CERN-TH.4101/85

NUMERICAL CALCULATIONS IN QUANTUM FIELD THEORY ^{*)}

C. Rebbi ⁺⁾

CERN - Geneva

ABSTRACT

The relevance of numerical calculations for quantum field theories in general, and in particular for quantum chromodynamics, is illustrated. The evaluation of the force between static quark and antiquark is presented in some detail as exemplification.

*) Invited talk presented at the meeting on "New Avenues in Quantum Field Theory and General Relativity", Amalfi, May 1984.

+) Permanent address: Brookhaven National Laboratory, Upton, NY 11973, USA.

The advent of modern computers, capable today of many millions of operations per second, has opened an entirely new field of scientific investigation, a method of analysis which lies in between the more traditional ones of experimental and theoretical research. The scientist can reproduce by computer the dynamics of very complex systems and study their behaviour numerically. One can thus explore the implications of hypotheses formulated at a fundamental level, whose consequences are made evident through the measurement of the "observables" produced by the computer. The name "computational physics" has recently been introduced to characterize this new kind of research. Computational physics is interdisciplinary; many areas of science have already derived benefits from it.

In the domain of elementary particles, computational physics has been used to obtain predictions from quantum field theories in circumstances where the standard methods based on perturbative expansions cannot be applied. Although not restricted to it, the majority of applications have dealt with the theory of strong interactions known as quantum chromodynamics (QCD).

Quantum chromodynamics explains the structure of hadrons (baryons and mesons) and their behaviour in terms of fundamental fields of quarks and gluons. Although quarks are a basic ingredient of matter, many important properties of QCD can be derived from the study of the dynamics of gluons alone and, if I am to present at least some concrete results in the time allowed by this talk, it will be convenient to restrict our attention to the gluonic sector of QCD.

In its purely gluonic sector, QCD is a quantum gauge field theory based on the gauge group SU(3) (the group of local "colour" transformations). Dynamical variables are the gauge potentials $A_\mu^\alpha(x)$ (α , colour index = 1-8). From these one derives the field strength

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + g f_{\beta\gamma}^\alpha A_\mu^\beta A_\nu^\gamma, \quad (1)$$

where g is the bare coupling constant, $f_{\beta\gamma}^\alpha$ are the structure constants of SU(3), and the action

$$S = \int d^4x \quad \frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu}_\alpha. \quad (2)$$

Finally, after a Wick rotation to the imaginary time axis, $t \rightarrow it$, and using units with $\hbar = c = 1$, the expectation values of quantum observables are given by functional averages over all possible values of the dynamical variables $A_\mu^\alpha(x)$:

$$\langle O \rangle = \frac{\int DA O(A) e^{-S(A)}}{\int DA e^{-S(A)}} \quad (3)$$

The expressions on the right-hand side of Eq. (3) are of course purely formal. The ultra-violet divergences present in Eq. (3) must be regularized. A rigorous mathematical meaning must be given to the functional integrals. This is normally done in the framework of a perturbative expansion; however, there are observables which cannot be evaluated by perturbative methods. This is because there are phenomena which are governed by an intrinsically strong coupling constant or, at an even more fundamental level, because the expressions for some observables have essential singularities for $g = 0$, which defy a perturbative expansion. Indeed, let us assume that the regularization of ultra-violet divergences has been achieved by imposing a cut-off on the wavelengths. Only fluctuations over domains of size $\lambda > a$ will be kept in the functional averages. Eventually, a must be sent to zero to recover the physics of the continuum. In this limiting procedure, the bare coupling constant g must also be changed (renormalization), according to a definite functional relationship $g = g(a)$ or, equivalently, $a = a(g)$, in order to keep physical length scales fixed. From a renormalization group analysis, one can establish that $g \rightarrow 0$ as $a \rightarrow 0$ and that the relationship between a and g must take for $g \rightarrow 0$ the limiting form

$$a(g) = \frac{1}{\Lambda} \left(\frac{16\pi^2}{11g^2} \right)^{\frac{51}{121}} e^{-\frac{8\pi^2}{11g^2}} (1 + O(g^2)), \quad (4)$$

Λ being a scale parameter with dimensions of a mass. Suppose we are to calculate a mass m . In the regularized theory, m will be given by an expression

$$m = f(g) a^{-1}. \quad (5)$$

(The dependence on the cut-off must be trivial on dimensional grounds.) For m to have a definite continuum limit, $f(g)$ must also behave as

$$f(g) = f_0 \left(\frac{16\pi^2}{11g^2} \right)^{\frac{51}{121}} e^{-\frac{8\pi^2}{11g^2}} (1 + O(g^2)). \quad (6)$$

This shows that f has an essential singularity for $g = 0$; similar arguments can be applied to other observables of QCD which carry physical dimensions. Numerical techniques have been applied to QCD to circumvent the limitations of the perturbative methods implicit in the above considerations. Their implementation requires, to begin with, that the theory be regularized in a non-perturbative fashion. A very elegant, non-perturbative regularization of a quantum gauge field theory is achieved by formulating the theory on a lattice¹⁾.

The continuum of points of Euclidean space-time is replaced by the set of points in a discrete lattice^{*)}. In most applications, this is taken to be hypercubical. The cut-off a is given by the lattice spacing. The dynamical variables are SU(3) group elements U_x^μ assigned to the oriented links of the lattice from x to $x + \hat{\mu}a$ ($\hat{\mu}$ denotes a unit vector in direction μ). This corresponds to the fact that in the continuum theory the gauge potentials $A_\mu^\alpha(x)$ parametrize an infinitesimal colour transformation of the form

$$\hat{U} = \exp \left\{ ig A_\mu^\alpha(x) \lambda_\alpha dx^\mu \right\} \quad (7)$$

[λ_α stand for the generators of the SU(3) group], in the transport between "neighbouring" points x^μ and $x^\mu + dx^\mu$. However, because a finite distance a separates the nearest neighbour points of the lattice, the dynamical variables U_x^μ are now finite, rather than infinitesimal elements of the group.

*) For a detailed account of the lattice formulation of a gauge theory and of many of the topics touched upon in this talk, the reader may consult, for example, Refs. 2) and 3).

Corresponding to the field strength, one defines variables

$$U_x^{\mu\nu} = U_x^{\nu\dagger} U_{x+a\hat{\mu}}^{\mu\dagger} U_{x+a\hat{\mu}}^{\nu} U_x^{\mu} \quad (8)$$

These are associated to the elementary squares of the lattice, or plaquettes, with vertices in x , $x + a\hat{\mu}$, $x + a\hat{\mu} + a\hat{\nu}$, $x + a\hat{\nu}$ and are obtained by multiplying the U_x^{μ} variables defined over the oriented sides of the plaquette. With elementary algebra, one shows that, assuming for U_x^{μ} a form as in Eq. (7) with $dx^{\mu} = a$, $U_x^{\mu\nu}$ reduces to

$$\exp \{ i g F_{\mu\nu}^{\alpha} \lambda_{\alpha} a^2 \} \quad (9)$$

in the limit $a \rightarrow 0$.

As action S for the lattice system, one then defines the quantity

$$S = \frac{6}{g^2} \sum_{\mu < \nu} (1 - \frac{1}{3} \text{Re Tr } U_x^{\mu\nu}) \quad (10)$$

Again, it is straightforward to show that in the limit $a \rightarrow 0$, S reduces to the continuum form of Eq. (2). S , which is frequently referred to as Wilson's action, does not represent the only possible form for the lattice action. An infinite variety of lattice actions will reduce to the form of Eq. (2) in the $a \rightarrow 0$ limit. Barring pathological choices, all these lattice actions should produce the same physics in the continuum limit. This is what is referred to as universality with respect to the choice of the (lattice) action. Notice that assuming different lattice actions is equivalent to following different renormalization prescriptions.

The quantum expectation values of the observables are given by averages over all possible values of the dynamical variables:

$$\langle O \rangle = \frac{\int \prod_{x,\mu} dU_x^\mu O(U) e^{-S(U)}}{\int \prod_{x,\mu} dU_x^\mu e^{-S(U)}} \quad (11)$$

For a system of finite volume V (V will be let to ∞ at the end of the calculations), the right-hand side is a well-defined mathematical expression. Notice how the lattice formulation achieves a non-perturbative, gauge-invariant regularization of the ultra-violet divergences.

The right-hand side of Eq. (11) bears a remarkable resemblance to the defining expressions of thermal averages in statistical mechanics. The analogy becomes even clearer if we define an "internal energy" of the plaquettes (although it is, in reality, a non-normalized action density)

$$E_x^{\mu\nu} = 1 - \frac{1}{3} \text{ReTr } U_x^{\mu\nu} \quad (12)$$

and a "coupling parameter"

$$\beta = \frac{6}{g^2} \quad (13)$$

We see then that the right-hand side of Eq. (11) is formally identical to the expression giving the thermal average of O in a system with internal energy $E = \sum_{\mu, \nu, x} E_x^{\mu\nu}$ and inverse temperature β . If one performs a low temperature expansion, i.e., an expansion for large β or small g^2 , one recovers the standard perturbative expansion of a quantum field theory (now made more complicated by the loss of Poincaré invariance). But in the lattice formulation, one can also perform strong coupling expansions, i.e., expansions for small β , corresponding to the high temperature expansions of statistical mechanics. In the domain where strong coupling expansions are valid, one can easily calculate non-perturbative observables and demonstrate properties of QCD, such as quark confinement, which are expected to be true in the continuum limit. Yet even strong coupling expansions are not, per se, a solution to the problem of determining observables of a non-perturbative nature in QCD. The point is that, in order to recover eventually a continuum limit, one must be able to proceed to the limit $a \rightarrow 0$, which requires that, at the same time, g be sent to zero or, equivalently, β to ∞ . Thus, results obtained in the strong coupling domain (β small) must be extrapolated to the limit $\beta = \infty$. As a matter of fact, one does not need to proceed all

the way to the continuum limit to obtain relevant information about the continuum theory. It is enough being able to extrapolate to sufficiently small values of a (or sufficiently large values of β), so that a scaling behaviour, such as the one expressed by Eqs. (5) and (6), is determined for the observables. If Eqs. (5) and (6) are seen to hold, then the continuum value of m is given by $f_0\Lambda$, where f_0 is a numerical constant, and Λ is the scale introduced in the renormalization Eq. (4). Λ , which per se is a renormalization-dependent, unphysical parameter, can be expressed in terms of any physical quantity q , and then all other observables are uniquely determined in units of q .

In more intuitive terms, the meaning of the above considerations is that it is enough to make, by the renormalization procedure, the lattice spacing a smaller than the typical length scales of the phenomena under consideration. From that point on, the physics on the lattice must reproduce the physics of the continuum. On the basis of the calculation of a definite observable (the string tension, see later), one can argue that the lattice spacing should equal approximately 0.25 fm for $\beta = 5.6$, 0.11 fm for $\beta = 6$ and 0.07 fm for $\beta = 6.4$. Since the typical length scale for hadronic phenomena is of the order of 1 fm, we might expect to be able to see the features of the continuum system if we can perform calculations at $\beta \gtrsim 6$. Such a value of the coupling parameter lies beyond the range of validity of strong coupling expansions, but meaningful calculations for this intermediate range of values of β can be performed by numerical techniques.

The numerical calculation of the observables proceeds through a computer-based simulation of the quantum fluctuations, i.e., through an approximate evaluation of the integrals on the right-hand side of Eq. (11) by methods of stochastic sampling. In the Monte Carlo algorithm, which constitutes the simulation technique most frequently followed, one stores in the memory of the computer all the U_x^μ variables. Their collection defines a configuration C of the system. The computer then generates a very large sequence of configurations

$$\dots \rightarrow C^i \rightarrow C^{i+1} \rightarrow C^{i+2} \rightarrow \dots, \quad (14)$$

where the passage between one configuration and the next is not deterministic but stochastic, and based on the extraction of (pseudo) random numbers. The sequence is thus a Markovian chain, with a definite transition matrix $p(C \rightarrow C')$.

Typically, the passage between one configuration and the next involves the replacement of just one of the dynamical variables in memory U_x^μ with a new, or upgraded, value $U_x^{\mu'}$. $U_x^{\mu'}$ is selected as follows. First of all, a new candidate value U_x^μ for U_x^μ is proposed. U_x^μ is also obtained from U_x^μ in a stochastic manner, e.g., multiplying U_x^μ by a matrix in group space chosen according to some distribution peaked around the identity. The width of the distribution is a parameter of the calculation, which can be adjusted for maximum efficiency; the final results do not depend on it. The probability distribution for the selection of U_x^μ must, however, satisfy the condition

$$p_0(U_x^\mu \rightarrow \hat{U}_x^\mu) = p_0(\hat{U}_x^\mu \rightarrow U_x^\mu). \quad (15)$$

After U_x^μ is determined, one calculates the variation in action which would be induced by the change

$$\Delta S = S(\hat{U}_x^\mu, \dots) - S(U_x^\mu, \dots). \quad (16)$$

Because the total action is a sum of local terms, the calculation of ΔS involves only the contribution from a small number of plaquettes (six in four dimensions) and can be executed very rapidly by the computer. Then, if ΔS is negative, the proposed change is accepted and

$$U_x^{\mu'} = \hat{U}_x^\mu \quad (17)$$

If ΔS is positive, the change is accepted with relative probability $\exp(-\Delta S)$: a (pseudo) random number r , uniformly distributed between 0 and 1, is extracted and

$$\begin{aligned} U_x^{\mu'} &= \hat{U}_x^\mu & \text{if } r \leq e^{-\Delta S}, \\ U_x^{\mu'} &= U_x^\mu & \text{otherwise.} \end{aligned} \quad (18)$$

This procedure implies that $p(C \rightarrow C')$ satisfies

$$\frac{p(C \rightarrow C')}{p(C' \rightarrow C)} = \frac{e^{-S(C')}}{e^{-S(C)}} \quad (19)$$

Equation (19), in turn, has as a consequence that the Boltzmann distribution

$$P(C) \propto e^{-S(C)} \quad (20)$$

is an eigenstate of the Markovian chain. Barring pathological circumstances, the sequence (14) will tend to the Boltzmann distribution. When statistical equilibrium is reached, one can then approximate the exact averages in Eq. (11) by averaging over the configurations in the sequence:

$$\langle O \rangle \approx \frac{1}{n} \sum_{i=i_0+1}^{i_0+n} O(C^i). \quad (21)$$

In actual calculations, one averages over subsets of all configurations in the sequence. The variation between C^i and C^{i+1} is too small to warrant a recalculation of the observables. After the upgrading of a definite dynamical variable, the Monte Carlo simulation proceeds to the upgrading of another one, and so on, until all U_x^μ have been upgraded. This can be done by scanning the lattice in a regular fashion, or by selecting the U_x^μ to be upgraded at random, in which case we can say that they have all been upgraded only in the average sense that as many steps $C \rightarrow C'$ have been performed as there are variables. In any case, one says that a Monte Carlo iteration has been completed when all U_x^μ have been upgraded (exactly or in the above average sense). The configurations included in the averages for the quantum observables are usually those obtained at the end of a full Monte Carlo iteration, or every few iterations. This achieves a better balance, from a computational point of view, between the time spent in generating configurations and that spent in evaluating observables.

From this rapid description of the calculational method, we can get some ideas about the computational resources that are required. If, for definiteness, we assume that we are simulating an $SU(3)$ gauge theory, as in QCD, on a lattice

extending for ten sites in each of the four dimensions, the number of U_x^μ dynamical variables is 40,000. Each is represented by a 3×3 complex matrix. It is computationally more convenient to store the full matrix in the memory rather than the eight real parameters which can be used to represent an $SU(3)$ transformation. The total number of real variables to be kept in memory in order to represent a configuration is then 720,000. The upgrading of an individual U_x^μ requires something of the order of 4000 arithmetic operations. A full Monte Carlo iteration implies therefore approximately 160 million arithmetic operations. In a typical calculation, one may have to proceed through thousands of Monte Carlo iterations to obtain an approximation to the observables with sufficient statistical accuracy. Clearly, then, the calculation is feasible if those 160 million arithmetic operations can be performed in a few seconds. Both the required memory capacity and computational speed are within the grasp of the supercomputers which have become available during the last few years. In the near future, we can expect major advances in the capabilities of computers, which will make even more demanding calculations possible. The resources required by numerical computations in quantum field theories are, however, formidable, and this explains why the development in this domain of investigations is only recent, whereas numerical simulations of thermodynamical systems, where the lower dimensionality implies smaller numbers of dynamical variables, have been successfully performed for several years.

I would like to proceed by outlining the calculation of a very fundamental observable of QCD, the force F between static quark and antiquark held fixed at a separation r . $F(r)$ can be deduced from the vacuum expectation value of another observable, the so-called Wilson loop factor W_{mn} . Let us consider a rectangular path extending for m lattice spacings in any spatial direction and for n lattice spacings in the temporal one. W_{mn} is given by the real part of the trace of the transport operator along sides of the rectangle (usually normalized by a factor of $1/3$), i.e., the product of all U_x^μ variables associated with the links on the path:

$$W_{mn} = \frac{1}{3} \text{ReTr} \left\{ U_x^{\nu\dagger} \dots U_{x+(m-1)\hat{\mu}}^{\nu\dagger} U_{x+m\hat{\mu}}^\nu \dots U_x^\nu \right\}. \quad (22)$$

In the continuum limit, W_{mn} reduces to the modification induced in the exponentiated action by a quark-current loop circulating around the rectangle. Thus,

$\langle W_{mn} \rangle$ is related to the variation in the vacuum energy due to the presence of a quark-antiquark pair, created at some time in a gauge-invariant manner at separation ma , propagating for a time na , and annihilated afterwards. On general grounds, one expects the large n behaviour for W_{mn} :

$$W_{mn} \underset{n \rightarrow \infty}{\sim} \exp \{ -na E(ma) \}, \quad (23)$$

where $E(ma)$ is the energy of the static quark-antiquark pair. From Eq. (23) we deduce:

$$E(ma) = \lim_{n \rightarrow \infty} \left\{ -\frac{1}{a} \ln \frac{W_{mn}}{W_{m, n-1}} \right\}. \quad (24)$$

$E(ma)$ is not an observable with a well-defined continuum limit, because it contains the self-energy of the quarks, which diverges in the limit of a pointlike source. Such self-energy, however, will cancel out in the expression for the force $F(r)$, which we approximate by the finite difference:

$$F \approx \frac{1}{a} \{ E(ma) - E((m-1)a) \}. \quad (25)$$

It is convenient to associate a definite value to the argument of F . Guided by the identity

$$E(r) - E(r-a) = a F(\sqrt{r(r-a)}), \quad (26)$$

which applies to the case where E is the superposition of a Coulombic term and a linear term, we shall set

$$\begin{aligned} F(\sqrt{m(m-1)a}) &= \frac{1}{a} \{ E(ma) - E((m-1)a) \} = \\ &= -\frac{1}{a^2} \lim_{n \rightarrow \infty} \ln \left\{ \frac{W_{mn} W_{m-1, n-1}}{W_{m, n-1} W_{m-1, n}} \right\}. \end{aligned} \quad (27)$$

In Refs. 4) and 5), Wilson loop factors of size up to 8×8 were evaluated by Monte Carlo methods, working with a large ($16^3 \times 32$) lattice at several values of the

coupling parameter β and with two different forms of the lattice action. Equation (27) was then used to calculate the force. The lattice calculation gives F in lattice units, i.e., one obtains values for the dimensionless quantity $(a^2 F)$ as a function of the two variables $\sqrt{m(m-1)}$ and β . Thus, the lattice results for $(a^2 F)$ depend on two arguments. Yet, if the physics of the lattice approximates well the physics of the continuum, rescaling the dimensionless quantities $(a^2 F)$ and $\sqrt{m(m-1)}$ in terms of a single function $a(\beta)$ (renormalization) should produce an expression for the force $F(r)$ independent of the coupling parameter β .

The final results of the calculation are displayed in the Figure. The circles and crosses represent values for F calculated with two different lattice actions and at six different values of the lattice parameter for each action [ranging between $\beta \equiv (6/g^2) = 5.6$ and $\beta = 6.6$ with Wilson's action, and between $\beta' \equiv (9/g^2) = 6.6$ and $\beta' = 8.6$ with a lattice action which includes the trace of the plaquette variables $U_x^{\mu\nu}$ in the adjoint representation]. The graph shows that the results for the force, when re-expressed in physical units, do become independent from the value of the coupling parameter; the independence from the choice of the lattice action (universality) is also manifest. Thus, the numerical results confirm the validity of the whole renormalization procedure and allow one to derive information about observables in the continuum limit.

As the separation goes to infinity, the force does not go to zero; the numerical results are compatible with a constant limiting value

$$\lim_{r \rightarrow \infty} F(r) = K, \quad (28)$$

where K is the so-called string tension, responsible for quark confinement. Evidence for a non-vanishing string tension was indeed one of the first important results obtained by Monte Carlo calculations⁶⁾.

In the Figure, force and separation are expressed in units of K . From phenomenology, one infers $K \approx (420 \text{ MeV})^2$; thus the unit interval on the horizontal axis corresponds to $(420 \text{ MeV})^{-1}$ or, very approximately, half a Fermi. The relationship between a and β is found to conform with Eq. (4), with no appreciable $O(g^2)$ correction, for $\beta \gtrsim 6$. Also, the scale parameter Λ (for Wilson's action) turns out to be $\Lambda \approx 9.6 \times 10^{-3}/K$. The solid line in the Figure represents a fit to the numerical results, aimed at providing a more manageable analytic expression for the force, for use in further calculations. The behaviour of the curve at short distances is, however, not adjustable; rather, the interpolating formula has been chosen so as to incorporate the asymptotically free

behaviour of the force for small r , which one can calculate exactly from perturbative QCD. Thus, the agreement between the solid line and the numerical data at short distances is independent of the fitting procedure and constitutes an important check on the validity of the numerical calculation. The analytic approximation offered by the interpolating formula has been used in Ref. 7) as input into a calculation of energy levels of heavy quark systems. After incorporating vacuum polarization effects due to creation of virtual $q\bar{q}$ pairs, one finds remarkable agreement with spectroscopic data. As final considerations on the calculation of $F(r)$, let me observe that the error bars and scattering of points in the numerical results remind us of the approximate, statistical nature of the calculations, and that further sources of systematic error may be introduced by the extrapolation needed to derive F [see Eq. (27)]. With these limitations in mind, the outcome of the calculation is still remarkable, since it provides the determination of a very fundamental observable of QCD in terms of first principles only. The results presented in Refs. 4) and 5) are in agreement with other recent Monte Carlo studies of the static quark-antiquark system^{8),9)}.

The calculation of the $q\bar{q}$ force has been presented above in order to illustrate, in some detail, the methodology of lattice numerical calculations and the results that can be obtained. Monte Carlo simulations have been used during the last few years to obtain a variety of results for QCD and for quantum field theories in general. I do not have the time here to review all the important results which have been achieved. Let me just mention that the calculation performed for QCD can be broadly divided into three categories.

- i) Calculations where one retains only the degrees of freedom of the quantized gauge field. The evaluation of the force, illustrated above, belongs to this category. Other important observables, which have been calculated within this approximation, are the masses of the lowest-lying excitations of the gauge medium (glueballs) and the deconfinement temperature (i.e., the temperature above which, because of Debye screening effects, the interaction due to the gauge fields loses its confining properties).
- ii) Calculations where quarks are included and propagate in the presence of the self-interacting gauge medium. The measure which governs the contributions to the functional integrals is, however, always taken as $\exp\{-S\}$, where S is the action of the gauge field. In other words, the corrections to the functional measure which would be induced by the integration over the

quark degrees of freedom are neglected. This calculational scheme has been used especially for evaluating the masses of lowest-lying mesons and baryons in quark model spectroscopy. The approximation consists of neglecting the effects of vacuum polarization due to virtual quark-antiquark pairs; various arguments of phenomenological and analytical nature have been used to justify the validity of such an approximation. From a computational point of view, the analysis becomes more demanding because, beyond generating gauge field configurations as described above, one must also calculate the quark propagators, which is equivalent to the solution of a very large but sparse set of linear equations.

- iii) Calculations which try to reproduce numerically the whole dynamics of QCD. The gauge field configurations must be generated according to a measure which includes the determinant of the lattice Dirac operator for the fields of the quarks. This modification to the pure gauge measure accounts for the vacuum polarization effects due to virtual quark-antiquark pairs. Contrary to the pure gauge action, however, the determinant of the Dirac operator is non-local, and to compute exactly the variation of the full measure at each upgrading step would be too time-consuming. Various schemes to overcome this difficulty have been proposed, but the simulations remain computationally rather demanding. Calculations have nevertheless been performed and others are in progress, albeit with smaller lattices and more limited statistics than in points i) and ii) above. The inclusion of the full dynamical effects of the quarks is particularly important in problems such as the spontaneous breaking of chiral symmetry and of the symmetry restoration with temperature, or of the presence of a deconfining phase transition at high temperature and/or high baryon density, where the polarization of the vacuum may substantially alter the properties of the pure gauge medium.

Thus we see that, as in other areas of science, computers permit today also in the domain of quantum field theories the derivation of results which were undreamt of only a few years ago. With the rapid progress in the development of computers, one may expect that even more formidable results will be attainable in the near future. All of this does not mean that the computer can replace the analytical insight of the theorist, but it has certainly become an invaluable tool for exploring the consequences of fundamental assumptions.

ACKNOWLEDGEMENTS

It is a pleasure to thank Gerardo Marotta, President of the "Istituto Italiano di Studi Filosofici", whose sponsorship has made this meeting possible.

REFERENCES

- 1) K. Wilson, Phys. Rev. D10 (1974) 2445.
- 2) M. Creutz, L. Jacobs and C. Rebbi, Phys. Reports 95 (1983) 201.
- 3) "Lattice Gauge Theories and Monte Carlo Simulations", C. Rebbi ed., World Scientific, Singapore (1983).
- 4) D. Barkai, K. Moriarty and C. Rebbi, Phys. Rev. D30 (1984) 1293.
- 5) D. Barkai, K. Moriarty and C. Rebbi, Phys. Rev. D30 (1984) 2201.
- 6) M. Creutz, Phys. Rev. D21 (1980) 2308.
- 7) M. Campostini, BNL preprint (1984), to be published in Phys. Lett. B.
- 8) N.A. Campbell, C. Michael and P.E.L. Rakow, Phys. Lett. 139B (1984) 188.
- 9) S. Otto and J. Stack, Phys. Rev. Lett. 52 (1984) 2328.

FIGURE CAPTION

Force between static quark and antiquark as a function of their separation.