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GAUGE DEPENDENCE OF THE NICOLAI MAP
IN SUPER YANG-MILLS THEORY

D. Amati and G. Veneziano
CERN-Geneva

ABSTRACT

Four-dimensional super Yang-Mills theory is reformulated in terms of the self-dual field strengths by deriving an effective action in an arbitrary gauge. With the exception of the light cone gauges, this action is in general non-Gaussian and, typically, non-local.

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1. - The $N = 1$ supersymmetric (SUSY) Yang-Mills theory has been recently proposed¹⁾ as a first example of a bona fide four-dimensional field theory possessing a local Nicolai map. Furthermore, that conclusion could only be reached in a very special gauge, the Wess-Zumino supersymmetric gauge complemented by a light cone axial gauge.

It has been subsequently suggested that a similar property could hold in a more general class of gauges²⁾ while, very recently, other authors have argued that the ordinary (i.e., non-SUSY) Yang-Mills theory could reduce to a Gaussian theory in the $A_0 = 0$ gauge³⁾.

In this note, we shall try to clarify this confused issue of gauge dependence by giving the Jacobian of the transformation between the original A_μ variables and the candidate stochastic variables F_i in a general gauge^{*)}.

Our results confirm the claim that the light cone gauge has a special status among all the others. Indeed, our expression for the Jacobian is shown to give rise to an effective action in terms of the variables F_i which is, in general, non-Gaussian and even non-local. A few examples are worked out to the first non-trivial order.

In order to put our results in the right perspective, we recall that the possibility of defining stochastic variables in a gauge theory rests on the fulfilment of the following conditions:

- i) The bosonic action must be expressible as a quadratic form in the candidate stochastic variables (SV) which should be local functionals of the original bosonic variables.
- ii) The Jacobian of the transformation between the original functional integration volume (including gauge fixing and Faddeev-Popov determinant) and the one given in terms of the stochastic variables must be well defined (implying that the correct number of SV must exist) and should cancel against the fermion determinant.

The above conditions i) and ii) are necessary for obtaining a stochastic theory. They should hold once the "correct" rules for computing determinants and Jacobians are used. This implies a prescription for singularities of propagators

*) We shall stick though to the Wess-Zumino supergauge.

and - as is usual in gauge theories - of spurious singularities implied by gauge fixing. The latter are related to residual gauge invariances left over by gauge fixing or, equivalently, by the necessity of imposing further constraints (such as Gauss's law) on the physical states.

In this paper, we shall not deal with these latter subtleties, limiting ourselves to the calculation of points i) and ii) at a lowest significant order where prescriptions are known and for which the gauge dependence of the stochastic character of the theory clearly emerges. Thus, our results will tell which gauges may have a chance of leading to a stochastic theory. Explicit calculations now being performed by several groups⁴⁾ will clarify whether or not residual gauge effects could jeopardise stochasticity even for light-like gauges.

The same question can also be studied by using the method of "stochastic identities" introduced in Ref. 1), and by paying more attention to boundary conditions as a possible new source of supersymmetry breaking.

2. - It is well known⁵⁾ how to satisfy the requirement i) in a Yang-Mills theory. Defining the three self-dual field strengths F_1^a by

$$F_i^a \equiv F_{4i}^a + \frac{1}{2} \epsilon_{ijk} F_{jk}^a \quad i, j, k = 1, 2, 3 \quad (1)$$

it is easy to see that the Yang-Mills action in the Euclidean formulation can be written as

$$\Gamma_{YM} = - \frac{1}{4} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a = \int d^4x F_i^a F_i^a + \text{surface terms} \quad (2)$$

The YM action is thus quadratic in the F_1 variables if we neglect surface terms^{*)}. The crucial question at this point is the functional integration volume which, ideally, should end up being $\int \prod dF_1$. The original gauge invariant integration

*) Keeping the surface term is equivalent to working in a $\theta \neq 0$ vacuum. This is immaterial for the super YM theory since the massless gluino allows us to rotate θ away.

volume is, however, $\int \prod_{\mu} dA_{\mu}$, and we have to compute the Jacobian of the transformation from the A_{μ} 's to the F_i 's. The 3×4 matrix $\delta F_i / \delta A_{\mu}$ can be written conveniently¹⁾ by choosing the basis

$$F_{\pm} = \frac{F_1 \pm i F_2}{2} \quad \begin{cases} A_{\pm} = A_1 \pm i A_2 \\ A_R = A_3 \pm i A_4 \end{cases} \quad (3)$$

where it takes the form

$$\frac{\delta F_h^a}{\delta A_{\alpha}^b} = \begin{pmatrix} -D_L & 0 & 0 & D_+ \\ D_- & -D_L & -D_+ & D_R \\ 0 & -D_- & D_R & 0 \end{pmatrix}, \quad \begin{aligned} h &= +, 3, - \\ \alpha &= +, R, -, L \end{aligned} \quad (4)$$

The covariant derivatives D_{α}^{ab} in (4) are all in the adjoint representation, i.e.,

$$D_{\alpha}^{ab} = \partial_{\alpha} \delta^{ab} + g f^{abc} A_{\alpha}^c \quad (5)$$

In order to define a new integration volume, we need a fourth variable besides the F_i . Let us choose one of the light-cone^{*)} A_{α} , say A_L . From Eq. (4) we find¹⁾

$$d^4 A_{\mu} = \frac{d^3 F_i \, dA_L}{\det \mathcal{D} \cdot \det D_L} \quad (6)$$

In order to define a (non-trivially divergent) functional integral over the gauge fields A_{μ} , we must introduce a gauge fixing term and, correspondingly, a Faddeev-Popov determinant:

*) The convenience of this choice is probably related to the light cone structure of the F 's themselves, which are linear combinations of $F_{\mu\nu}$'s with relative weights ± 1 .

$$\prod_{\mu,a} dA_{\mu}^a \Rightarrow d[A] = \prod_{\mu,a} dA_{\mu}^a \delta(\chi^a) \Delta_{\chi}$$

where $\chi^a = \chi^a(A_{\mu}) = 0$ is the gauge condition and

$$\Delta_{\chi} \equiv \det \frac{\delta \chi^a}{\delta \Omega^b}$$

with Ω^b an infinitesimal gauge transformation parameter is the FP determinant.

Clearly, for any gauge invariant functional I , we have

$$\begin{aligned} \int d[A] I &= \int \frac{dF_i^a dA_L^a \delta(\chi^a) \Delta_{\chi}}{\det \mathcal{D} \det D_L} I \\ &= \int \frac{dF_i^a d\chi^a \delta(\chi^a)}{\det \mathcal{D} \det D_L} \Delta_{\chi} \det \left(\frac{\delta A_L}{\delta \chi} \right)_F \cdot I \end{aligned} \quad (7)$$

The last determinant in Eq. (7) is the Jacobian of the transformation from A_L to χ at constant F , i.e., the inverse of the Jacobian from A_L to

$$\chi = \chi(A_L, F) \quad (8)$$

This Jacobian can be conveniently rewritten using the fact that

$$\begin{aligned} D_L^{ab} = \frac{\delta A_L^a}{\delta \Omega^b} &= \left(\frac{\delta A_L^a}{\delta \chi^c} \right)_F \frac{\delta \chi^c}{\delta \Omega^b} + \left(\frac{\delta A_L^a}{\delta F_i^c} \right)_{\chi} \frac{\delta F_i^c}{\delta \Omega^b} = \\ &= \left(\frac{\delta A_L^a}{\delta \chi^c} \right)_F \frac{\delta \chi^c}{\delta \Omega^b} + g f^{cab} F^c \left(\frac{\delta A_L^a}{\delta F_i^c} \right)_{\chi} \end{aligned} \quad (9)$$

Thus

$$\Delta_\chi \det \left(\frac{\delta A_L}{\delta \chi} \right)_F = \det \left[\frac{\delta \chi}{\delta \Omega} \left(\frac{\delta A_L}{\delta \chi} \right)_F \right] = \det \left[D_L - g \not{F} \left(\frac{\delta A_L}{\delta F} \right)_\chi \right] \quad (10)$$

Inserting (10) into (7), we finally get

$$\begin{aligned} \int d[A] I &= \int \frac{dF_i^a d\chi^a \delta(\chi^a)}{\det \not{D} \det D_L} \det \left(D_L - g \not{F} \left(\frac{\delta A_L}{\delta F} \right)_\chi \right) \cdot I = \\ &= \int \frac{dF_i^a}{\det \not{D}} \det \left(1 - g D_L^{-1} \left(\frac{\delta A_L}{\delta F_i} \right)_\chi \not{F}_i \right)_{\chi=0} \cdot I \end{aligned} \quad (11)$$

where $\delta A_L / \delta F_i$ is taken at χ fixed and equal to zero. Such a derivative is obviously zero if $\chi = A_L$.

In the case of $N = 1$ supersymmetric YM theory, there is a factor $\det \not{D}$ which comes from the integration over the gaugino field, and one is left with the following effective action written in terms of the F 's:

$$\begin{aligned} &\int dF_i^a \exp \left[\Gamma_{\text{eff}}(F_i^a) \right] \\ \Gamma_{\text{eff}}(F_i^a) &= \int d^4x \left[F_i^a{}^2 + \frac{1}{2} \log \left(1 - g D_L^{-1} \frac{\delta A_L}{\delta F_i} \not{F}_i \right)_{\chi=0} \right] \end{aligned} \quad (12)$$

where A_L (which also appears in D_L) stands for $A_L(F_i, \chi)_{\chi=0} \equiv A_L(F_i)$. Equation (12) is our main result, exhibiting an explicit, generally non-local action in the candidate stochastic variables F_i .

3. - We shall now discuss the computation of the effective action, and therefore the correlation functions of the F 's, to the lowest non-trivial order in some particular gauges.

In the $\chi = A_L = 0$ gauge^{*)}, the tr log in Eq. (12) vanishes so $\Gamma_{\text{eff}}(\vec{F})$ is purely Gaussian. The same is, of course, true for any other light cone gauge $\chi = n \cdot A$, $n^2 = 0$. This may be checked by going back to Eq. (7) and verifying that

$$\frac{\Delta \chi}{\det D_L} \det \left(\frac{\delta A_L^a}{\delta \chi^b} \right)_F = 1$$

a procedure which is equivalent to choosing $n \cdot A$ instead of A_L as the fourth variable in Eq. (6). It is easy to realize that gauge conditions of the form

$$\chi^a = n \cdot A^a - \phi(F) \quad , \quad n^2 = 0 \quad (13)$$

where $\phi(F)$ is gauge invariant also lead to a stochastic result. This is not true for more general forms of ϕ (non-local, non-gauge invariant functional) which we have looked at.

Let us now turn to other axial gauges $n \cdot A = 0$, $n^2 \neq 0$. In order to show that the tr log expression in Eq. (12) is a non-trivial functional of the F_i , let us compute it to order g^2 . We thus need $A_L(F)$ to order g , which, in momentum space, may be written as

$$A_L^a(p) = K_p^i F_i^a(p) + g f^{abc} \int d^4 q F_i^b(p-q) O_{p,q,p-q}^{ij} F_j^c(q) \quad (14)$$

where the expressions for K^i and O^{ij} are easy to obtain from Eq. (4) and will be given explicitly later for some particular gauges. Equation (14) represents the infradiagrams⁶⁾ which give A_L as functions of F . At order g , these are represented by

$$A \xrightarrow{p} \times F \quad + \quad A \xrightarrow{p} \begin{array}{l} \nearrow q \times F \\ \searrow p-q \times F \end{array}$$

By introducing now the expression (14) for A_L in D_L and in $(\delta A_L / \delta F)$ in the tr log of Eq. (12), it is easy to see that to order g^2 ,

*) Light-like gauge conditions are not well suited for a Euclidean formulation. An appropriate metric is discussed in Ref. 1).

$$\Gamma_{\text{eff}} = g^2 F_i^a(q) \Omega_{ij}^{(q)} F_j^a(q)$$

$$\Omega_{ij}^{(q)} = \frac{1}{2} \int \frac{d^4 p}{p_L (q-p)_L} K_p^i K_{q-p}^j - \int \frac{d^4 p}{p_L^2} K_p^i K_{q-p}^j -$$

$$- \int \frac{d^4 p}{p_L} \left(O_{p,q,p-q}^{ij} - O_{p,p-q,q}^{ji} \right) \quad (15)$$

The integrals of Eq. (15) are typical of one-loop gauge theory calculations and imply a prescription for the singularities. This should not present any problems, at least at the single-loop level, the p_L^{-1} having to be defined with the Mandelstam prescription. We shall come back to this question later.

Let us now discuss the forms of K^i and O^{ij} and, therefore, of Γ_{eff} for some specific gauges. We first consider a gauge which is infinitesimally close to the light cone

$$\chi^a = n \cdot A^a \quad n = (1-\epsilon, 0, 0, 1+\epsilon) \quad , \quad n^2 = 4\epsilon$$

$$\therefore \chi^a = A_L - \epsilon A_R = 0 \quad (16)$$

We shall here use ϵ as an expansion parameter. The $n^2 \rightarrow 0$ behaviour of gauge theory calculations has been discussed in the literature. It has been noted that, while this limit leads to singularities if a principal value prescription for $(n \cdot p)^{-1}$ is used, the same limit is smooth⁷⁾ if

$$(n \cdot p)^{-1} = \lim_{\delta \rightarrow 0} \frac{1}{n \cdot p + i\delta} \quad , \quad n^* \equiv (n_0, -\vec{n}) \quad (17)$$

For $n^2 \rightarrow 0$, Eq. (17) gives the Leibbrandt light cone prescription which - at least as far as we have used it - leads to results analogous to that of Mandelstam⁸⁾. Clearly

$$A_L(\vec{F}) \Big|_{\chi=0} = A_L(\vec{F}) \Big|_{A_L=\epsilon A_R} = \epsilon A_R(\vec{F}) \Big|_{A_L=0} + O(\epsilon^2)$$

where $A_R(\vec{F})|_{A_L=0}$ is the light cone gauge expression for A_R which can be found in Ref. 1). It has the form of Eq. (14) with

$$K_p^i = \epsilon \Delta_{Ri}(p)$$

$$O_{p,q,p-q}^{ij} = \epsilon \Delta_{Rh}^{(p)} \Delta_{\alpha i}^{(q)} \Delta_{\beta}^{(p-q)} \eta^{k\alpha\beta} \quad (18)$$

where indices i, j, k run over $+, 3, -$ and indices α, β over $+, R, -$ with

$$\Delta_{\alpha i}(p) = \begin{pmatrix} p_L^{-1} & 0 & 0 \\ \frac{p-p_R}{p_L p^2} & \frac{p_R}{p^2} & \frac{p_+}{p^2} \\ \frac{p_-^2}{p_L p^2} & \frac{p_-}{p^2} & -\frac{p_L}{p^2} \end{pmatrix} \quad \begin{matrix} \alpha=+ \\ \alpha=R \\ \alpha=- \end{matrix} \quad (19)$$

and $\eta^{k\alpha\beta}$ given by

$$\eta^{+\alpha\beta} = 0, \quad \eta^{3\alpha\beta} = \delta_{\alpha-} \delta_{\beta+} - \delta_{\alpha+} \delta_{\beta-}, \quad \eta^{-\alpha\beta} = \delta_{\alpha R} \delta_{\beta-} - \delta_{\alpha-} \delta_{\beta R} \quad (20)$$

To order ϵ in Γ_{eff} , only the last integral in Eq. (15) contributes. By using the explicit expression (18) for O^{ij} , it is a simple matter to write $\Omega^{ij}(q)$. It will contain typical one-loop integrals of the form

$$\int \frac{dp \, p_{\alpha} (p-q)_{\beta}}{p_L p^2 (p-q)^2} \quad (21)$$

with α, β being $+, -, R, L$ indices. As stated before, we notice the typical $1/p_L$ light cone gauge singularity which requires a prescription, as extensively discussed in the literature. Requirements such as the fulfilment of Ward identities, lead⁸⁾ to the prescriptions of Mandelstam and Liebrandt in accordance with Eq. (17). With those prescriptions, the integrals of Eq. (15) are explicitly calculable^{7), 8)}. Using dimensional regularization they are finite for $\alpha, \beta = +, R, -$ while they give a $(4-d)^{-1}$ pole otherwise. This implies the logarithmically

divergent wave function renormalization typical of supersymmetric gauge theories. We find therefore a non-local effective action, bilinear in F at order g^2 , showing explicitly that in the gauge (16) the theory is not stochastic.

The advantage of the gauge (16) was merely the presence of the parameter ε that allowed the exclusion of the integrals with K^1 in Eq. (15). Their contribution, for other gauges, may be evaluated analogously. For the $A_0 = 0$ gauge, for instance, in the basis $i, j = 1, 2, 3$, one finds

$$K_p^i = T_{(p)}^{3i}, \quad O_{p,q,p-q}^{ij} = T_{(p)}^{3i} T_{(q)}^{mj} T_{(p-q)}^{un} \varepsilon_{lmn} \quad (22)$$

where

$$T_{(p)}^{mi} = \frac{1}{p_0 p^2} \begin{pmatrix} p_0^2 + p_1^2 & -p_0 p_3 - p_1 p_2 & p_1 p_3 - p_0 p_2 \\ p_0 p_3 - p_1 p_2 & p_0^2 + p_2^2 & -p_0 p_1 - p_2 p_3 \\ p_1 p_3 + p_0 p_2 & p_0 p_1 - p_2 p_3 & p_0^2 + p_3^2 \end{pmatrix} \quad (23)$$

The insertion of (22) and (23) into (15) leads to one-loop integrals similar to those discussed before. As before, this gives a non-local expression for Γ_{eff} at order g^2 , showing the non-stochasticity of the theory in the $A_0 = 0$ gauge⁹⁾.

Let us remark that the Jacobian of the transformation from the initial A_μ to the F_i is what gives rise to this effective action. This is what remains after cancellation of the Faddeev-Popov determinant and the $\det \mathcal{D}$ coming from the fermion integration. For pure QCD, i.e., in the absence of fermions, $\text{tr} \log \mathcal{D}$ would have to be added to the effective action. Neither contribution vanishes, nor do they cancel each other, in contrast to a recent claim by Claudson and Halpern³⁾. Their result in the $A_0 = 0$ gauge is due to a retarded prescription for singularities that automatically kills the expansion of the $\text{tr} \log$ expression for Γ_{eff} beyond the trivial first order. From our point of view, the prescription used in Ref. 3) is unwarranted for calculating Green functions as correlation functions in the physical vacuum. More recently, the same authors have argued⁹⁾ that their prescription is actually justified for expectation values in a "null state" which differs from the usual Fock vacuum.

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