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THE OCTONIONIC INSTANTON

S. Fubini and H. Nicolai

CERN - Geneva

ABSTRACT

The existence of self-dual instanton solutions of $d = 4$ Yang-Mills theories is related to the fact that the action can be expressed as the square of the self-dual field $\vec{E} + \vec{B}$. In this letter, we prove a similar result for $d = 8$ Yang-Mills theories. By use of the octonionic algebra the action can be represented as the sum of the squares of seven combinations of the field strengths. As a consequence, we construct a classical solution which is the eight dimensional analogue of the $d = 4$ instanton.

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Supersymmetric theories can be characterized by the existence of a transformation of the bosonic variables which renders the bosonic part of the action Gaussian and whose Jacobian equals the Matthews-Salam determinant obtained by integrating out the fermionic variables¹⁾. Of particular interest are those theories for which the new bosonic variables ξ_α are local functions of the original variables ϕ_α , and their space-time derivatives $\partial_\mu \phi_\alpha$, i.e.

$$S_B(\phi) = \int \xi_\alpha \xi_\alpha dx + \text{topol. term} \quad (1)$$

$$\xi_\alpha = \xi_\alpha(\phi, \partial_\mu \phi)$$

since they admit a reinterpretation in terms of stochastic differential equations²⁾. There is only a limited number of theories for which an explicit "quadrature" of the bosonic action is possible. The cases known so far are supersymmetric quantum mechanics, for which the variable transformation (1) is given by $\xi = \dot{q} \pm V(q)$ [where $V(q)$ is the derivative of the "superpotential"]^{3),4)}, the $N = 2$ Wess-Zumino model in two dimensions^{2),4)} and the $N = 1$ super Yang-Mills theory in four dimensions. For the latter, the Gaussian variables are just⁵⁾

$$\xi_i = E_i \pm B_i \quad i = 1, 2, 3 \quad (2)$$

where the electric and magnetic fields are, of course, defined as

$$E_i \equiv F_{i4} \quad , \quad B_i \equiv \frac{1}{2} \epsilon_{ijk} F_{jk} \quad (3)$$

These three cases are easily seen to be related to the real numbers $\mathbb{R}$ (supersymmetric quantum mechanics), the complex numbers $\mathbb{C}$ ($d = 2$, $N = 2$ Wess-Zumino model) and the quaternions $\mathbb{H}$ ($N = 1$ super Yang-Mills), as ϵ_{ijk} in (3) are nothing but the quaternionic structure constants. In this paper, we present a discussion of the fourth remaining case corresponding to the last division algebra, the octonions $\mathbb{O}$. The central observation is that the quadrature of the Yang-Mills action in eight dimensions can be achieved by replacing (3) by

$$E_i \equiv F_{i8} \quad , \quad B_i \equiv \frac{1}{2} c_{ijk} F_{jk} \quad (4)$$

where c_{ijk} are the octonionic structure constants⁶⁾ and the indices $i, j, k, \dots$ run from one to seven from now on. However, we will not explore the possible

implications of this observation for supersymmetry here, but rather concentrate on another (classical) aspect which may also have important consequences. Namely, for the cases treated so far, it is known that the equation

$$\xi_{\alpha}(\phi) = 0 \quad (5)$$

leads to solutions of the corresponding equations of motion with non-trivial topological properties and a vanishing energy momentum tensor⁷⁾ [for a discussion of supersymmetric quantum mechanics, see Ref. 4)]. We will demonstrate in this paper that this is also true in eight dimensions by deriving a new "instanton" solution for eight dimensional Yang-Mills theories^{*)}. Although this solution has many features reminiscent of the well-known $d = 4$ instanton solutions⁷⁾, there are some important differences which make a separate study worthwhile. We also emphasize that the possibility of "taking the square root" of the Yang-Mills action in eight dimensions is an isolated phenomenon and unlikely to recur in yet higher dimensions.

The instanton solution of Ref. 7) was based on the gauge group $SU(2)$. The relevant group in our case turns out to be one of the non-canonically embedded $SO(7)$ subgroups of $SO(8)$ which we denote by $SO(7)^{\pm}$, and we therefore begin with a brief discussion of its properties. Our presentation closely adheres to Section 3 of Ref. 9) to which we refer the reader for further details. For the sake of simplicity, we will restrict our attention to the $SO(7)^+$ subgroup throughout most of this paper. The basic quantity is the following $SO(7)^{\pm}$ invariant tensor C_{MNPQ} in eight dimensions, which is constructed out of the octonionic structure constants by defining⁹⁾⁻¹¹⁾

$$\begin{aligned} C^{mnp8} &\equiv C^{mnp} \\ C^{mnpq} &\equiv \frac{1}{6} \eta \epsilon^{mnpqrst} C_{rst} \end{aligned} \quad (6)$$

This tensor is self-dual with respect to the eight index Levi Civita symbol with the duality phase $\eta = \pm 1$, and obeys the relation [Ref. 9), see also Ref. 10)]

*) The "self-duality" condition based on (4) rather than (3) was already considered in Ref. 8) where the special role of eight dimensions was emphasized.

$$C^{MNPT} C_{QRST} = 6 \delta_{QRS}^{MNP} - 9 \eta \delta_{[Q}^{[M} C^{NP]}_{RS] \quad (7)$$

where capital indices run from one to eight. From (6), one obtains the projection operators⁹⁾

$$P_1^{MN}{}_{PQ} \equiv \frac{3}{4} \left(\delta_{PQ}^{MN} + \frac{1}{6} \eta C^{MNPQ} \right) \quad (8)$$

$$P_2^{MN}{}_{PQ} \equiv \frac{1}{4} \left(\delta_{PQ}^{MN} - \frac{1}{2} \eta C^{MNPQ} \right) \quad (9)$$

onto the orthogonal 21- and 7-dimensional subspaces of the 28-dimensional vector space whose elements are labelled by antisymmetric index pairs $[MN], \dots$. If Γ_{MN} are the generators of $SO(8)$, the generators

$$G_{MN} = P_1^{MN}{}_{PQ} \Gamma^{PQ} \quad (10)$$

span the Lie algebra of $SO(7)^+$ and $SO(7)^-$ for the choices $\eta = +1$ and $\eta = -1$, respectively. We will henceforth put $\eta = 1$. The commutation relations of the $SO(7)^+$ Lie algebra are then given by

$$\begin{aligned} [G^{MN}, G_{PQ}] &= 6 G_{[M}^{[Q} \delta_{N]}^{P]} + \\ &+ \frac{1}{2} \left(C_{PQR} [^M G^{N]R} - C^{MNR} [^P G_{Q]R} \right) \end{aligned} \quad (11)$$

Note the appearance of the second term on the right-hand side which has no analogue in the ordinary $SO(7)$ Lie algebra.

The Yang-Mills Lagrangian is

$$\mathcal{L} = - \frac{1}{4} \text{Tr} F_{MN} F^{MN} \quad (12)$$

where the trace over Yang-Mills indices is appropriately normalized. Using the formulae above it is not difficult to see that

$$\mathcal{L} = -\frac{1}{2} \text{Tr} \xi_m \xi_m - \frac{1}{2} C_{MNPQ} \text{Tr} F^{MN} F^{PQ} \quad (13)$$

where, in accordance with our previous discussion, we have defined

$$\xi_m \equiv F_{m8} - \frac{1}{2} C_{mnp} F_{np} \quad (14)$$

[if $SO(7)^+$ is replaced by $SO(7)^-$, the second terms in (13) and (14) appear with a plus sign]. The second term in (13) is a total derivative as

$$\text{Tr} F_{[MN} F_{PQ]} = \partial_{[M} \phi_{NPQ]} \quad (15)$$

with⁷⁾

$$\phi_{NPQ} \equiv 4 \text{Tr} \left\{ A_{[N} \partial_P A_{Q]} + \frac{2}{3} A_{[N} A_P A_{Q]} \right\} \quad (16)$$

To construct the new instanton solution we start from the ansatz

$$A_M(x) = G_{MP} f_P(x) \quad (17)$$

where $f_P(x) \equiv \partial_P f(x)$ ^{*}; we shall also employ the notation $f_{PQ} \equiv \partial_P \partial_Q f$. It is crucial that the matrix in (17) is G_{MP} and not Γ_{MP} . The definition (10) ensures that G_{MN} has only 21 independent components⁹⁾ instead of the 28 components of Γ_{MN} . From (11) and (17), one readily computes the field strength

^{*}) This assumption may actually be too restrictive unlike in the case of ordinary instantons. If one imposes the Landau gauge, it follows that $G_{MN} \partial_M f_N = 0$ which implies only $\partial[M f_N] + 1/6 C_{MNPQ} \partial_P f_Q = 0$ but not $\partial[M f_N] = 0$.

$$\begin{aligned}
 F_{MN} &= \partial_M A_N - \partial_N A_M + [A_M, A_N] = \\
 &= (2 f_{P[M} - 3 f_P f_{[M} - \frac{3}{2} f_Q^2 \delta_{P[M}) G_{N]P} - \\
 &\quad - \frac{1}{2} f_P f_Q C_{MNPR} G_{QR}
 \end{aligned} \tag{18}$$

which, by means of the projection operators (8) and (9), can be rewritten in the form

$$\begin{aligned}
 F_{MN} &= P_{1MN}{}^{PQ} \phi_{PR}^{(1)} G_{QR} + \\
 &\quad + P_{2MN}{}^{PQ} \phi_{PR}^{(2)} G_{QR}
 \end{aligned} \tag{19}$$

with

$$\phi_{PQ}^{(1)} = 2(f_{PQ} - f_P f_Q) - \frac{3}{2} f_R^2 \delta_{PQ} \tag{20}$$

$$\phi_{PQ}^{(2)} = 2(f_{PQ} - 3 f_P f_Q) - \frac{3}{2} f_R^2 \delta_{PQ} \tag{21}$$

It is crucial for the following that any term proportional to δ_{PQ} can be added to (21) because it drops out in (19) since

$$\begin{aligned}
 G^{MN} - \frac{1}{2} C^{MNPQ} G_{PQ} &= \\
 &= P_2{}^{MN}{}_{PQ} G^{PQ} = 0
 \end{aligned} \tag{22}$$

by the orthogonality of the projectors (8) and (9). Demanding the field strength to be self-dual¹⁰⁾ is equivalent to requiring $\xi_m = 0$ in (14). Thus,

$$F_{MN} = \frac{1}{2} C_{MNPQ} F^{PQ} \tag{23}$$

now amounts to putting $\phi_{PQ}^{(2)} = 0$ up to terms proportional to δ_{PQ} . Hence,

$$\ddot{f}_{PQ} - 3 \dot{f}_P \dot{f}_Q = h \delta_{PQ} \quad (24)$$

where h is an arbitrary and as yet undetermined function. To solve this differential equation, we substitute

$$\ddot{f}(x) = -\frac{1}{3} \log \phi(x) \quad (25)$$

into (24) and take the trace with respect to δ_{PQ} to determine h . In this way, we arrive at the equation

$$\partial_M \partial_N \phi = \frac{1}{8} \delta_{MN} \partial_P^2 \phi \quad (26)$$

which, for a specific choice of co-ordinates, is solved by $\phi(x) = 1 + x^2$. Thus, the gauge potential

$$A_M(x) = -\frac{2}{3} \frac{1}{1+x^2} G_{MN} x_N \quad (27)$$

gives rise to a field strength F_{MN} which is "self-dual" in the sense of (23). More explicitly, F_{MN} is given by

$$\begin{aligned} F_{MN} = & \frac{2}{3} \frac{2+x^2}{(1+x^2)^2} G_{MN} + \frac{4}{3} \frac{1}{(1+x^2)^2} x_{[M} G_{N]P} x_P \\ & - \frac{2}{9} \frac{1}{(1+x^2)^2} x^P x^Q C_{MNPQ} G_{QR} \end{aligned} \quad (28)$$

It is easy to verify that the solution (27) also satisfies the Yang-Mills field equations by virtue of the Bianchi identities and the self-duality constraint (23). This also follows directly from formula (13) as the variation of (13) with respect to A_M is linear in ξ_m . If, on the other hand, one were to impose the requirement

$$P_1{}^{MN}{}_{PQ} F^{PQ} = 0 \quad (29)$$

which is equivalent to putting $\phi_{PQ}^{(1)} = 0$, a calculation similar to the one just performed shows that (29) admits only the trivial solution $F_{MN} = 0$. This is

because no term proportional to δ_{PQ} can be added to $\phi_{PQ}^{(1)}$ in (20). The reason is, of course, that condition (23) requires only seven combinations to vanish whereas (29) would require 21 combinations to vanish. Thus, unlike in the case of ordinary instantons, there is an asymmetry between "self-dual" and "antiself-dual" solutions; for the latter, the $SO(7)^-$ invariant tensor must be used with $\eta = -1$ in (6). "Multi-instantons" will be more difficult to find.

Just as for ordinary instantons, one may enquire as to the topological meaning of our new solution. While this problem deserves further study, we here only remark that one may either consider the quantity $\text{Tr } F_{[MN}^F F_{PQ}^F F_{RS}^F F_{TU}]$, which is a total derivative or instead the four-form

$$\begin{aligned} q &\equiv \text{Tr } F \wedge F = \\ &= \text{Tr } F_{MN} F_{PQ} dx^M dx^N dx^P dx^Q = d\phi \end{aligned} \quad (30)$$

where, for the field strength (28), the three form ϕ is given by

$$\phi_{MNP} = -\frac{2}{27} \frac{3+x^2}{(1+x^2)^3} C_{MNPQ} x^Q \quad (31)$$

Since q is a four-form, it must be integrated over a four-dimensional hypersurface. As an example, we consider the hyperplane where the co-ordinates $(y_1, y_2, y_3, y_4) = (x_M, x_N, x_P, x_Q)$ are varied while the remaining four co-ordinates $z = (z_1, z_2, z_3, z_4)$ are kept fixed. One finds

$$\begin{aligned} Q &= \frac{2}{27} C_{MNPQ} \int_{\mathbb{R}^4} \frac{3+z^2(4+y^2+z^2)}{(1+y^2+z^2)^4} dy = \\ &= \frac{\pi^2}{27} C_{MNPQ} \end{aligned} \quad (32)$$

which is independent of z . Alternatively, one could have used Gauss' theorem to reduce (32) to a surface integral at infinity. The result is, of course, the same and does not depend on the shape of the hypersurface away from infinity. We also note that (32) is invariant under rotations in the $SO(7)^+$ subgroup of $SO(8)$ because of the invariance properties of the tensor C_{MNPQ} .

Let us finally mention another important difference with $d = 4$ instanton. It is well known that the usual Yang-Mills action is conformally invariant only in four dimensions. As shown in Ref. 12), the existence of the standard $d = 4$ instanton can be deduced directly from the conformal invariance of the theory. It is therefore clear that the formalism of Ref. 12) works only in four dimensions. The existence of an instanton in eight dimensions follows from the explicit quadrature of the action in (13). One may, of course, adopt a different attitude and consider the conformally invariant action in eight dimensions which is of fourth order in the field strength. In this case, the formalism of Ref. 12) may be applied. Solutions based on such an action have been found recently¹³⁾.

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