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A COMMENT ON THE PHENOMENOLOGY OF THE SU(3) SKYRME MODEL

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ABSTRACT

We find that the realistic baryon mass spectrum in the SU(3) Skyrme model with massive 0^- mesons can be obtained only for $F_\pi < 70$ MeV. We fit m_Λ and m_Σ^* and obtain baryon masses within the limits of $\pm 16\%$ of the experimental values, but at the expense of very low $F_\pi = 46.32$ MeV.

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The simplest low-energy effective theory of QCD in the limit of $N_c \rightarrow \infty$ (N_c - number of colours) is the sigma model describing the interactions of pseudoscalar mesons. The observation that such a model has non-trivial topological solutions - solitons - led already in the Sixties to the idea that the solitons can be interpreted as baryons¹⁾ (Skyrme model). There have already been several calculations of the properties of baryons in the Skyrme model with the symmetry groups $SU(2)$ ²⁾⁻⁶⁾ and $SU(3)$ ⁷⁾⁻¹¹⁾. The numerical results are generally within 30% of the experimental values. The calculations in the $SU(2)$ case provide links between the mesonic and baryonic "faces" of the model, in the sense that all baryonic properties are expressed in terms of the mesonic parameters like F_π or m_π .

The purpose of the present note is to calculate baryon masses in terms of F_π , m_π , m_K and parameter e , which we introduce later. We find that it is impossible to fit baryon masses with all of the above parameters taking their experimental values; in such a case, one obtains numbers orders of magnitude too high. As in the $SU(2)$ case, one can adjust F_π and e in order to get the right masses: this can also be achieved in our case, but at the expense of very small $F_\pi = 46.32$ MeV. This unpleasant result led us to some more detailed analysis of the energy scales appearing in the model. We found that the bound on F_π can be obtained: $F_\pi < 70$ MeV. The similar bound for the $SU(2)$ model is not so restrictive since a) the pion mass is of the order of F_π and b) the $SU(2)$ Casimir operators are smaller than the corresponding $SU(3)$ operators.

The paper is organized as follows. First we start from the Lagrangian density for the massive pseudoscalar mesons of the $SU(3)$ 0^- nonet, which is then quantized and the Hamiltonian for the baryons is obtained. Knowing the wave functions of baryons^{7),11)}, we can calculate the action of the Hamiltonian on the baryon states, hence baryon masses. At the end, we discuss the numerics.

The Lagrangian L for the $SU(3)$ Skyrme model takes the following form:

$$L = L_{\text{chiral}} + \int d^3x \mathcal{L}_{\text{mass}} \quad , \quad (1)$$

where

$$L_{\text{chiral}} = \int d^3x (\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{st}}) + L_{\text{WZ}}, \quad (2)$$

$$\mathcal{L}_{\text{kin}} = \frac{F_\pi^2}{16} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger), \quad (3a)$$

$$\mathcal{L}_{\text{st}} = \frac{1}{32e^2} \text{Tr}([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2), \quad (3b)$$

$$\mathcal{L}_{\text{mass}} = a \text{Tr}(U + U^\dagger - 2) + b \text{Tr}(\lambda_8 (U + U^\dagger)), \quad (3c)$$

$$L_{\text{WZ}} = -i \frac{N_c}{240\pi^2} \int d\Sigma^{\mu\nu\sigma\tau} \text{Tr}(U^\dagger \partial_\mu U U^\dagger \partial_\nu U U^\dagger \partial_\sigma U U^\dagger \partial_\tau U) \quad (3d)$$

and $U = U(\vec{x}, t)$ is the $SU(3)$ matrix. The Wess-Zumino term (3d) is given as the integral over the fifth-dimensional disc in the $SU(3)$ group. For a more detailed discussion of L_{WZ} see Refs. 7) and 9)-15). L_{chiral} describes the interactions of massless 0^- pseudoscalar mesons (π , K , η), while $\mathcal{L}_{\text{mass}}$ breaks explicitly the chiral invariance. By expanding $U = \exp((2i/F_\pi) \vec{\lambda} \cdot \vec{\phi})$ (λ_a : Gell-Mann matrices), we find:

$$a = \frac{F_\pi^2}{32} (m_\pi^2 + m_\eta^2), \quad b = \frac{\sqrt{3}}{24} F_\pi^2 (m_\pi^2 - m_K^2). \quad (4)$$

Equation (4) implies that the quadratic mass sum rule holds:

$$m_\pi^2 + 3m_\eta^2 - 4m_K^2 = 0. \quad (5)$$

Equation (5) is experimentally satisfied at the 3% level.

For the meson theory it is well known that (3a) already gives the desired current algebra results with $F_\pi = 186$ MeV. However, since we are interested in the solitonic solutions, we have [following Skyrme^{1),4)}] to add $\mathcal{L}_{st}$ to $\mathcal{L}_{kin}$ to make the classical solitons energetically stable. $\mathcal{L}_{st}$ is by no means a unique stabilizing term^{5),6),16)}, therefore we treat the dimensionless constant e as a free parameter, although it can be related to the π - π scattering length^{3),16)}; then it is bound to be $1.5 < e < 6.5$.

The crucial novelty of the SU(3) model is the presence of the Wess-Zumino term L_{WZ} , whose topological origin was explained by Witten¹²⁾. The remarkable result is that the coefficient N_c is quantized and equals the number of colours.

Some other terms can be added to L : a) $\text{Tr}(\lambda_3(U+U^\dagger))$ breaks the isospin invariance and can be responsible for the charge splittings within the multiplets⁷⁾; b) $\text{Tr}(\lambda_6 \partial_\mu U \partial^\mu U^\dagger)$ accounts for the weak interactions¹⁶⁾; c) some higher derivative terms can be responsible for the difference of the decay constants which we take to be equal: $F_\pi = F_K = F_\eta$. In the following, we neglect the above terms, since their effects on the hadron masses which we are going to calculate are expected to be negligible.

Now we have to specify the soliton solution. For the SU(2) model, the Skyrme ansatz^{1),4)} turns out to be the only known possibility. For SU(3) there are many more possibilities¹⁷⁾; however, the right quantum numbers for baryons are obtained if one assumes that¹³⁾:

$$U_0(\vec{r}) = \exp \left\{ i \sum_{A=1}^3 \hat{r}^A \lambda_A \cdot \theta(r) \right\}; \quad \hat{r} \equiv \frac{\vec{r}}{|\vec{r}|}, \quad (6)$$

where the function $\theta(r)$ is subject to the boundary conditions

$$\theta(0) = \pi, \quad \theta(\infty) = 0, \quad (7)$$

so that $U_0 = \mathbb{1}$ at spatial infinity. This requirement breaks the $SU(3) \times SU(3)$ symmetry

$$U_0 \rightarrow U'_0 = A U_0 B^\dagger ; \quad A, B = \text{const} \in \text{SU}(3) \quad (8)$$

to its diagonal subgroup $\text{SU}(3)_D$, i.e., $B = A$.

In order to quantize the model, we have to introduce the time-dependent co-ordinates - this can be achieved by observing that the zero modes of (6) are just the $\text{SU}(3)$ rotations. Time-dependent $\text{SU}(3)$ matrices $A(t)$

$$U(\vec{x}, t) = A(t) U_0(\vec{x}) A^\dagger(t) \quad (9)$$

define the general co-ordinates a^α ($\alpha = 1, \dots, 8$)

$$A^\dagger dA = \frac{i}{2} \sum_{\alpha=1}^8 \lambda_\alpha da^\alpha. \quad (10)$$

Since our ansatz $U_0(\vec{x})$ commutes with $g = e^{i\lambda_8(\phi/2)}$, the co-ordinates $A(t)$ are defined up to the right multiplication by g . Therefore, the eight-co-ordinate a^8 is in fact not independent; in other words, the system is constrained.

Before we write the Lagrangian, let us observe that there is the following freedom in defining $A(t)$

$$A \rightarrow g_L A ; \quad g_L \in \text{SU}(3)_L, \quad (11a)$$

$$A \rightarrow A k_R ; \quad k_R \in \text{SU}(2)_R. \quad (11b)$$

The transformations (11a) and (11b) do not change the Lagrangian which depends on $A(t)$ through $\text{Tr}(A^\dagger \dot{A} \dots A^\dagger \dot{A})$. These symmetries are interpreted as flavour $\text{SU}(3)$ (11a) and spin $\text{SU}(2)$ (11b)^{7),9)10)}.

Now we can write the chiral Lagrangian:

$$L = -M + \frac{1}{2} \alpha^2 \{ (\dot{a}^1)^2 + (\dot{a}^2)^2 + (\dot{a}^3)^2 \} + \frac{1}{2} \beta^2 \{ (\dot{a}^4)^2 + (\dot{a}^5)^2 + (\dot{a}^6)^2 + (\dot{a}^7)^2 \} + \frac{N_c}{213} \dot{a}^8, \quad (12)$$

where

$$\alpha^2 = \frac{2\pi}{3} \frac{1}{e^3 F_{\pi}} \int_0^{\infty} dx \sin^2 \theta \left(x^2 + 4 \left(\sin^2 \theta + x^2 \left(\frac{d\theta}{dx} \right)^2 \right) \right), \quad (13a)$$

$$\beta^2 = \pi \frac{1}{e^3 F_{\pi}} \int_0^{\infty} dx \sin^2 \theta \left(x^2 + \left(2 \sin^2 \theta + x^2 \left(\frac{d\theta}{dx} \right)^2 \right) \right), \quad (13b)$$

$$M = 4\pi \frac{F_{\pi}}{e} \int_0^{\infty} dx \left\{ \frac{1}{8} \left(x^2 \left(\frac{d\theta}{dx} \right)^2 + 2 \sin^2 \theta \right) + \frac{1}{2} \sin^2 \theta \left(\frac{\sin^2 \theta}{x^2} + 2 \left(\frac{d\theta}{dx} \right)^2 \right) \right\}. \quad (13c)$$

The redundant a^8 co-ordinate describes the motion generated by the right hypercharge transformation. Formally we can define the momentum conjugated to a^8 7), 11)

$$\frac{\partial L}{\partial \dot{a}^8} = \frac{N_c}{2\sqrt{3}}, \quad (14)$$

which can be interpreted as the right hypercharge generator, $(\sqrt{3}/2)Y_R$ 11). Because of (14), we see that

$$Y_R = 1. \quad (15)$$

The constraint (15) will be used while constructing the wave functions of baryons.

The quantization of L_{chiral} has been described in detail in Refs. 10) and 14) [see also Refs. 7), 9) and 11)]. First one evaluates the classical constants of motion [as in Eq. (14)] for the constrained system [i.e., with a^8 eliminated from Eq. (12)], and then one promotes them to the quantum operators. There has been some doubt whether such a procedure is correct 18), but we leave this question aside, relying on the results of Ref. 19).

The Hamiltonian H_{chiral} takes the following form¹⁰⁾:

$$H_{\text{chiral}} = M + \frac{1}{2} \beta^{-2} \left\{ C_2(SU(3)) - \frac{3}{4} \right\} + \frac{1}{2} (\alpha^{-2} - \beta^{-2}) C_2(SU(2)) . \quad (16)$$

C_2 are Casimir operators for $SU(3)_L$ and $SU(2)_R$. It is straight forward to derive the Hamiltonian corresponding to L_{mass} :

$$H_{\text{mass}} = \frac{8\pi}{3} \frac{m_K^2 - m_\pi^2}{e^3 F_\pi} D_8^8(A) \gamma^2 + 4\pi \frac{F_\pi}{e} \mu^2 \gamma^2 , \quad (17)$$

where the adjoint representation is defined as $\lambda_\alpha D_\beta^\alpha(A) = A^\dagger \lambda_\beta A$ and

$$\mu = \frac{\tilde{m}}{e F_\pi} , \quad \tilde{m} = \sqrt{\frac{m_\pi^2 + m_\eta^2}{2}} \quad (18)$$

and

$$\gamma^2 = \frac{1}{4} \int dx (1 - \cos \theta) . \quad (19)$$

Now we have to specify the baryonic wave functions. They are defined on the coset manifold $SU(3)/U(1)$, but because of the constraint (15) they can be extended to the whole group $SU(3)$. In fact⁷⁾:

$$\psi(A) = \sqrt{\dim(R)} \langle Y I I_3 | D^{(R)}(A) | 1 S - S_3 \rangle , \quad (20)$$

where Y , I , I_3 , S and S_3 are the baryon quantum numbers: Y is the hypercharge, I , I_3 stand for isospin; S , S_3 for spin, and $D^{(R)}(A)$ is the operator of the representation R of $SU(3)$. The minus sign for S_3 comes from the commutation rules for the $SU(2)_R$ spin group^{7), 10), 14)}. Now we observe that in the notation of Eq. (20):

$$D_8^8(A) = \langle 000 | D^{(8)}(A) | 000 \rangle. \quad (21)$$

Equation (21) allows us to calculate the action of $D_8^8(A)$ on the wave functions (20) only by means of the Clebsch-Gordan coefficients⁷⁾ [see Eq. (25) and the Table].

In order to calculate the masses, one has to solve numerically the differential equation for $\theta(x)$ ^{4),10)}:

$$\begin{aligned} & \left(\frac{1}{4}x^2 + 2\sin^2\theta \right) \frac{d^2\theta}{dx^2} + 2\sin\theta\cos\theta \left(\frac{d\theta}{dx} \right)^2 \\ & - 2\sin\theta\cos\theta \left(\frac{1}{4} + \frac{\sin^2\theta}{x^2} \right) - \frac{\mu^2}{4}x^2\sin\theta = 0, \end{aligned} \quad (22)$$

where $x = eF_\pi r$. This equation is exactly the same for the SU(2) and SU(3) models. We have solved Eq. (22) for a wide range of the parameter μ ($0.2 < \mu < 3.5$) by means of the fourth-order Runge-Kutta method. The numerical solution and its derivative were adjusted to fit the asymptotics

$$\theta(x) \longrightarrow A \frac{e^{-\mu x}}{x}. \quad (23)$$

The severe check on the solutions is provided by the Derrick theorem

$$\begin{aligned} & \int_0^\infty dx \left\{ \frac{1}{8} \left(x^2 \left(\frac{d\theta}{dx} \right)^2 + 2\sin^2\theta \right) + \frac{3}{4} \mu^2 x^2 (1 - \cos\theta) \right\} \\ & = \int_0^\infty dx \frac{1}{2} \sin^2\theta \left(\frac{\sin^2\theta}{x^2} + 2 \left(\frac{d\theta}{dx} \right)^2 \right), \end{aligned} \quad (24)$$

which in our case has been satisfied at the 1⁰⁰/₀ level.

We have chosen m_π , m_K , F_π and μ (instead of e) as free parameters. The formula for the mass of the hadron B in the representation R of SU(3) reads:

$$M_B = \frac{F_\pi^2}{\tilde{m}} f(\mu) + \frac{\tilde{m}^3}{F_\pi^2} g_R(\mu) - \frac{F_\pi^2}{\tilde{m}^3} (m_K^2 - m_\pi^2) d_B^R h(\mu), \quad (25)$$

The numbers d_B^R are displayed in the Table; functions $f(\mu)$ and $g(\mu)$ are plotted in Figs. 1a and 1b. The first observation to be made is that if one takes the physical values for F_π , m_π and m_K , then it is impossible to fit baryon masses by adjusting μ , since $M_B(\mu)$ is a U-shaped function of μ with the minimum order of magnitude too high. Adopting the method of Ref. 4), one can adjust μ and F_π to fit the baryonic data. The result is presented in the Table. The octet and decuplet masses can be fit reasonably well at the expense of very small $F_\pi = 46.32$. The resulting $\mu = 1.958$ corresponds to $e = 5.1$.

We found it interesting to trace the origin of such bad numerical behaviour of the model. Let us observe that the first two terms in formula (25) should add up to ~ 1.2 GeV for $R = 8$ (the average octet mass). Hence

$$\frac{F_\pi^2}{\tilde{m}} f(\mu) \approx (1-\varepsilon) \cdot 1.2 \text{ GeV}, \quad \frac{\tilde{m}^3}{F_\pi^2} g_8(\mu) = \varepsilon \cdot 1.2 \text{ GeV}, \quad (26)$$

where $\varepsilon \in (0,1)$. One can see immediately from (26) that

$$f(\mu) \cdot g_8(\mu) \leq \frac{0.36 \text{ GeV}^2}{\tilde{m}^2}. \quad (27)$$

Now

$$\tilde{m} = \begin{cases} 0.139 \text{ GeV} & \text{for SU(2)} \\ 0.412 \text{ GeV} & \text{for SU(3)} \end{cases} \quad (28)$$

and therefore (27) implies that

$$f(\mu) \cdot g_8(\mu) \leq \begin{cases} 18.63 & \text{for SU(2)} \\ 2.12 & \text{for SU(3)}. \end{cases} \quad (29)$$

As can be seen from Fig. 2, for SU(3) the above limit yields $\mu > 1.75$ and, as a consequence, $f(\mu) > 100$. This can be translated through Eq. (26) to the following bound for F_π :

$$F_\pi \leq 70 \text{ MeV}. \quad (30)$$

The bounds for SU(2) are not so restrictive, as seen from Figs. 1 and 2.

It is of course possible to change the bound (30) by changing the pion and kaon masses. For example, in order to obtain F_π greater than 100 MeV, one has to assume $\tilde{m} \approx 200$ MeV, a value certainly incompatible with the data.

The recent revival of the Skyrme idea that baryons can be interpreted as the solitons of the sigma model described by the Lagrangian (1) is due mainly to the remarkable numerical results of Adkins, Nappi and Witten⁴⁾ obtained for the SU(2) Skyrmions. In the case of SU(3), the bound (30) destroys the nice phenomenological behaviour of the model. It seems that the further we are from the chiral limit, the worse numerical behaviour can be expected. Whether this makes the Skyrme model less attractive remains, however, a question of taste; its beautiful topological properties and many other features will surely be widely explored in the literature. Nevertheless, the bound (30) significantly weakens the phenomenological power of the model which has created the widespread interest in Skyrmions during the last two years.

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NOTE ADDED

After this paper was completed, we became aware of the work by M. Chemtob, Saclay preprint SphT 84/100, where the phenomenology of the SU(3) Skyrme model is discussed in great detail. His conclusion that for the reasonable value of the parameter eF_π one is not able to fit the baryonic masses (with the mesonic masses taking their physical values) is just the reflection of the existence of the bound on F_π discussed in our note.

Representation R	Baryon B	d_B^R	Masses (MeV)	
			This Model	Experiment
8	N	3/10	1097 (+17%)	939
	Λ	1/10	1116 (input)	1116
	Σ	-1/10	1135 (-5%)	1193
	Ξ	-1/5	1144 (-13%)	1318
10	Δ	1/8	1373 (+11%)	1232
	Σ^*	0	1385 (input)	1385
	Ξ^*	-1/8	1399 (-9%)	1533
	Ω	-1/4	1409 (-16%)	1672

Table

Numerical results of the fit of the formula (25). Numbers in brackets indicate the relative error with respect to the physical values. $m_\pi = 139$ MeV and $m_K = 495$ MeV were taken as an input. As the result, $F_\pi = 46.32$ MeV and $\mu = 1.958$ ($e = 5.1$) were obtained.

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FIGURE CAPTIONS

Fig. 1 : a) Plot of the function $f(\mu)$ defined by Eq. (25);
b) Plot of the functions $g_R(\mu)$ defined by Eq. (25) for $R = 8, 10$ for the SU(3) and SU(2) models. The SU(2) curve was obtained by putting $\beta^{-2} = 0$ in Eq. (16).

Fig. 2 : Plot of the function $f(\mu) \cdot g(\mu)$ for SU(2) and SU(3) models (for $R=8$).

Fig. 1a

Fig. 1b

Fig. 2