CERN - DATA HANDLING DIVISION
DD/84/22
December 1984

REDUCE PROCEDURES FOR THE MANIPULATION OF
GENERALIZED POWER SERIES

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ABSTRACT

Based on methods discussed by Życzkowski some time ago, the paper presents procedures, written in REDUCE, for manipulating generalized power series. These series include ordinary ^{ow}paper series as special cases. The procedures handle the multiplication and division of two series, the raising to a power of a series, the substitution of one series into another, the reversion of a series, and the elimination of a common parameter from two series. A procedure is also given for the algorithm proposed by Thacher for a more general inversion problem for ordinary power series. Examples are discussed, and listings are given for the procedures, as well as for the output of the examples.

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1. INTRODUCTION

Power series appear in a wide range of applications, and their manipulation with respect to the more elementary operations, such as addition, multiplication and division, is in principle straightforward. Operations of a more advanced nature, for example raising a power series to a power, substituting one series into another, or reversion of a power series, are considerably more complicated, and the computation of the coefficients in the resulting series is in general quite tedious. The handbooks on mathematical functions usually give only rather limited information about such operations. On the other hand, Knuth [1, pp. 444-450] gives a thorough discussion, dealing in particular with raising a series to a power and reversion, including a certain generalization of the latter operation. He also gives recurrence relations for the coefficients of the series which are obtained by exponentiation or by taking the logarithm of a power series. A systematic discussion of operations on power series can also be found in Henrici [2, pp. 35-65]. Comtet [3, pp. 183-197] gives many useful identities and expansions related to operations on such series. Because these discussions usually leave aside the question of convergence, the operations are often called operations on "formal" power series.

Nearly twenty years ago, Życzkowski [4] systematically investigated the manipulation of so-called generalized power series (gps)

$$y(x) = \sum_{j=0}^{\infty} a_j x^{\mu+\nu j}, \quad (1.1)$$

where μ and ν are in principle arbitrary numbers. However, most of the operations considered impose certain restrictions on these values if the resultant series is to be a gps. Series of this type appear in a wide range of theoretical and practical applications. For μ and ν rational, they are often called Puiseux series, in particular in algebraic geometry.

It is clear that ordinary power series (with non-negative integer μ and ν) are special cases of (1.1), and one might ask whether the application of formulae

for generalized power series to ordinary power series do not give rise to unnecessary complications. Życzkowski [4] insists that it is often the contrary which is true, and that these formulae can make problems involving ordinary power series more transparent and their solution more economical.

Życzkowski's original intention [5,6] was to facilitate the tedious computation by hand of the coefficients $A_j(\xi)$ in the ξ -th power of a gps,

$$[y(x)]^\xi = \left(\sum_{j=0}^{\infty} a_j x^{\mu+\nu j} \right)^\xi = \sum_{j=0}^{\infty} A_j(\xi) x^{\mu\xi+\nu j}, \quad (1.2)$$

where ξ is an arbitrary real number. These coefficients are also needed for other operations, and can be calculated by using Faà di Bruno's formula [7, No. 24.1.2], written as

$$A_j(\xi) = \sum_{\rho=1}^{R(j)} \frac{\xi(\xi-1) \dots (\xi-m(j)+1)}{k_{1\rho}! k_{2\rho}! \dots k_{j\rho}!} a_0^{\xi-m(j)} a_1^{k_{1\rho}} \dots a_j^{k_{j\rho}}, \quad (1.3)$$

where the non-negative integers $k_{i\rho}$ are the $R(j)$ solutions of the diophantine equation

$$k_1 + 2k_2 + \dots + jk_j = j,$$

and where

$$m(j) = k_1 + k_2 + \dots + k_j.$$

This preliminary computation is in itself tedious, and Życzkowski [6] provided a table of the "weights" in formula (1.3) for $j = 1(1)4$ and 70 rational values of $\xi = \pm p/q$.

The arrival of computer formula manipulation systems has made such tabulations unnecessary. In fact, these systems can easily be used to compute the coefficients $A_j(\xi)$ by a well-known recurrence relation (see subsection 3.2).

As formula manipulation systems developed, several authors took up the problem of manipulating power series. For example, Norman [8] wrote a power series package for the SCRATCHPAD system, which handles, amongst other things, the raising to a power and reversion. Procedures for power series are also available in MACSYMA.

Harrington [9,10] discussed several aspects of manipulating power series and their implementation in REDUCE. However, these procedures do not form part of the recently released version of REDUCE [11].

It is therefore the purpose of the present paper, using a pragmatic approach, to present procedures, written in REDUCE, which will provide an easy-to-use tool for the manipulation of generalized power series. We also wish to draw attention to a paper [4] which treated this subject in a form appropriate to symbolic computation long before any such systems were generally available, and which does not seem to be well known.

The present report is intended to represent a long write-up for the different procedures. It therefore starts by explaining in some detail the mathematical background as developed by Życzkowski.

2. NOTATION

It is easy to see that the representation (1.1) is not unique. Życzkowski [4] therefore introduces four operations involving renumbering. Two of these allow a gps to be represented by a unique notation, and the two others are essential when performing operations on gps.

2.1 Contraction

In the case where $a_j = 0$ for $j < n$ ($n > 0$), we first write

$$y(x) = \sum_{j=n}^{\infty} a_j x^{\mu+\nu j} = \sum_{j=0}^{\infty} a_{j+n} x^{\mu+\nu(j+n)} = \sum_{j=0}^{\infty} \tilde{a}_j x^{\tilde{\mu}+\nu j}, \quad (2.1)$$

where $\tilde{a}_j = a_{j+n}$, $\tilde{\mu} = \mu+\nu n$. Further, if it happens that $\tilde{a}_j = 0$ for $j \neq mN$, $m = 0, 1, 2, \dots, N$, ($N > 1$), we set

$$y(x) = \sum_{j=0}^{\infty} \tilde{a}_j x^{\tilde{\mu}+\nu j} = \sum_{j=0}^{\infty} \tilde{a}_{jN} x^{\tilde{\mu}+\nu Nj} = \sum_{j=0}^{\infty} \tilde{\tilde{a}}_j x^{\tilde{\mu}+\tilde{\nu}j}, \quad (2.2)$$

where $\tilde{\tilde{a}}_j = \tilde{a}_{jN}$, $\tilde{\nu} = \nu N$.

As an example, we consider the series

$$\sin x^{1/3} = x^{1/3} - \frac{1}{3!} x + \frac{1}{5!} x^{5/3} - + \dots$$

Here $a_0 = a_2 = \dots = 0$, $a_1 = 1$, $a_3 = -1/3!$, ..., $\mu = 0$, and $\nu = 1/3$. Applying (2.1) with $n = 1$ and then (2.2) with $N = 2$ gives $\tilde{a}_0 = 1$, $\tilde{a}_1 = -1/3!$, ..., $\tilde{\mu} = 1/3$, $\tilde{\nu} = 2/3$, and hence

$$\sin x^{1/3} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{1/3+2/3j}$$

in "normalized" notation. This notation is characterized by the fact that $\tilde{a}_0 \neq 0$ and that there is no number $N > 1$ such that $\tilde{a}_j = 0$ for $j \neq mN$.

2.2 Extension

This is in a sense the inverse operation to contraction. Unlike contraction, extension is always possible. We can write

$$y(x) = \sum_{j=0}^{\infty} a_j x^{\mu+\nu j} = \sum_{j=n}^{\infty} a_{j-n} x^{\mu+\nu(j-n)} = \sum_{j=0}^{\infty} \tilde{a}_j x^{\tilde{\mu}+\tilde{\nu}j}, \quad (2.3)$$

where $\tilde{a}_j = 0$ for $j < n$, $\tilde{a}_j = a_{j-n}$ for $j \geq n$, and $\tilde{\mu} = \mu - \nu n$, n being an arbitrary positive integer. Further,

$$y(x) = \sum_{j=0}^{\infty} \tilde{a}_j x^{\tilde{\mu}+\tilde{\nu}j} = \sum_{j=0}^{\infty} \tilde{a}_{j/N} x^{\tilde{\mu}+(\nu/N)j} = \sum_{j=0}^{\infty} \tilde{\tilde{a}}_j x^{\tilde{\mu}+\tilde{\tilde{\nu}}j}, \quad (2.4)$$

where $\tilde{\tilde{a}}_j = \tilde{a}_{j/N}$ for $j = mN$, $m = 0, 1, 2, \dots$; $\tilde{\tilde{a}}_j = 0$ for $j \neq mN$, and $\tilde{\tilde{\nu}} = \nu/N$, N being an arbitrary positive integer.

3. OPERATIONS ON GENERALIZED POWER SERIES

Of the operations on gps described by Życzkowski [4], addition is of least interest and can be handled directly by any formula manipulation system. We therefore start the description of the procedure with

3.1 Multiplication

Let

$$y(x) = \sum_{j=0}^{\infty} a_j x^{\mu_1+\nu_1 j}, \quad z(x) = \sum_{k=0}^{\infty} b_k x^{\mu_2+\nu_2 k} \quad (3.1)$$

be two given gps (in normalized form), then formally

$$y(x)z(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_j b_k x^{\mu_1 + \mu_2 + \nu_1 j + \nu_2 k}. \quad (3.2)$$

In order to write this expression in the form of a gps according to formula (1.1), we have to find a way to replace $\nu_1 j + \nu_2 k$ in the power of x by a term $\nu_3(j+k)$. This is possible if the ratio ν_1/ν_2 is rational and positive. In this case, we can apply the extension procedure of subsection 2.2 and obtain

$$y(x)z(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_j \tilde{b}_k x^{\mu_1 + \mu_2 + \nu_3(j+k)}, \quad (3.3)$$

where

$$\begin{aligned} \tilde{a}_j &= a_{j/n_1} \quad \text{for } j = mn_1 \quad (m = 0, 1, 2, \dots), \\ \tilde{b}_k &= b_{k/n_2} \quad \text{for } k = mn_2 \quad (m = 0, 1, 2, \dots), \\ \tilde{a}_j &= \tilde{b}_k = 0 \quad \text{otherwise;} \\ \frac{n_1}{n_2} &= \frac{\nu_1}{\nu_2}, \quad \gcd(n_1, n_2) = 1, \end{aligned}$$

and $\nu_3 = \nu_1/n_1 = \nu_2/n_2$. Introducing a new index of summation p yields

$$y(x)z(x) = \sum_{j=0}^{\infty} \sum_{p=j}^{\infty} \tilde{a}_j \tilde{b}_{p-j} x^{\mu_1 + \mu_2 + \nu_3 p}$$

and hence, by changing the order of summation,

$$y(x)z(x) = \sum_{p=0}^{\infty} c_p x^{\mu_1 + \mu_2 + \nu_3 p}; \quad c_p = \sum_{j=0}^p \tilde{a}_j \tilde{b}_{p-j}. \quad (3.4)$$

3.2 Raising to a power

This operation is straightforward. Starting from

$$y(x) = \sum_{j=0}^{\infty} a_j x^{\mu + \nu j}$$

we obtain for arbitrary real ξ , and with $t = x^{\nu}$,

$$[y(x)]^\xi = x^{\mu\xi} \left(\sum_{j=0}^{\infty} a_j x^{\nu j} \right)^\xi = x^{\mu\xi} \left(\sum_{j=0}^{\infty} a_j t^j \right)^\xi .$$

The last series on the right-hand side is an ordinary power series, and we can apply the well-known recurrence relation due to Miller^{*)} (see, for example, Knuth [1, p. 445], Henrici [2, p. 42]), namely $A_0(\xi) = a_0^\xi$,

$$A_j(\xi) = \frac{1}{j a_0} \sum_{q=1}^j (\xi q + q - j) a_q A_{j-q}(\xi) , \quad (3.5)$$

in order to find the coefficients $A_j(\xi)$ in

$$\left(\sum_{j=0}^{\infty} a_j t^j \right)^\xi = \sum_{j=0}^{\infty} A_j(\xi) t^j ,$$

which finally yields

$$[y(x)]^\xi = \left(\sum_{j=0}^{\infty} a_j x^{\mu+\nu j} \right)^\xi = \sum_{j=0}^{\infty} A_j(\xi) x^{\mu\xi+\nu j} . \quad (3.6)$$

3.3 Division

The division of two gps can be treated as a raising to the power -1 followed by multiplication. Let the series for $y(x)$ and $z(x)$ be given by (3.1). We then have, according to (3.2) and (3.6),

$$\frac{y(x)}{z(x)} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_j B_k(-1) x^{\mu_1-\mu_2+\nu_1 j+\nu_2 k} ,$$

where the coefficients $B_k(-1)$ can be computed by a simplified recurrence (3.5), namely

*) Note that the restriction "n is a natural number" in [12, No. 0.314] is superfluous.

$$B_k(-1) = -\frac{1}{b_0} \sum_{q=1}^k b_q B_{k-q}(-1), \quad B_0(-1) = 1/b_0. \quad (3.7)$$

By extension and the use of formula (3.4) we obtain, in the case of v_1/v_2 positive rational,

$$\frac{y(x)}{z(x)} = \sum_{p=0}^{\infty} c_p x^{\mu_1 - \mu_2 + v_3 p}, \quad c_p = \sum_{j=0}^p \tilde{a}_j \tilde{B}_{p-j}(-1), \quad (3.8)$$

where n_1, n_2, v_3 , and $\tilde{a}_j$ are defined as in the case of multiplication (subsection 3.1), and where $\tilde{B}_{p-j}(-1) = B_{(p-j)/n_2}(-1)$ for $j = mn_2$ ($m = 0, 1, 2, \dots$), $\tilde{B}_{p-j}(-1) = 0$ otherwise.

3.4 Substitution

We consider the two gps

$$z(y) = \sum_{j=0}^{\infty} a_j y^{\mu_1 + v_1 j} \quad (3.9)$$

and

$$y(x) = \sum_{k=0}^{\infty} b_k x^{\mu_2 + v_2 k}, \quad (\mu_2 \neq 0), \quad (3.10)$$

and derive a gps for $z(x) = z(y(x))$. Using (3.6), we obtain

$$\begin{aligned} z(x) &= \sum_{j=0}^{\infty} a_j \left(\sum_{k=0}^{\infty} b_k x^{\mu_2 + v_2 k} \right)^{\mu_1 + v_1 j} \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_j B_k(\mu_1 + v_1 j) x^{\mu_1 \mu_2 + \mu_2 v_1 j + v_2 k}, \end{aligned}$$

which is in the form of a product to which we can apply the results of subsection 3.1. Under the assumption that $\mu_2 v_1/v_2$ is positive rational (note that $\mu_2 \neq 0$ is necessary), we define

$$\frac{n_1}{n_2} = \mu_2 \frac{v_1}{v_2}, \quad \gcd(n_1, n_2) = 1, \quad (3.11)$$

and

$$v_3 = v_2/n_2 = \mu_2 v_1/n_1,$$

and obtain by extension

$$z(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_j \tilde{B}_k \left(\mu_1 + \frac{\nu_1}{n_1} j \right) x^{\mu_1 \mu_2 + \nu_3 (j+k)},$$

where

$$\begin{aligned} \tilde{a}_j &= a_{j/n_1} \quad \text{for } j = mn_1 \quad (m = 0, 1, 2, \dots), \\ \tilde{B}_k(\mu_1 + \nu_1 j/n_1) &= B_{k/n_2}(\mu_1 + \nu_1 j/n_1) \quad \text{for } k = mn_2 \quad (m = 0, 1, 2, \dots), \\ \tilde{a}_j &= \tilde{B}_k(\mu_1 + \nu_1 j/n_1) = 0 \quad \text{otherwise.} \end{aligned}$$

By changing the order of summation, we finally get

$$z(x) = \sum_{p=0}^{\infty} c_p x^{\mu_1 \mu_2 + \nu_3 p}, \quad (3.12)$$

where

$$c_p = \sum_{j=0}^p \tilde{a}_j \tilde{B}_{p-j} \left(\mu_1 + \frac{\nu_1}{n_1} j \right). \quad (3.13)$$

3.5 Reversion

The reversion of a power series is an important operation, and has received considerable attention in the past. Życzkowski [4] obtained a solution to the problem in the case of a gps by noticing that if one puts $z(y) = x$ in (3.9), where the series (3.10) for $y(x)$ is taken as given, the substitution procedure in subsection 3.4 can be used to obtain the coefficients a_j in (3.9), and hence those in the reversed series $x = x(y)$ of $y = y(x)$. From (3.12) we have

$$x = \sum_{p=0}^{\infty} c_p x^{\mu_1 \mu_2 + \nu_3 p}. \quad (3.14)$$

Thus $\mu_1 \mu_2 = 1$, $c_0 = 1$, and $c_p = 0$ ($p \geq 1$), with ν_3 arbitrary. Writing μ for μ_2 , ν for ν_2 , and $\bar{b}_j(\mu, \nu)$ for a_j , it follows that

$$\mu_1 = 1/\mu, \quad (\mu \neq 0)^*.$$

*) Note that the condition $\mu = 0$ is not really restrictive. It can be overcome by setting $Y(x) = y(x) - b_0$ and renumbering appropriately.

Because v_3 is arbitrary, we can choose $n_1 = n_2 = 1$. From (3.11), this gives

$$v_1 = v/\mu$$

and, by substitution into (3.9),

$$x = \sum_{j=0}^{\infty} \bar{b}_j(\mu, v) y^{1/\mu + (v/\mu)j} \quad (3.15)$$

as the reversed power series corresponding to the gps (3.10). From (3.5) and (3.13) we obtain, comparing with (3.14),

$$\bar{b}_0(\mu, v) b_0^{1/\mu} = 1, \quad \sum_{j=0}^p \bar{b}_j(\mu, v) B_{p-j} \left(\frac{1}{\mu} + \frac{v}{\mu} j \right) = 0.$$

This system of linear equations for $\bar{b}_j(\mu, v)$ can be solved recursively to give

$$\bar{b}_0(\mu, v) = b_0^{-1/\mu}, \quad (3.16)$$

$$\bar{b}_p(\mu, v) = -b_0^{-1/\mu - (v/\mu)p} \sum_{j=0}^{p-1} \bar{b}_j(\mu, v) B_{p-j} \left(\frac{1}{\mu} + \frac{v}{\mu} j \right).$$

Życzkowski [4] gives a closed formula for $\bar{b}_j(\mu, v)$ which is similar to formula (1.3) for the power of a gps. He remarks, however, that this formula had not yet been proved.

3.6 Elimination of a common parameter

Using results obtained in previous sections, Życzkowski [4] was able to solve, with relatively little effort, the complicated problems of eliminating a common parameter t from two gps. Let

$$y(t) = \sum_{j=0}^{\infty} a_j t^{\mu_1 + v_1 j}, \quad x(t) = \sum_{k=0}^{\infty} b_k t^{\mu_2 + v_2 k} \quad (3.17)$$

($\mu_1 \neq 0, \mu_2 \neq 0$)

be the two gps, and suppose that we wish to express $y = y(x)$ by a new gps. We introduce the variable $t^{\mu_1 + v_1 j}$ in (3.17) and write

$$x(t) = \sum_{k=0}^{\infty} b_k \left(t^{\mu_1 + \nu_1 j} \right)^{\mu_2 / (\mu_1 + \nu_1 j) + [\nu_2 / (\mu_1 + \nu_1 j)]k}$$

which gives, by reversion according to subsection 3.5,

$$t^{\mu_1 + \nu_1 j} = \sum_{k=0}^{\infty} \bar{b}_k \left(\frac{\mu_2}{\mu_1 + \nu_1 j}, \frac{\nu_2}{\mu_1 + \nu_1 j} \right) x^{(\mu_1 + \nu_1 j) / \mu_2 + (\nu_2 / \mu_2)k},$$

($\mu_2 \neq 0$) .

Substituting this result into (3.17) yields

$$y(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_j \bar{b}_k \left(\frac{\mu_2}{\mu_1 + \nu_1 j}, \frac{\nu_2}{\mu_1 + \nu_1 j} \right) x^{\mu_1 / \mu_2 + (\nu_1 / \mu_2)j + (\nu_2 / \mu_2)k}.$$

As in the case of multiplication (subsection 3.1), we see that a reduction to a gps is possible if ν_1 / ν_2 is positive rational, and we define

$$\frac{n_1}{n_2} = \frac{\nu_1}{\nu_2}, \quad \gcd(n_1, n_2) = 1,$$

and

$$\nu_3 = \frac{\nu_1}{n_1 \mu_2} = \frac{\nu_2}{n_2 \mu_2}.$$

Hence, after appropriate extension,

$$y(x) = \sum_{p=0}^{\infty} c_p x^{\mu_1 / \mu_2 + \nu_3 p}, \tag{3.18}$$

where

$$c_p = \sum_{j=0}^p \tilde{a}_j \tilde{b}_{p-j} \left(\frac{\mu_2}{\mu_1 + \frac{\nu_1}{n_1} j}, \frac{\nu_2}{\mu_2 + \frac{\nu_1}{n_1} j} \right) \tag{3.19}$$

and

$$\begin{aligned} \tilde{a}_j &= a_{j/n_1} \text{ for } j = mn_1 \quad (m = 0, 1, 2, \dots). \\ \tilde{b}_{p-j} &= \bar{b}_{(p-j)/n_2} \text{ for } p-j = mn_2 \quad (m = 0, 1, 2, \dots) \\ \tilde{a}_j &= \tilde{b}_{p-j} = 0 \text{ otherwise.} \end{aligned}$$

4. A GENERALIZED REVERSION OF ORDINARY POWER SERIES

For reasons of completeness, we add an algorithm for solving the problem of reversion for the more general case

$$\sum_{j=1}^{\infty} v_j y^j = \sum_{j=1}^{\infty} u_j x^j, \quad (4.1)$$

where the given series are ordinary power series with the special condition $v_1 = 1$. The result is given in the form of an ordinary power series,

$$y(x) = \sum_{k=1}^{\infty} c_k x^k. \quad (4.2)$$

This algorithm was established by Thatcher [13] and is reproduced in Knuth [1, pp. 449-450]. Note that the condition $v_1 \neq 0$ involves a loss of generality. For example, the series $f(x) = x^2 + x^6$ can be written as $f(t) = t + t^3$, by substituting $t = x^2$, and the algorithm can be applied. However, for $f(x) = x^2 + x^3$ this would be impossible.

For convenience, we give here the first five coefficients c_k :

$$c_1 = u_1$$

$$c_2 = -u_1^2 v_2 + u_2$$

$$c_3 = u_1^3 (2v_2^2 - v_3) - 2u_1 u_2 v_2 + u_3$$

$$c_4 = u_1^4 (-5v_2^3 + 5v_2 v_3 - v_4) + 3u_1^2 (2u_2 v_2^2 - v_2 v_3) - 2u_1 u_3 v_2 - u_2^2 v_2 + u_4$$

$$c_5 = u_1^5 (14v_2^4 - 21v_2^2 v_3 + 6v_2 v_4 + 3v_3^2 - v_5) + 4u_1^3 (-5u_2 v_2^3 + 5u_2 v_2 v_3 - u_2 v_4) + 3u_1^2 (2u_3 v_2^2 - u_3 v_3) + u_1 (6u_2^2 v_2^2 - 3u_2^2 v_3 - 2u_4 v_2) - 2u_2 v_3 v_2 + u_5.$$

We also note that Kamber [14] has treated the problem of reversing the series

$$x^\mu = y^\mu [1 + \mu(c_1 y + c_2 y^2 + \dots)]$$

to obtain

$$y^v = x^v [1 + v(b_{v,1}(\mu)x + b_{v,2}(\mu)x^2 + \dots)],$$

where μ and v are complex numbers. He gives the recurrence relation for the coefficients $b_{v,k}(\mu)$, as well as a closed expression in the form of a k -th order determinant.

5. THE REDUCE PROCEDURES

In the following description of the procedures, N represents a non-negative integer, and X a dummy variable which can be replaced by any other (valid) kernel. A and B are one-dimensional arrays in which $A(j)$ ($j = 0, 1, \dots, N$) and $B(k)$ ($k = 0, 1, \dots, N$) contain the first $N + 1$ coefficients a_j and b_k , respectively. The procedures return as result the expression consisting of the first $N + 1$ terms of the series in X , or an error message.

The program package consists of the following six procedures:

i) MPYPOW($N, A, \mu_1, \nu_1, B, \mu_2, \nu_2, X$)

multiplies the (normalized) gps

$$y(x) = \sum_{j=0}^N a_j x^{\mu_1 + \nu_1 j}, \quad z(x) = \sum_{k=0}^N b_k x^{\mu_2 + \nu_2 k},$$

giving the result $y(x)z(x)$.

ii) POTPOW(N, A, μ, ν, ξ, X)

raises the (normalized) gps

$$y(x) = \sum_{j=0}^N a_j x^{\mu + \nu j}$$

to the power ξ , giving the result $[y(x)]^\xi$.

iii) DIVPOW($N, A, \mu_1, \nu_1, B, \mu_2, \nu_2, X$)

divides the (normalized) gps

$$y(x) = \sum_{j=0}^N a_j x^{\mu_1 + \nu_1 j} \quad \text{by} \quad z(x) = \sum_{k=0}^N b_k x^{\mu_2 + \nu_2 k},$$

giving the result $y(x)/z(x)$.

iv) SUBPOW(N,A,MU1,NU1,B,MU2,NU2,X)

substitutes

$$y(x) = \sum_{k=0}^N b_k x^{\mu_2 + \nu_2 k} \text{ into } z(y) = \sum_{j=0}^N a_j y^{\mu_1 + \nu_1 j},$$

giving the result $z(y(x))$. Both series are (normalized) gps.

v) REVPOW(N,B,MU,NU,X)

reverses the (normalized) gps

$$y(x) = \sum_{k=0}^N b_k x^{\mu + \nu k},$$

giving the result $x(y)$.

vi) DEPPOW(N,A,MU1,NU1,B,MU2,NU2,X)

eliminates the parameter t from the (normalized) gps

$$y(t) = \sum_{j=0}^N a_j t^{\mu_1 + \nu_1 j}, \quad x(t) = \sum_{k=0}^N b_k t^{\mu_2 + \nu_2 k},$$

giving the result $y(x)$.

vii) GRVPOW(N,A,B,X)

solves the problem of reversing the two (ordinary) power series

$$\sum_{j=1}^N v_j y^j = \sum_{j=1}^N u_j x^j \quad (v_1 = 1),$$

giving the result

$$y(x) = \sum_{k=1}^N c_k x^k.$$

6. ERROR EXITS

The following error exists can occur:

- i) For MPYPOW: $NU2 = 0, \quad NU1/NU2 \leq 0.$
- ii) For POTPOW: None.
- iii) For DIVPOW: $NU2 = 0, \quad NU1/NU2 \leq 0.$

- iv) For SUBPOW: $NU2 = 0, MU2*NU1/NU2 \leq 0$.
- v) For REVPOW: $MU = 0$.
- vi) For DEPPOW: $NU2 = 0, NU1/NU2 \leq 0$.
- vii) For GRVPOW: $U(0) \neq 0, V(0) \neq 0, V(1) \neq 1$.

In each case the procedure returns a message, for example

$NU1/NU2 \leq 0$ IN DIVPOW .

Note, however, that there is no test on whether $NU1/NU2$ in MPYPOW, DIVPOW, or DEPPOW, or $MU2*NU1/NU2$ in SUBPOW, is a rational number.

7. EXAMPLES

We now present a few examples to illustrate the use of the routines. For simplicity, we consider mainly series with numerical coefficients. It is clear that these coefficients can in general be expressions.

Example 1 (in part from Życzkowski [4])

Let

$$z(y) = ye^{\sqrt{y}} = \sum_{j=0}^{\infty} \frac{1}{j!} y^{1+\frac{1}{2}j} = y + y^{3/2} + \frac{1}{2} y^2 + \frac{1}{6} y^{5/2} + \dots,$$

where $y = y(x)$ is defined by the equation

$$y^3 + y^4 = x^2 + x^6. \quad (7.1)$$

We look for a gps $z = z(x)$ in the neighbourhood of $(x,y) = (0,0)$, $x > 0$.

By setting $t = x^2$ in (7.1) we obtain

$$y^3 + y^4 = t + t^3,$$

which allows us to apply the procedure GRVPOW, with $a_3 = a_4 = b_1 = b_3 = 1$,

$a_j = b_k = 0$ otherwise. This results in

$$t(y) = y^3 + y^4 - y^9 - 3y^{10} - 3y^{11} - y^{12} + 3y^{15} + 15y^{16} + \dots$$

By reversion with REVPOW we obtain, with $\mu = 3$, $\nu = 1$,

$$y(t) = t^{1/3} - \frac{1}{3} t^{2/3} + \frac{1}{3} t - \frac{35}{81} t^{4/3} + \frac{154}{243} t^{5/3} + \dots$$

Performing the substitution with SUBPOW and remembering that $t = x^2$, yields, with $\mu_1 = 1$, $\nu_1 = 1/2$, $\mu_2 = 2/3$, $\nu_2 = 2/3$,

$$z(x) = x^{2/3} + x + \frac{1}{6} x^{4/3} - \frac{1}{3} x^{5/3} + \frac{1}{24} x^2 + \dots$$

Example 2 (Knuth [1, pp. 114-115])

Knuth considers, for $x \rightarrow \infty$, the integral

$$I = \int_0^\infty e^{-xv} \left(1 + \frac{1}{u}\right) dv,$$

where

$$v = u - \ln(1 + u). \quad (7.2)$$

In order to get an asymptotic expansion for I , he develops $1 + 1/u$ in terms of $v^{1/2}$.

From (7.2), we get

$$2v = u^2 \left(1 - \frac{2}{3} u + \frac{1}{2} u^2 - \frac{2}{5} u^3 + \dots\right).$$

Taking the square root of the series with POTPOW ($\mu = 0$, $\nu = 1$) yields

$$w = \sqrt{2v} = u - \frac{1}{3} u^2 + \frac{7}{36} u^3 - \frac{73}{540} u^4 + \frac{1331}{12960} u^5 - \dots$$

By inversion with REVPOW ($\mu = 1$, $\nu = 1$),

$$u = w + \frac{1}{3} w^2 + \frac{1}{36} w^3 - \frac{1}{270} w^4 + \frac{1}{4320} w^5 + \dots,$$

and, by applying the POTPOW with $\xi = -1$ we obtain the reciprocal

$$\begin{aligned} 1 + \frac{1}{u} &= \frac{1}{w} + \frac{2}{3} + \frac{1}{12} w - \frac{2}{135} w^2 + \frac{1}{864} w^3 + \frac{1}{2835} w^4 + \dots \\ &= \frac{1}{\sqrt{2}} v^{-1/2} + \frac{2}{3} + \frac{\sqrt{2}}{12} v^{1/2} - \frac{4}{135} v + \frac{\sqrt{2}}{432} v^{3/2} + \frac{4}{2835} v^2 + \dots \end{aligned}$$

With the help of

$$\int_0^{\infty} e^{-xv} v^{\alpha} dv = \frac{\Gamma(\alpha+1)}{x^{\alpha+1}} \quad (\alpha > -1) ,$$

the desired expansion can then be easily calculated.

Example 3 (Życzkowski [4])

Given

$$y(x) = \frac{1}{2} (\sinh x \cos x + \cosh x \sin x) ,$$

to find $x = x(y)$.

We start with the known power series for $\sinh x$, $\cosh x$, $\sin x$, and $\cos x$ and obtain, by using MPYPOW twice,

$$y(x) = \sum_{j=0}^{\infty} b_j x^{1+4j} = x - \frac{1}{30} x^5 + \frac{1}{22680} x^9 - \frac{1}{97297200} x^{13} + \dots$$

With the help of REVPOW, we get

$$x(y) = y + \frac{1}{30} y^4 + \frac{25}{4536} y^9 + \frac{1655}{1297296} y^{13} + \dots \quad (7.3)$$

Życzkowski uses this example to show why his method, applied to the special case of ordinary power series, is usually more transparent and economical than classical methods. To illustrate this, consider the procedure for reversion of an ordinary power series given in the handbook [7, No. 3.6.25]. There the coefficients $A, B, \dots, G$ are given explicitly as functions of $a, b, \dots, g$, where

$$y(x) = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + gx^7 + \dots$$

and

$$x(y) = Ay + By^2 + Cy^3 + Dy^4 + Ey^5 + Fy^6 + Gy^7 + \dots$$

In order to obtain (7.3), instead of computing only $\bar{b}_0, \bar{b}_1, \bar{b}_2$, and $\bar{b}_3$ according to (3.16), one would have to compute thirteen coefficients, of which only seven are given in [7], and most of them would evaluate to zero.

Example 4 (Kamke [15, p. 374])

Kamke gives the solution of the differential equation

$$yy'^2 + 2xy' - 9y = 0$$

in the parametric form

$$x = \frac{1}{14} t + Ct^{1/8}, \quad y^2 = \frac{1}{9} tx + \frac{1}{36} t^2.$$

We shall express the solution as a power series $y = y(x)$, valid in a neighbourhood of $(x,y) = (0,0)$, $x > 0$, for the case $C = 1$. From

$$y^2 = t^{9/8} \left(\frac{1}{9} + \frac{1}{28} t^{7/8} \right)$$

and using POTPOW with $a_0 = 1/9$, $a_1 = 1/28$, $a_j = 0$ for $j \geq 2$, $\mu = 9/8$, $\nu = 7/8$, $\xi = 1/2$, we obtain

$$y(t) = t^{9/16} \left(\frac{1}{3} + \frac{3}{56} t^{7/8} - \frac{27}{6272} t^{7/4} + \frac{243}{351232} t^{21/8} + \dots \right).$$

The use of DEPPPOW with $b_0 = 1$, $b_1 = 1/14$, $b_j = 0$ for $j \geq 2$, $\mu_1 = 9/16$, $\nu_1 = 7/8$, $\mu_2 = 1/8$, $\nu_2 = 7/8$, gives finally

$$y(x) = x^{9/2} \left(\frac{1}{3} - \frac{3}{56} x^7 + \frac{165}{6272} x^{14} - \frac{6451}{351232} x^{21} + \dots \right).$$

Example 5 (Thacher [13])

As an example of the application of his algorithm for series reversion, Thacher considers the equation

$$Y^{Y/\alpha} = X, \quad \text{or} \quad \frac{1}{\alpha} Y \ln Y = \ln X, \quad (\alpha \neq 0)$$

for $\alpha = 1$. Substituting $Y = 1 + y$, $X = 1 + x$ yields

$$(1 + y) \ln(1 + y) = \alpha \ln(1 + x),$$

and we require a power series $y = y(x)$ valid in a neighbourhood of $(x,y) = (0,0)$.

Writing

$$\sum_{j=1}^{\infty} v_j y^j = \sum_{j=1}^{\infty} u_j x^j ; \quad y(x) = \sum_{k=1}^{\infty} c_k x^k ,$$

it is easy to see that

$$u_j = -\alpha(-1)^j/j ,$$

and

$$v_1 = 1, \quad v_j = \frac{(-1)^j}{j(j-1)} , \quad (j \geq 2) .$$

Applying GRVPOW yields

$$c_1 = \alpha$$

$$c_2 = -\frac{1}{2} \alpha(\alpha+1)$$

$$c_3 = \alpha \left(\frac{2}{3} \alpha^2 + \frac{1}{2} \alpha + \frac{1}{3} \right)$$

$$c_4 = -\alpha \left(\frac{9}{8} \alpha^3 + \alpha^2 + \frac{11}{24} \alpha + \frac{1}{4} \right)$$

$$c_5 = \alpha \left(\frac{32}{15} \alpha^4 + \frac{9}{4} \alpha^3 + \frac{7}{6} \alpha^2 + \frac{5}{12} \alpha + \frac{1}{5} \right)$$

$$c_6 = -\alpha \left(\frac{625}{144} \alpha^5 + \frac{16}{3} \alpha^4 + \frac{51}{16} \alpha^3 + \frac{5}{4} \alpha^2 + \frac{137}{360} \alpha + \frac{1}{6} \right)$$

....

For $\alpha = 1$, we obtain

$$y(x) = x \left(1 - x + \frac{3}{2} x^2 - \frac{17}{6} x^3 + \frac{37}{6} x^4 - \frac{1759}{120} x^5 + \frac{13279}{360} x^6 - \dots \right)$$

in accordance with [13].

Example 6 (Życzkowski [4])

The following problem occurs in plasticity theory. Let

$$\beta(u) = 2 \sqrt{u - u^2} = 2 \sum_{j=0}^{\infty} c_j u^{1/2+j} ,$$

where $c_0 = 1$,

$$c_j = \frac{2j-3}{2j} c_{j-1} , \quad (j \geq 1) ,$$

and

$$y(t) = \int_0^t u\beta(u) du = \sum_{j=0}^{\infty} a_j t^{5/2+j}$$

$$x(t) = \int_0^t \beta(u) du = \sum_{j=0}^{\infty} b_j t^{3/2+j},$$

where

$$a_j = 2c_j / \left(\frac{5}{2} + j\right), \quad b_j = 2c_j / \left(\frac{3}{2} + j\right).$$

We look for a gps $y = y(x)$. Using DEPPOW, with $\mu_1 = 5/2$, $\nu_1 = 1$, $\mu_2 = 3/2$, $\nu_2 = 1$, we obtain

$$y(x) = \sum_{p=0}^{\infty} c_p x^{5/3+2/3p}$$

$$= \frac{3^{5/3}}{5.2^{4/3}} x^{5/3} + \frac{3^{7/3}}{35.2^{8/3}} x^{7/3} + \frac{33}{1400} x^3 + \frac{823.3^{5/3}}{9625.2^{16/3}} x^{11/3}$$

$$+ \frac{150653.3^{7/3}}{4379375.2^{17/3}} x^{13/3} + \dots$$

Example 7

An electrostatics problem arising in the design of particle detectors requires the power series of the function

$$z(x) = \frac{e^{\sqrt{1+x}-1}}{\sqrt{1+x}} = 1 + a_1 x + a_2 x^2 + \dots$$

for small values of $|x|$. The square root can be computed, either by the binomial series, or by using POTPOW with $a_0 = a_1 = 1$, $a_j = 0$ ($j > 1$), $\mu = 0$, $\nu = 1$, $\xi = \frac{1}{2}$.

Using POTPOW and then SUBPOW, we obtain with $\mu_1 = 0$, $\nu_1 = 1$, $\mu_2 = 1$, $\nu_2 = 1$,

$$e^{\sqrt{1+x}-1} = 1 + \frac{1}{2} x + \frac{1}{48} x^3 - \frac{5}{384} x^4 + \frac{3}{320} x^5 + \dots$$

and hence, using DIVPOW ($\mu_1 = \mu_2 = 0$, $\nu_1 = \nu_2 = 1$),

$$z(x) = 1 + \frac{1}{8} x^2 - \frac{5}{48} x^3 + \frac{3}{32} x^4 - \frac{329}{3840} x^5 + \dots$$

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```

PROCEDURE MPYPOW(N,A,MU1,NU1,B,MU2,NU2,X);
BEGIN SCALAR J,P,H,N1,N2,NU3;
  H:=NU1/NU2;
  IF H <= 0 THEN RETURN "NU1/NU2 <= 0 IN MPYPOW";
  N1:=NUM(H);
  N2:=DEN(H);
  NU3:=NU1/N1;
  RETURN X**((MU1+MU2)*(FOR P:=0:N SUM
    (FOR J:=0:P SUM IF FIXP(J/N1) AND FIXP((P-J)/N2)
      THEN A(J/N1)*B((P-J)/N2) ELSE 0))*X**(NU3*P));
END;

```

```

PROCEDURE POTPOW(N,A,MU,NU,XI,X);
BEGIN SCALAR J,P,K1,C(N); ARRAY C(N);
  K1:=XI+1;
  C(0):=A(0)**XI;
  FOR J:=1:N DO
    C(J):=(FOR P:=1:J SUM (K1*P-J)*A(P)*C(J-P))/(J*A(0));
  RETURN X**((MU*XI)*(FOR J:=0:N SUM C(J)*X**(NU*J)));
END;

```

```

PROCEDURE DIVPOW(N,A,MU1,NU1,B,MU2,NU2,X);
BEGIN SCALAR J,K,P,H,N1,N2,NU3,C(N); ARRAY C(N);
  H:=NU1/NU2;
  IF H <= 0 THEN RETURN "NU1/NU2 <= 0 IN DIVPOW";
  N1:=NUM(H);
  N2:=DEN(H);
  NU3:=NU1/N1;
  C(0):=1/B(0);
  FOR K:=1:N DO
    C(K):=-(FOR P:=1:K SUM B(P)*C(K-P))/B(0);
  RETURN X**((MU1-MU2)*(FOR P:=0:N SUM
    (FOR J:=0:P SUM IF FIXP(J/N1) AND FIXP((P-J)/N2)
      THEN A(J/N1)*C((P-J)/N2) ELSE 0))*X**(NU3*P));
END;

```

```

PROCEDURE SUBPOW(N,A,MU1,NU1,B,MU2,NU2,X);
BEGIN SCALAR J,K,P,H,N1,N2,NU3,K1,B1(N,N); ARRAY B1(N,N);
  H:=MU2*NU1/NU2;
  IF H <= 0 THEN RETURN "MU2*NU1/NU2 <= 0 IN SUBPOW";
  N1:=NUM(H);
  N2:=DEN(H);
  NU3:=NU2/N2;
  FOR K:=0:N DO
    << K1:=MU1+NU1*K/N1+1;
    B1(0,K):=B(0)**(K1-1);
    FOR J:=1:N DO
      B1(J,K):=(FOR P:=1:J SUM (K1*P-J)*B(P)*B1(J-P,K))/(J*B(0)) >>;
  RETURN X**((MU1*MU2)*(FOR P:=0:N SUM
    (FOR J:=0:P SUM IF FIXP(J/N1) AND FIXP((P-J)/N2)
      THEN A(J/N1)*B1((P-J)/N2,J) ELSE 0))*X**(NU3*P));
END;

```

```

PROCEDURE REVPOW(N,B,MU,NU,X);
BEGIN SCALAR J,K,P,K1,B1(N,N),C(N); ARRAY B1(N,N),C(N);
  IF MU = 0 THEN RETURN "MU = 0 IN REVPOW";
  C(0):=1/B(0)**(1/MU);
  FOR P:=0:N DO
    << K1:=(1+NU*P)/MU+1;
      B1(0,P):=B(0)**(K1-1);
      FOR J:=1:N DO
        B1(J,P):=(FOR K:=1:J SUM (K1*K-J)*B(K)*B1(J-K,P))/(J*B(0)) >>;
  FOR P:=1:N DO
    C(P):=-(FOR J:=0:P-1 SUM C(J)*B1(P-J,J))/B(0)**((1+NU*P)/MU);
  RETURN X**(1/MU)*(FOR P:=0:N SUM C(P)*X**(NU*P/MU));
END;

```

```

PROCEDURE DEPPOW(N,A,MU1,NU1,B,MU2,NU2,X);
BEGIN SCALAR J,K,P,Q,H,N1,N2,NU3,MUK,NUK,K1,B1(N,N),C(N,N);
  ARRAY B1(N,N),C(N,N);
  IF MU1 = 0 THEN RETURN "MU1 = 0 IN DEPPOW";
  IF MU2 = 0 THEN RETURN "MU2 = 0 IN DEPPOW";
  H:=NU1/NU2;
  IF H <= 0 THEN RETURN "NU1/NU2 <= 0 IN DEPPOW";
  N1:=NUM(H);
  N2:=DEN(H);
  NU3:=NU1/(N1*MU2);
  FOR K:=0:N DO
    << MUK:=MU2/(MU1+NU1*K/N1);
      NUK:=NU2/(MU1+NU1*K/N1);
      B1(0,K):=1/B(0)**(1/MUK);
      FOR P:=0:N DO
        << K1:=(1+NUK*P)/MUK+1;
          C(0,P):=B(0)**(K1-1);
          FOR J:=1:N DO
            C(J,P):=(FOR Q:=1:J SUM (K1*Q-J)*B(Q)*C(J-Q,P))/(J*B(0)) >>;
      FOR P:=1:N DO
        B1(P,K):=-(FOR J:=0:P-1 SUM B1(J,K)*C(P-J,J))/
          B(0)**((1+NUK*P)/MUK) >>;
  RETURN X**(MU1/MU2)*(FOR P:=0:N SUM
    (FOR J:=0:P SUM IF FIXP(J/N1) AND FIXP((P-J)/N2)
      THEN A(J/N1)*B1((P-J)/N2,J) ELSE 0)*X**(NU3*P));
END;

```

```

PROCEDURE GRVPOW(N,U,V,X);
BEGIN SCALAR J,K,C(N,N); ARRAY C(N,N);
  IF U(0) NEQ 0 THEN RETURN "U(0) NEQ 0 IN GRVPOW";
  IF V(0) NEQ 0 THEN RETURN "V(0) NEQ 0 IN GRVPOW";
  IF V(1) NEQ 1 THEN RETURN "V(1) NEQ 1 IN GRVPOW";
  C(1,1):=U(1);
  FOR J:=2:N DO
    << C(1,J):=U(J);
      FOR K:=2:J DO
        << C(K,J):=FOR I:=1:J-K+1 SUM C(1,I)*C(K-1,J-I);
          C(1,J):=C(1,J)-V(K)*C(K,J) >> >>;
  RETURN FOR K:=1:N SUM C(1,K)*X**K;
END;

```

```

COMMENT EXAMPLE 1;

ARRAY A(20),B(20),U(20),V(20);

ON DIV;

U(3):=1$
U(4):=1$
V(1):=1$
V(3):=1$

TY:=GRVPOW(20,U,V,Y);


$$TY := Y^3 \times (3 \times Y^{17} + 15 \times Y^{16} + 30 \times Y^{15} + 30 \times Y^{14} + 15 \times Y^{13} + 3 \times Y^{12} - Y^9 - 3 \times Y^8 - 3 \times Y^7 - Y^6 + Y + 1)$$


ON RATIONAL;

COEFF(TY/Y**3,Y,B)$

OFF RATIONAL;

YT:=REVPOW(10,B,3,1,H**3);

*** ARRAY C REDEFINED


$$YT := H \times (83472494/4782969 \times H^{10} - 15237130/1594323 \times H^9 + 16/3 \times H^8 - 60287/19683 \times H^7 + 13055/6561 \times H^6 - H^5 + 154/243 \times H^4 - 35/81 \times H^3 + 1/3 \times H^2 - 1/3 \times H + 1)$$


ON RATIONAL;

COEFF(YT/H,H,B)$

OFF RATIONAL;

A(0):=1$

FOR J:=1:10 DO A(J):=A(J-1)/J;

Z:=SUBPOW(10,A,1,1/2,B,2/3,2/3,X);

*** ARRAY B1 REDEFINED


$$Z := 58223/68040 \times X^{(2/3)}^3 - 541/6480 \times X^{(2/3)}^2 - 1/3 \times X^{(2/3)} \times X + X^{(2/3)} + 157051/1088640 \times X^{(1/3)}^3 \times X^3 + 37/90 \times X^{(1/3)}^2 \times X^2 + 1/6 \times X^{(1/3)} \times X - 897269/3628800 \times X^4 - 178/315 \times X^3 + 1/24 \times X^2 + X$$


END;

```



```

COMMENT EXAMPLE 2;

ARRAY A(20),B(20);
ON DIV;

FOR J:=2:10 DO A(J-2):=2*(-1)**J/J;
W:=POTPOW(6,A,2,1,1/2,U);

W := U*(372571/5443200*U6 - 22409/272160*U5 + 1331/12960*U4 - 73/540*U3 + 7/36*U2 - 1/3*U + 1)
ON RATIONAL;
COEFF(W/U,U,B)$
OFF RATIONAL;
U:=REVPOW(6,B,1,1,W1);
*** ARRAY C REDEFINED

U := W1*( - 139/5443200*W16 + 1/17010*W15 + 1/4320*W14 - 1/270*W13 + 1/36*W12 + 1/3*W1 + 1)
ON RATIONAL;
COEFF(U/W1,W1,A)$
OFF RATIONAL;
LET SQR(2)**2=2;
R:=1+POTPOW(6,A,1,1,-1,(2*V)**(1/2));
*** ARRAY C REDEFINED

R := - 139/97200*SQR(2)(-1) *SQR(V)(-1) *V3 + 1/216*SQR(2)(-1) *SQR(V)(-1) *V2 + 1/6*SQR(2)(-1) *
SQR(V)(-1) *V + SQR(2)(-1) *SQR(V)(-1) + 4/2835*V2 - 4/135*V + 2/3
END;

```

COMMENT EXAMPLE 3;

ARRAY A(20),B(20);

ON DIV;

FOR J:=0:9 DO << A(J):=1/FCT(2*J+1); B(J):=(-1)**J/FCT(2*J) >>;

Z:=MPYPOW(9,A,1,2,B,0,2,X);

Z := X*($-1/237588086736000X^{18} + 1/1389404016000X^{16} + 1/10216206000X^{14} - 1/97297200X^{12} - 1/1247400X^{10} +$

$1/22680X^8 + 1/630X^6 - 1/30X^4 - 1/3X^2 + 1)$

FOR J:=0:9 DO << A(J):=1/FCT(2*J); B(J):=(-1)**J/FCT(2*J+1) >>;

Z:=1/2*(Z+MPYPOW(9,A,0,2,B,1,2,X));

Z := X*($1/1389404016000X^{16} - 1/97297200X^{12} + 1/22680X^8 - 1/30X^4 + 1)$

ON RATIONAL;

COEFF(SUB(X=T*(1/4),Z/X),T,B)\$

OFF RATIONAL;

X:=REVPOW(4,B,1,4,Y);

X := Y*($32633/95001984Y^{16} + 1655/1297296Y^{12} + 25/4536Y^8 + 1/30Y^4 + 1)$

END;

```

COMMENT EXAMPLE 4;

ARRAY A(20),B(20);

ON DIV;

A(0):=1/9$
A(1):=1/28$
YS:=POW(4,A,9/8,7/8,1/2,T);
YS := T(9/16) * (3/56 * T(7/8) + 243/351232 * T(5/8) * T2 - 27/6272 * T(3/4) * T - 10935/78675968 * SQRT(T) * T3 + 1/3)
YS:=SUB(T=T1*(8/7),YS/T*(9/16))$
ON RATIONAL;
COEFF(YS,T1,A)$
OFF RATIONAL;
B(0):=1$
B(1):=1/14$
Y:=DEPPOW(4,A,9/16,7/8,B,1/8,7/8,X);
*** ARRAY C REDEFINED
Y := SQRT(X)*X4 * (171951/11239424 * X28 - 6451/351232 * X21 + 165/6272 * X14 - 3/56 * X7 + 1/3)
END;

```

```

COMMENT EXAMPLE 5;

ARRAY A(20),U(20),V(20);
ON DIV;

V(1):=1$
FOR J:=2:7 DO V(J):=(-1)**J/(J*(J-1));
FOR J:=1:7 DO U(J):=-ALFA*(-1)**J/J;
Y:=GRVPOW(6,U,V,X)$
ON RATIONAL;
NC:=COEFF(Y,X,A)$
OFF RATIONAL;
FOR N:=0:NC DO WRITE A(N):=A(N);
A(0) := 0
A(1) := ALFA
A(2) := - (1/2*ALFA)*(ALFA + 1)
A(3) := ALFA*(2/3*ALFA2 + 1/2*ALFA + 1/3)
A(4) := ALFA*( - 9/8*ALFA3 - ALFA2 - 11/24*ALFA - 1/4)
A(5) := ALFA*(32/15*ALFA4 + 9/4*ALFA3 + 7/6*ALFA2 + 5/12*ALFA + 1/5)
A(6) := ALFA*( - 625/144*ALFA5 - 16/3*ALFA4 - 51/16*ALFA3 - 5/4*ALFA2 - 137/360*ALFA - 1/6)
Y:=SUB(ALFA=1,Y);
Y := X*( - 1759/120*X5 + 37/6*X4 - 17/6*X3 + 3/2*X2 - X + 1)
END;

```

```

COMMENT EXAMPLE 6;

ARRAY A(20),B(20),C(20);

ON DIV;

C(0):=1$

FOR J:=1:4 DO C(J):=-((3/2-J)/J)*C(J-1);

FOR J:=0:4 DO A(J):=2*C(J)/(5/2+J);

FOR J:=0:4 DO B(J):=2*C(J)/(3/2+J);

LET 3*(1/3)=C3, 4*(1/3)=C4;

Y:=DEPPON(4,A,5/2,1,B,3/2,1,X);

*** ARRAY C REDEFINED

Y := X*(2469/154000*X(2/3) *X(-2) *C32 + 3/5*X(2/3) *C4(-2) *C32 + 1355877/140140000*X(1/3) *X(-1) *C3 + 9/
140*X(1/3) *X(-1) *C4(-1) *C3 + 33/1400*X2)

END;

```

```

COMMENT EXAMPLE 7;

ARRAY A(20),B(20);
ON DIV;

A(0):=1$
A(1):=1$
Y:=POTPOW(7,A,0,1,1/2,X)-1;

Y := X*(33/2048*X6 - 21/1024*X5 + 7/256*X4 - 5/128*X3 + 1/16*X2 - 1/8*X + 1/2)
ON RATIONAL;
COEFF(Y/X,X,B)$
OFF RATIONAL;
FOR J:=0:6 DO A(J):=1/FCT(J);
Z:=SUBPOW(6,A,0,1,B,1,1,X);

Z := - 329/46080*X6 + 3/320*X5 - 5/384*X4 + 1/48*X3 + 1/2*X + 1
ON RATIONAL;
COEFF(Z,X,A)$
COEFF(Y+1,X,B)$
OFF RATIONAL;
Z:=DIVPOW(6,A,0,1,B,0,1,X);
*** ARRAY C REDEFINED

Z := 731/9216*X6 - 329/3840*X5 + 3/32*X4 - 5/48*X3 + 1/8*X2 + 1
END;

```