very pronounced. In fact, for $ma=0$ and 1.0 , it is all but negligible. Only for $ma=10$, do we see an interesting effect.

Let us now try to compare our calculation with Barnes's treatment of the MIT Bag model^[12]. First note that also in the three-dimensional case the momentum eigenvalues become equally spaced when $mR \rightarrow \infty$ (R is the Bag radius). We have: $Rk_+ = n\pi$ for the $J^{P=1/2^+}$, and $Rk_- = (2n+1)\pi/2$ for the $J^{P=1/2^-}$ -modes ($n>0$); that is, all Rk values are shifted upwards by $\pi/2$ as compared with the corresponding one-dimensional values [comparing $J^{P=1/2^+}$ ($1/2^-$) modes with the positive (negative) parity modes]. The resulting radial densities $\rho(r) = r^{d-2} \psi^\dagger(r) \psi(r)$ are rather different at small r because of the factor r^{d-2} . Taking into account, however, that the Coulomb propagator behaves as $D_C(r, r') = |r-r'|^{3-d}$, the combinations $C = \int dr \rho(r) D_C(r, r')$ entering the matrix elements are qualitatively not too different. The analogy cannot be made more precise, since Barnes in his calculation makes two approximations. The diagonal $O(g^2)$ interaction terms are neglected, and only mixing between $J^{P=1/2^+}$ states is included. In this connection, three points can be made about our calculation:

1. At least for large mass, the result changes very little if one omits the self-energy contributions to the off-diagonal matrix elements.
2. Omitting the $O(g^2)$ interaction terms in the diagonal matrix elements makes a large difference. As seen from Figs.4a and 3d, the mixing effects in this case act delocalizing.
3. Including only mixing between states in which the fermions have the same parity, enhances the localization of the ground state (see Fig.4b). If we take the analogy between the two models at all seriously, this shows that the assumptions made by Barnes can indeed change the result, but we would also conclude that the "Coulomb spike" would remain, albeit with a different strength, after including all $O(g^2)$ interactions and mixing with other than the $J^{P=1/2^+}$ modes.

3.2 The $F\text{-}\bar{f}$ system

Let us first consider a system containing a massive fermion F and a massless antifermion $\bar{f}$. In Fig. 5, we plot the energy spectrum for the

first few levels of the system for $g^2=1.0$ (as in Fig.2 no mixing between the levels is considered here). One sees that when $m_F a$ becomes large, the $F-\bar{f}$ spectrum separates into bands. The emergence of bands is caused by the fact that the kinetic energy of the heavy fermion goes to zero, whereas that of the massless fermion stays the same. As a consequence, each $F\bar{f}$ level separates into individual states, each of which is a rather pure $F\bar{f}$ basis state (see Table 1). The bands, therefore, can be labeled for large ma by the mode of the light fermion.

Our results for the density of the heavy fermion in the ground state of the $F\bar{f}$ system are not significantly different from those in Fig.3. We, therefore, do not plot them separately. Also the mixing in the $F\bar{f}$ system is still rather weak for the values of the masses of Fig.3. It is only at the level of the composition of the wave function that important differences show up. The reason for this becomes clear when we compare Figs.2 and 5. In the limit of $m \rightarrow \infty$ only states which lie in the same band will mix appreciably with each other, and then only if they have the same parity. The $F\bar{f}$ ground state thus will dominantly mix with even-parity modes and become localized. In the $F\bar{F}$ case there is just one band and becoming localized is much harder in this situation. If we compare the wave functions for the $F\bar{f}$ and the $F\bar{F}$ ground state for $m_F a=100.$, we see that there is 0.8 % probability that the (light) antifermion is in another mode than the lowest one, compared to 6.1 % for the fermion. These numbers are 34.1 % and 2.2 % for the $F\bar{F}$ ground state.

When we compare Figs.3d and 6, we see that the heavy fermion in the $F-\bar{f}$ system is subject to stronger interactions than the fermion in the $F-\bar{F}$ system. This can be understood more easily by comparing with the result of the Born-Oppenheimer approach. We then see that, in the $F-\bar{f}$ case, the equivalent problem is that of a particle of mass ma moving in a harmonic oscillator potential $V(r)=e^2 r^2/a$ [20]. In the $F-\bar{F}$ case, the equivalent problem is that of a particle of mass $ma/2$ moving in a potential $V'(r)=e^2 r$.

3.3 Other configurations

Note that the situation described in the previous paragraphs and depicted in Fig.5 is a simplification, when we consider the third and higher bands. For $g=0$, and large m_F , each level in the third band is two-fold degenerate, because the light fermion part of the system can be made up of an antifermion in mode (5) or an $f\bar{f}^2$ combination in a

$(1+\bar{1}+\bar{3})$ configuration. (Only the $F\bar{f}$ configuration has been considered so far.) Letting g become nonzero, this degeneracy will be lifted and the two levels will be split by an energy $\Delta a = 1.5g^2/\pi^2$ (see Table 2, level 2). Since this splitting is comparable to the matrix elements for mixing within the band, levels in the third band will in general have two different light fermion components. Consequently, localization will not be as efficient as for lower bands.

It is easy to generalize our results to other fermion systems, such as Ff^2 and $F^2\bar{f}^2$. If the system contains only one heavy fermion, e.g. F of Ff^2 , the prospects for localization of its ground state are those found for the $F\bar{f}$ system. If, on the other hand, there are two or more heavy fermions, as in the $F^2\bar{f}^2$ system, the localization of the ground state wave function will be as in the $F\bar{F}$ system.

Level	Fermions	Bosons
1	$1+\bar{1}$	1
2	$1+\bar{3} \ 3+\bar{1}$	$1^2 \ 2$
3	$1+\bar{5} \ 3+\bar{3} \ 5+\bar{1}$	$1^3 \ 1+2 \ 3$
4	$1+\bar{7} \ 3+\bar{5} \ 5+\bar{3} \ 7+\bar{1} \ 1+\bar{1}+3+\bar{3}$	$1^4 \ 1^2+2 \ 1+3 \ 2^2 \ 4$
5	$1+\bar{9} \ 3+\bar{7} \ 5+\bar{5} \ 7+\bar{3} \ 9+\bar{1}$	$1^5 \ 1^3+2 \ 1^2+3 \ 1+2^2 \ 1+4$
	$1+\bar{1}+3+\bar{5} \ 1+\bar{1}+5+\bar{3}$	$2+3 \ 5$

Table 1. $Q=0$ Energy levels. The total energy (column 1) and the eigenenergies of the boson modes are given in units of $\pi/2a$, and the eigenenergies of the fermion modes are in units of $\pi/4a$. Bars indicate anti fermion modes, and powers the multiplicity of the boson modes.

Level	Shift Δ	Wave function
1	1	1
2	2	1 -1 * 1/√2
	$1/2$	1 1 * 1/√2
3	3	1 2 1 * 1/√6
	$3/2$	1 0 -1 * 1/√2
	$1/3$	1 -1 1 * 1/√3
4	4	1 -3 3 -1 2 * 1/√24
	$5/2$	1 -1 -1 1 0 * 1/√4
	$4/3$	1 0 0 -1 -1 * 1/√3
	1	1 1 -1 -1 2 * 1/√8
	$1/4$	1 1 1 1 0 * 1/√4
5	5	1 4 6 4 1 5 -5 * 1/√120
	$7/2$	1 2 0 -2 -1 1 1 * 1/√12
	$7/3$	1 1 0 1 1 -1 1 * 1/√6
	2	1 0 -2 0 1 1 -1 * 1/√8
	$5/4$	1 0 0 0 -1 -1 -1 * 1/√4
	$5/6$	1 -1 0 1 -1 1 1 * 1/√6
	$1/5$	1 -1 1 -1 1 0 0 * 1/√5

Table 2. Level splittings Δ in units of $(ea)^2/\pi^2a$. The eigenvectors are given in terms of the basis states listed in Table 1.

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FIGURE CAPTIONS

- Figure 1: Diagrams contributing to $O(e^2)$.
- Figure 2: The energy eigenvalues $\omega_a - m_a$ for the first few levels of the $F - \bar{F}$ system as a function of m_a .
- Figure 3: Density $\rho(x)$ of the fermion in the ground state of an $F - \bar{F}$ system for $m_a = 0.0$ (a), 1.0 (b), 10.0 (c), and 100.0 (d) and $g^2 = 0.0, 1.0$ and 2.0.
- Figure 4: Density $\rho(x)$ of the fermion in the ground state of an $F - \bar{F}$ system for $m_a = 100.0$ and $g^2 = 1.0$. Fig.4a shows the effect of omitting the diagonal $O(g^2)$ terms and Fig.4b the result of retaining only the off-diagonal transitions between like-parity modes. The integers denoted the number of levels included in the mixing.
- Figure 5: The energy eigenvalues $\omega_a - m_a$ for the first few levels of the $F - \bar{f}$ system as a function of m_a .
- Figure 6: Density $\rho(x)$ of the fermion in the ground state of an $F - \bar{f}$ system for the full calculation. Parameter values and notation are as in Fig.4.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6