STOCHASTIC DIAGRAMS AND FEYNMAN DIAGRAMSH. Hüffel and P.V. Landshoff^{*)}

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ABSTRACT

We study the relationship between ordinary perturbation theory and perturbation theory obtained from stochastic quantization. We give a simple proof that, except in gauge theories, the several stochastic diagrams of a given topology are together equivalent to the corresponding Feynman diagram. Our analysis is presented in Minkowski space, but most of it may readily be adapted to Euclidean space. The field propagator may be a non-diagonal matrix, such as is the case in real-time thermal field theory.

We present a new version of the Langevin equation which directly reproduces the usual axial-gauge perturbation theory. Otherwise, we find that the relationship between ordinary and stochastic perturbation theory is not simple, and we present a recursive method of reconstructing Feynman diagrams from stochastic diagrams, without the need explicitly to introduce ghost fields. We consider both the original Parisi-Wu version of the Langevin equation, and Zwanziger's modified version with its stochastic gauge-fixing term.

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1. - INTRODUCTION

In Parisi and Wu's stochastic approach to the quantization of field theory¹⁾, one introduces an additional co-ordinate t (a fictitious time), and a white-noise random source $\eta(x,t)$. A classical field $\phi(x,t)$ is defined as the solution of the Langevin equation

$$\frac{\partial}{\partial t} \phi(x,t) = i \frac{\delta S}{\delta \phi(x,t)} + \eta(x,t) \quad (1.1)$$

subject to a suitable boundary condition. One then calculates stochastic correlation functions by taking stochastic averages

$$\langle \phi(x_1, t) \phi(x_2, t) \dots \phi(x_N, t) \rangle \quad (1.2)$$

and finds that in the equilibrium limit these approach the quantum Green's functions

$$\langle 0 | T \phi(x_1) \phi(x_2) \dots \phi(x_N) | 0 \rangle \quad (1.3)$$

The equilibrium limit usually requires $t \rightarrow +\infty$, but with a suitable boundary condition it may already be achieved at finite t (see Section 2).

In this paper, ϕ may represent a whole set of fields, including fermionic fields (see Section 4) and matrix fields. We work in Minkowski space, which is why²⁾ the action term in the Langevin equation (1.1) appears with the factor i . However, most of our equations may readily be adapted to Euclidean space. If we write

$$i \frac{\delta S}{\delta \phi} = i D \left(i \frac{\partial}{\partial x} \right) \phi(x,t) + i \frac{\delta S^{INT}}{\delta \phi} \quad (1.4)$$

where the first term corresponds to the free-field part of the action, Feynman's prescription includes in D an infinitesimal imaginary part which is positive and, at least in each term of perturbation theory, it ensures convergence to equilibrium. In the case of zero-temperature field theory, this imaginary part is just a multiple of the unit matrix; at finite temperature³⁾, it is a specific non-diagonal infinitesimal matrix whose eigenvalues are positive.

The stochastic average (1.2) may be calculated as a perturbation expansion¹⁾, whose terms correspond to diagrams which we call stochastic diagrams. In the equilibrium limit several stochastic diagrams together correspond to a given

Feynman diagram of the perturbation expansion of the quantum Green's function (1.3). Various authors⁴⁾⁻⁷⁾ have explored by various methods the relationship between the two types of diagram.

In Section 2 we give a new, simple analysis of this relationship. We work with a Fourier transform with respect to the fictitious time t , which allows us rather easily to take the equilibrium limit early in the calculation. We require that D be invertible (its inverse is the Feynman propagator) and we show that in perturbation theory the limit of the stochastic average is indeed the quantum Green's function.

A particular interest of the stochastic approach lies in the quantization of gauge theories. Parisi and Wu¹⁾ have proposed that it is not necessary explicitly to include either gauge-fixing terms or ghosts in the Langevin equation. However, then D is not invertible and our analysis of Section 2 is not immediately applicable. In Section 3 we present a new version of the Langevin equation for which the work of Section 2 is valid; this equation yields in the equilibrium limit the usual perturbation theory in axial gauge.

As we describe in Section 4, a more general form of the Langevin equation includes an additional "kernel" operator K :

$$\frac{\partial}{\partial t} \phi(x, t) = i \int dy \ K(x-y) \frac{\delta S}{\delta \phi(y, t)} + \zeta(x, t) \quad (1.5)$$

If ϕ represents a set of fields, K will generally be a matrix operator. The white-noise function ζ satisfies

$$\begin{aligned} \langle \zeta(x, t) \rangle &= 0 \\ \langle \zeta(x, t) \zeta(y, t') \rangle &= [K(x-y) + K(y-x)] \delta(t-t') \end{aligned} \quad (1.6)$$

with appropriate relations for stochastic averages of more than two ζ 's. We show that when D is invertible, and K fulfils certain conditions given in Section 4, the analysis of Section 2 may easily be modified and in the equilibrium limit the stochastic average (1.2) again approaches the quantum Green's function (1.3), independent of K . Introducing a kernel allows one easily to handle fermions^{8),9)}. The corresponding components of ϕ must be anticommuting Grassmann fields, but this detail will not affect our work in any essential way. In Minkowski space, there are complications because the Dirac matrix-operator D contains complex γ matrices, so that convergence to equilibrium is not ensured by the $i\epsilon$ prescription. Introducing the same kernel as is used for Euclidean space⁸⁾ overcomes

this problem. Another interesting choice for the kernel is the free-field propagator itself; in this case each stochastic diagram becomes equal to the corresponding Feynman diagram times a constant fraction. In any case one needs a kernel in order to match the dimensions of the fictitious time t in the boson and fermion Langevin equations.

Our demonstration that one may introduce a kernel is not valid when D is not invertible. Nevertheless, our axial-gauge Langevin equation of Section 3 may be obtained formally from the Parisi-Wu form of the Langevin equation, by introducing a particular kernel. A different kernel, similarly applied, gives an equation which in the equilibrium limit yields the usual perturbation theory in the Landau gauge. However, ghost contributions are absent, so that this procedure is not valid for a non-Abelian gauge theory. In general, then, a kernel may not be used in non-Abelian gauge theory. We have no fundamental understanding of why it happens to work in the particular case of our axial-gauge equation.

In Section 5 we discuss Zwanziger's approach^{10),11)} in which a t -dependent gauge transformation is applied to the Parisi-Wu form of the Langevin equation. This is chosen so as to yield a new D that is invertible. However, in the non-Abelian case it also introduces a new interaction vertex into the theory. This vertex is not derivable from an action and we show that in consequence the relation between stochastic diagrams and Feynman diagrams is very complicated. We give a recursive method of constructing Feynman diagrams in this theory, which allows the calculation of gauge invariant quantities without the explicit inclusion of ghosts. By means of a simple example, we show that it is again not generally valid in the Zwanziger approach to introduce a kernel.

In Section 6, we discuss the direct use of the Parisi-Wu form of the Langevin equation without explicitly introducing gauge fixing or ghosts. We show that it is essential to take the limit $\epsilon \rightarrow 0^+$ before going to the equilibrium limit. The boundary condition applied to the solution of the Langevin equation now has an important effect. There is again a complicated relationship between stochastic and ordinary Feynman diagrams.

We should perhaps emphasize that all our work in this paper is within the framework of perturbation theory.

2. - STOCHASTIC QUANTIZATION AND FEYNMAN DIAGRAMS

In this section we develop a perturbation theory for calculating the equilibrium limit of the stochastic average (1.2). We show that, provided D in (1.4) is invertible, we retrieve the ordinary Feynman-diagram expansion of the matrix element (1.3).

For simplicity of presentation, we restrict ourselves to real scalar fields [so that D is symmetric and $D(p) = D(-p)$] and suppose that the interaction is trilinear and totally symmetric, with no derivatives:

$$\mathcal{L}^{\text{INT}} = \frac{1}{3!} \lambda_{ijk} \phi_i \phi_j \phi_k \quad (2.1)$$

These details are not essential to our analysis (fermions are discussed in Section 4).

We work with the Fourier transform of the field:

$$\phi(p, \omega) = \int d^n x \, dt \, e^{i p \cdot x + i \omega t} \phi(x, t) \quad (2.2)$$

The Fourier transform with respect to t was previously used⁷⁾ in a superfield formulation of stochastic quantization. We show its usefulness when applied directly to the basic Langevin equation (1.1). We have

$$-i(\omega I + D(p)) \phi(p, \omega) = \eta(p, \omega) + \frac{1}{2} i \lambda \phi^2(p, \omega) \quad (2.3)$$

where I is the unit matrix and

$$\begin{aligned} [\lambda \phi^2(p, \omega)]_i &= \frac{\lambda_{ijk}}{(2\pi)^{n+1}} \int d^n p_1 \, d^n p_2 \, d\omega_1 \, d\omega_2 \, \delta^{(n)}(p_1 + p_2 - p) \\ &\quad \delta(\omega_1 + \omega_2 - \omega) \phi_j(p_1, \omega_1) \phi_k(p_2, \omega_2) \end{aligned} \quad (2.4)$$

It is important to notice that by taking the Fourier transform with respect to t , we have suppressed that part of $\phi(x, t)$ which depends on whatever initial condition we impose on the solution of the Langevin equation. Because D implicitly contains Feynman's $i\epsilon$, this part of the solution is transient: it dies away exponentially like $e^{-\epsilon t}$ in the equilibrium limit. Because it explodes exponentially for large negative t , its Fourier transform does not exist and our

procedure does not take account of it. As we are interested only in the equilibrium limit, this removal represents a considerable simplification. A similar procedure is familiar in the analysis of alternating-current electric circuits, where the usual expression $V/Z(\omega)$ for the electric current in terms of the complex impedance is valid only in the steady-state limit, when the transients associated with the precise way in which the circuit is switched on have died away. Suppressing the transients amounts, at least formally, to imposing the initial condition¹²⁾

$$\phi(x, t) \rightarrow 0 \quad \text{as } t \rightarrow -\infty \quad (2.5)$$

so that the stochastic process relaxes to equilibrium already at finite fictitious time t . We shall indeed find that when D is invertible our procedure yields stochastic averages that are independent of t at finite t . When D is not invertible, as in the case of the Parisi-Wu form of the gauge-field Langevin equation, special care is needed; we discuss this in Section 6. In this section we suppose that D is invertible, and then there is no subtlety.

Introducing the stochastic Green's function

$$G(p, \omega) = [\omega I + D(p)]^{-1} \quad (2.6)$$

we may solve (2.3) iteratively and represent the resulting perturbation expansion for $\phi(p, \omega)$ graphically: see Fig. 1. With each line of a graph we associate a pair of variables (p, ω) and a stochastic Green's function $iG(p, \omega)$. There is an integration over all the (p, ω) other than those on the left-most line. Each vertex represents the interaction $\frac{1}{2} i\lambda/(2\pi)^{n+1}$; because of the δ -functions in (2.4) the total (p, ω) flowing from left to right at each vertex is conserved. Each cross represents a factor η , whose arguments are the values of (p, ω) on the line attached to it.

When, as in this section, K is the unit operator, the Fourier transforms of the relations (1.6) are

$$\begin{aligned} \langle \eta(p, \omega) \rangle &= 0 \\ \langle \eta(p, \omega) \eta(p', \omega') \rangle &= 2(2\pi)^{n+1} \delta^{(n)}(p+p') \delta(\omega+\omega') \end{aligned} \quad (2.7)$$

To calculate the stochastic average

$$\langle \phi(p_1, \omega_1) \phi(p_2, \omega_2) \dots \phi(p_N, \omega_N) \rangle \quad (2.8)$$

we first draw all possible stochastic diagrams, obtained by inserting the expansion of Fig. 1 for each ϕ . A typical example is drawn in Fig. 2. It is convenient to draw the diagrams such that the lines (p_1, ω_1) , (p_2, ω_2) , ..., (p_N, ω_N) all enter from the left. The rules for calculating these diagrams are the same as for Fig. 1, except that because of (2.7) a cross now denotes $2 \cdot (2\pi)^{n+1}$, with the pair of ω variables entering it from the left adding to zero, and likewise the pair of p variables. Also, if one draws each stochastic diagram of a given topology only once, the factor $\frac{1}{2}$ in the interaction vertex should be omitted. This takes account of the different ways in which the crosses in the expansion of ϕ can be paired in (2.8). (There are complications in the case of diagrams, such as bubble diagrams, that have internal symmetry. We do not go into these complications: they are just the same as in ordinary perturbation theory.)

Because of these rules, each diagram contains the δ -function

$$\delta(\omega_1 + \omega_2 + \dots + \omega_N) \quad (2.9)$$

and

$$V_N = \langle \phi(p_1, t) \phi(p_2, t) \dots \phi(p_N, t) \rangle, \quad (2.10)$$

which is obtained by applying to (2.8) the integration

$$\frac{1}{(2\pi)^N} \int d\omega_1 d\omega_2 \dots d\omega_N e^{-i(\omega_1 + \omega_2 + \dots + \omega_N)t} \quad (2.11)$$

is independent of t . From what we have said, it should therefore be equal to the matrix element

$$\langle 0 | T \phi(p_1) \phi(p_2) \dots \phi(p_N) | 0 \rangle \quad (2.12)$$

[by which we mean just the Fourier transform of the matrix element (1.3)]. Because our Fourier transform procedure suppresses the transients, the equilibrium limit has in effect already been taken.

Because of (2.11) and the δ -function (2.9), we must now integrate over all the ω variables, including those on the external lines of the stochastic diagram.

The integrand is a product of poles, so that the ω integrations just give a sum of residues. However, if the matrix $D(p)$ is not diagonal, this is not completely straightforward.

As a simple example, consider the lowest-order contribution to the three-point function [$N = 3$ in (2.10)]. There are three stochastic diagrams, Fig. 3 and two similar diagrams. Figure 3 contributes

$$V_3^{(1)} = (2\pi)^n \delta^{(n)}(p_1 + p_2 + p_3) \frac{1}{(2\pi)^2} \int d\omega_1 d\omega_2 d\omega_3 \delta(\omega_1 + \omega_2 + \omega_3) U_3^{(1)} \quad (2.13)$$

where

$$U_3^{(1)} = \begin{bmatrix} G(p_1, \omega_1) \\ 2 G(p_2, \omega_2) G(-p_2, -\omega_2) \\ 2 G(p_3, \omega_3) G(-p_3, -\omega_3) \end{bmatrix} \lambda \quad (2.14)$$

We have used a generalized matrix notation, in which the i^{th} row of the square bracket corresponds to the i^{th} line on the left of the diagram. From (2.6) it is trivial to derive the identity

$$D(p) G(p, \omega) = I - \omega G(p, \omega) \quad (2.15)$$

Hence

$$\begin{aligned} & \begin{bmatrix} D(p_1) \\ I \\ I \end{bmatrix} U_3^{(1)} + \begin{bmatrix} I \\ D(p_2) \\ I \end{bmatrix} U_3^{(1)} + \begin{bmatrix} I \\ I \\ D(p_3) \end{bmatrix} U_3^{(1)} \\ &= \begin{bmatrix} I \\ 2 G(p_2, \omega_2) G(-p_2, -\omega_2) \\ 2 G(p_3, \omega_3) G(-p_3, -\omega_3) \end{bmatrix} \lambda + \begin{bmatrix} G(p_1, \omega_1) \\ 2 G(-p_2, -\omega_2) \\ 2 G(p_3, \omega_3) G(-p_3, -\omega_3) \end{bmatrix} \lambda + \end{aligned}$$

$$+ \begin{bmatrix} G(p_1, \omega_1) \\ 2 G(p_2, \omega_2) G(-p_2, -\omega_2) \\ 2 G(-p_3, -\omega_3) \end{bmatrix} \lambda - (\omega_1 + \omega_2 + \omega_3) U_3^{(1)} \quad (2.16)$$

When the integration (2.13) is applied to this equation, the last term on the right-hand side vanishes because of the δ -function. We use the δ -function to perform the ω_1 integration in the other terms on the right-hand side. Then by closing the contour of the ω_2 integration with an infinite semicircle in the lower half complex plane, and remembering the $i\epsilon$ -prescription included in G , we see that the second term also vanishes. Similarly, the ω_3 integration makes the third term vanish. In the first term the ω_2 and ω_3 integrals are both of the form

$$\int d\omega \ 2 G(p, \omega) G(-p, -\omega) \quad (2.17)$$

Because we have said that we are assuming that $D(p) = D(-p)$, the matrices $G(p, \omega)$ and $G(-p, -\omega)$ commute and so the integral is straightforward: it is equal to $-2\pi i D^{-1}(p)$. (It is not immediately obvious that things work out the same way for fermion fields, but nevertheless it is true, as we indicate in Section 4.)

If we now add to $V_3^{(1)}$ the two other diagrams, obtained by permuting the left-hand lines in Fig. 3, we therefore have

$$\begin{bmatrix} D(p_1) \\ I \\ I \end{bmatrix} V_3 + \dots + \dots$$

$$= - (2\pi)^n \delta^{(n)}(p_1 + p_2 + p_3) \begin{bmatrix} I \\ D^{-1}(p_2) \\ D^{-1}(p_3) \end{bmatrix} \lambda + \dots + \dots \quad (2.18)$$

where the terms not written explicitly are obtained by cyclic permutation. If we now diagonalize each of the matrices $D(p)$, which is possible by means of an orthogonal transformation because we have said they are symmetric, and use the

fact that none of their eigenvalues vanish if they are invertible, we obtain finally (at least for almost all values of the momenta p)

$$V_3 = -(2\pi)^n \delta^{(n)}(p_1 + p_2 + p_3) \begin{bmatrix} D^{-1}(p_1) \\ D^{-1}(p_2) \\ D^{-1}(p_3) \end{bmatrix} \lambda \quad (2.19)$$

which is the correct three-point function in lowest order.

Notice the dual role of Feynman's ϵ prescription in this work. Not only does it ensure convergence towards the equilibrium limit; it also makes sure that we end up with the correctly-defined propagators.

We now derive the corresponding result for general diagrams. Our proof is by induction. It is along the lines described in Ref. 4), but it is simpler because of our ω -space technique. We assume first that (2.10) and (2.12) are equal up to some order in the coupling λ , and show that this implies that they are equal also up to the next order. This induction argument begins with the lowest-order result (2.19) above.

A general stochastic diagram gives a contribution which may be written in the form

$$V_N^{(\alpha)} = \int d\omega_1 d\omega_2 \dots d\omega_N \delta(\omega_1 + \omega_2 + \dots + \omega_N) \begin{bmatrix} G(p_1, \omega_1) \\ G(p_2, \omega_2) \\ \vdots \\ G(p_N, \omega_N) \end{bmatrix} \tilde{V}_N^{(\alpha)} \quad (2.20)$$

From the identity (2.15), applied in the same way as in (2.16) above,

$$\sum_{r=1}^N \begin{bmatrix} I \\ I \\ \vdots \\ I \\ D(p_r) \\ I \\ \vdots \\ I \end{bmatrix} V_N^{(\alpha)} = \int d\omega_1 \dots d\omega_N \delta(\omega_1 + \dots + \omega_N) \sum_{r=1}^N \begin{bmatrix} G(p_1, \omega_1) \\ \vdots \\ G(p_{r-1}, \omega_{r-1}) \\ I \\ G(p_{r+1}, \omega_{r+1}) \\ \vdots \\ G(p_N, \omega_N) \end{bmatrix} \tilde{V}_N^{(\alpha)} \quad (2.21)$$

Consider the r^{th} term on the right-hand side. If the (p_r, ω_r) line connects directly to a cross (as is the case for the bottom line in Fig. 2), the term vanishes. The argument is the same as for the second and third terms on the right-hand side of (2.16). There are two other cases shown in Fig. 4. In each of these, replacing $G(p_r, \omega_r)$ by I is equivalent to removing the corresponding external line from the diagram. For Fig. 4a if we remove also the vertex within the dotted box, that is the factor $i\lambda/(2\pi)^{n+1}$, and possibly also an integration over a loop momentum, we obtain just the integral corresponding to the residual stochastic diagram. Although this has one more incoming line than we started with, it has one less vertex. Hence if we sum over all stochastic diagrams α of the given order, we can apply our induction hypothesis. In the case of Fig. 4b, we have to remove the structure within the dotted box by also explicitly performing the ω_{r-1} integration, but the conclusion is similar.

What we have proved, then, is that if V_N is the sum of all stochastic diagrams of a given order in the coupling, and F_N is the corresponding sum of Feynman diagrams,

$$\sum_{r=1}^N \begin{bmatrix} I \\ \vdots \\ I \\ D(p_r) \\ I \\ \vdots \\ I \end{bmatrix} (V_N - F_N) = 0 \quad (2.22)$$

As in the discussion following (2.18), this implies that $V_N = F_N$.

We note that actually a slightly stronger result may be proved in the same way: the sum of all stochastic diagrams of a given topology is equal to the Feynman diagram of the same topology.

3. - AXIAL GAUGE

For a non-Abelian gauge theory, with Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \quad (3.1)$$

the Langevin equation (1.1) reads

$$\frac{\partial}{\partial t} A_\mu^a(x, t) = -i \Delta^{ab\nu} F_{\mu\nu}^b(x, t) + \eta_\mu^a(x, t) \quad (3.2)$$

where Δ is the covariant derivative

$$\Delta_\nu^{ab} = \partial_\nu \delta^{ab} - g f^{abc} A_\nu^c \quad (3.3)$$

and the white-noise function obeys

$$\langle \eta_\mu^a(x, t) \eta_\nu^b(y, t') \rangle = -2 \delta^{ab} g_{\mu\nu} \delta^{(n)}(x-y) \delta(t-t') \quad (3.4)$$

In momentum space (3.2) takes the form

$$\begin{aligned} \frac{\partial}{\partial t} A_\mu^a(p, t) = & i \left[(p^2 + i\epsilon) g_{\mu\nu} - p_\mu p_\nu \right] A^{\nu a}(p, t) \\ & + I_\mu^a(p, t) + \eta_\mu^a(p, t) \end{aligned} \quad (3.5)$$

where I_μ^a denotes the three-gluon and four-gluon interaction terms. We have explicitly written Feynman's $i\epsilon$. Note that in (3.2), (3.4) and (3.5) we have introduced a "kernel" $K = -1$ [see (1.5) and (1.6)] in order that the $i\epsilon$ gives convergence to the equilibrium limit. Less trivial kernels are discussed later in this paper.

The work of Section 2 is not applicable to this Langevin equation, because the coefficient of $A^{\nu a}$ on the right-hand side is not invertible when $\epsilon \rightarrow 0$. As is well known, this is connected with the fact that the field equations contain unphysical degrees of freedom. The Langevin equation (3.5) is supposed¹⁾ to yield the correct results only for gauge-invariant quantities, but if we attempt an analysis such as that associated with Fig. 4, described in the previous section, we encounter non-gauge-invariant subamplitudes. We have not been able to

set up a consistent induction procedure to deal with this. Ideally, one should try to build up a formulation that involves only gauge-invariant quantities, such as Wilson loops¹³⁾, but this seems to be very difficult.

The best we can do in this direction is to write down a Langevin equation which in the equilibrium limit yields Feynman perturbation theory in axial gauge. We define

$$N^{\mu\nu}(p) = g^{\mu\nu} - \frac{p^\mu n^\nu}{n \cdot p} \quad (3.6)$$

This has the properties

$$N^{\mu e}(p) N_e^{\nu}(p) = N^{\mu\nu}(p) \quad (3.7)$$

and

$$N^{\mu e}(p) N_e^{\nu}(p) = g^{\mu\nu} - \frac{p^\mu n^\nu + n^\mu p^\nu}{n \cdot p} + n^2 \frac{p^\mu p^\nu}{(n \cdot p)^2} \quad (3.8)$$

Our Langevin equation is

$$\frac{\partial}{\partial t} A_\mu^a = i(p^2 + i\epsilon) A_\mu^a + N_{\mu\nu} (N_e^{\nu} I_e^a + \eta^{a\nu}) \quad (3.9)$$

where I^a is the same interaction as in (3.5) and η^a is also the same white-noise function, obeying (3.4).

It is straightforward to work through the analysis of Section 2 using this Langevin equation. Because (3.8) is the numerator of the usual axial-gauge propagator, the stochastic perturbation theory based on (3.9) reproduces the usual axial-gauge perturbation theory in the equilibrium limit. We remark that we do not intend to discuss here problems connected with the zeros of the denominator in (3.6); whatever is the appropriate prescription¹⁴⁾ in the usual theory, whether it be principal value or otherwise, must also be applied in (3.6).

4. - INTRODUCTION OF A KERNEL

Under certain conditions, the Langevin equation may be modified by the introduction of a "kernel" operator K : see (1.5) and (1.6). If we impose certain conditions on K , it is straightforward to adapt the analysis of Section 2.

(These conditions may perhaps be unnecessary, but they are needed if we want to use the methods of Section 2).

In the Fourier-transformed Langevin equation (2.3), $D(p)$ will be replaced by $K(p) D(p)$ and the interaction term also is multiplied by $K(p)$. We must retain the usual Feynman prescription in D , in order that the propagators obtained in the equilibrium limit be correctly defined. But we must also ensure that $\text{Im } KD > 0$; otherwise there will be no equilibrium limit. Because the interaction is multiplied by $K(p)$, a vertex in a stochastic diagram now corresponds to $iK(p)\lambda/(2\pi)^{n+1}$, where p is the momentum flowing into the vertex from the left. Instead of (2.7) we require the Fourier transform of (1.6):

$$\langle \zeta(p, \omega) \zeta(p', \omega') \rangle = (K(p) + K(p')) (2\pi)^{n+1} \delta^{(n)}(p+p') \delta(\omega+\omega') \quad (4.1)$$

and then a cross in a stochastic diagram will denote $(K(p) + K(-p))(2\pi)^{n+1}$.

$D(p)$ is replaced by $K(p) D(p)$ in the Green's function $G(p, \omega)$ of (2.6) and in the identity (2.15) which it satisfies. When we calculate the three-point function V_3 , instead of (2.17) we encounter the integral

$$\int d\omega G(p, \omega) (K(p) + K(-p)) G(-p, -\omega) \quad (4.2)$$

In order to be able readily to evaluate this, and obtain a recognisable result, we now impose on $K(p)$ the condition that it commutes with $D(p)$:

$$[K(p), D(p)] = 0 \quad (4.3)$$

Because we are still assuming that $D(-p) = D(p)$, then $K(-p)$ will also commute with $D(p)$, and the three factors in the integrand of (4.2) commute with each other. Hence the integral is straightforward and equal to $-2\pi i D^{-1}(p)$, the same as (2.17). $D(p)$ is replaced by $K(p) D(p)$ also in the analogues of (2.18) and (2.22), so we must assume that KD is symmetric in order to be able to diagonalize it. If it is also invertible, so that it does not have zero eigenvalues, we again conclude that $V_N = F_N$, independent of K .

An interesting choice for K which satisfies the conditions we have imposed is

$$K(p) = i D^{-1}(p) \quad (4.4)$$

In this case the ω integrations in the stochastic diagrams decouple from the momentum integrations: each stochastic diagram is equal to the corresponding Feynman diagram times a constant fraction.

Formally, our axial-gauge Langevin equation (3.9) may be obtained by applying to the original Parisi-Wu equation (3.5) the kernel $N^{\mu\rho} N_\rho^\nu$ of (3.8), or in matrix notation

$$K = N N^T \quad (4.5)$$

We identify $\zeta = N\eta$; because η satisfies (3.4), ζ will satisfy (4.1). The stochastic Green's function is

$$G(p, \omega) = \frac{1}{\omega + N N^T D} = \frac{1}{\omega + N p^2} \quad (4.6)$$

where, from (3.5)

$$D^{\mu\nu} = p^2 g^{\mu\nu} - p^\mu p^\nu \quad (4.7)$$

Because of the factor N in ζ and because the interaction is multiplied by K , in stochastic diagrams G always appears multiplied on the right by N . But because of the identity (3.7),

$$G N = \frac{1}{\omega + p^2} N \quad (4.8)$$

That is, we may use $G = (\omega + p^2)^{-1}$ instead of (4.6). But this is just the stochastic Green's function corresponding to our modified Langevin equation (3.9).

We have not paid attention to the $i\epsilon$ prescription in this "derivation" because of course KD is not invertible, so that it does not satisfy the condition we imposed in our general discussion above. Thus, although we showed in Section 3 that (3.9) does lead to the correct axial-gauge perturbation series, we are not able to derive it from the original Parisi-Wu equation (3.5). It is easy to see that such a derivation is not generally possible. If in the equations (4.5) to (4.8) we use for N not (3.6) but rather $N = D/p^2$, the corresponding equation (3.9) would then reproduce perturbation theory in the Landau gauge. However, it would not include any ghost terms, and so would be wrong in non-Abelian gauge theory.

In order to include fermions in the formalism, it is necessary to introduce a kernel in the corresponding Langevin equations. If for no other reason, this is because otherwise the fictitious time t in the fermion equations would have a different dimension from that in the boson equations. More seriously, the Langevin equation for the fermion fields has

$$D(p) = \gamma \cdot p - m + i\epsilon \quad (4.9)$$

At least one of the γ matrices is inevitably complex, which means that the $i\epsilon$ is not sufficient to give convergence to equilibrium^{*)}. Furthermore, in our analysis we required $K(p)$ $D(p)$ to be a symmetric matrix. These difficulties are most easily overcome by introducing the kernel

$$K(p) = D(-p) \quad (4.10)$$

This choice is also made for a slightly different reason in work⁸⁾ on stochastic quantization of fermion fields in Euclidean space. [An alternative would be again the choice (4.4).]

Our general analysis is now valid, except for some technical points. First, to end up with Fermi statistics we must make the fermion field and its stochastic white-noise source anticommuting Grassmann functions. Secondly, when taking Fourier transforms it is convenient to treat ψ and $\bar{\psi}$ as independent, that is we use the definition (2.2) for each of them. In consequence, the momentum of the $\bar{\psi}$ fields is $-p$ rather than p , and its inverse propagator is

$$\bar{D}(p) = D(-p) \quad (4.11)$$

and of course we choose $\bar{K}(p) = K(-p)$. Then $\bar{G}(p, \omega) = G(-p, \omega) = G(p, \omega)$. Instead of (4.1) we impose

^{*)}One of us is grateful to F. Englert for a discussion on this.

$$\begin{aligned}
 \langle \xi(p, \omega) \xi(p', \omega') \rangle &= \langle \bar{\xi}(p, \omega) \bar{\xi}(p', \omega') \rangle = 0 \\
 \langle \xi(p, \omega) \bar{\xi}(p', \omega') \rangle &= \\
 2 D(p) (2\pi)^{n+1} \delta^{(n)}(p+p') \delta(\omega+\omega') &\quad (4.12)
 \end{aligned}$$

Everything is then straightforward. For example, if we evaluate the three-point function which couples ϕ and $\bar{\psi}$ to the gauge field, we encounter instead of (4.2) the integral

$$\int d\omega G(p, \omega) 2 D(p) \bar{G}(-p, -\omega) = \frac{-2\pi i}{8 \cdot p - m} \quad (4.13)$$

Finally in this section, we remark that if one chooses to handle a non-Abelian gauge theory by including in the action both a gauge-fixing term and a ghost field, and then writing a Langevin equation for each field, it is straightforward to see from our analysis that the correct matrix elements will be obtained in the equilibrium limit.

5. - STOCHASTIC GAUGE FIXING

Although stochastic quantization based on the Parisi-Wu form (3.2) of the Langevin equation is believed to be consistent without any gauge fixing, it is often useful to impose a generalized gauge fixing^{10),11)}. In the case of the Parisi-Wu equation, the ω integrations remain finite for gauge-invariant sums of diagrams in the equilibrium limit, but not for individual stochastic diagrams. With appropriate gauge fixing, they remain finite for all diagrams.

The basic idea¹⁰⁾ is to transform the field $A_\mu^a(x, t)$ into a new field so that in the Langevin equation for this new field the matrix D of (4.7) is replaced by an invertible matrix. The transformation is required to leave unchanged all gauge-invariant quantities, and so must be a gauge transformation which, in order to be useful, depends on t . The Langevin equation (3.2) then acquires¹⁰⁾ on its right-hand side an additional term

$$\Delta_{\mu}^{ab} \chi^b \quad (5.1)$$

where χ is an arbitrary function. With the simple choice

$$\chi^b(x, t) = -i \partial^\nu A_\nu^b(x, t) \quad (5.2)$$

the new Langevin equation becomes, when Fourier transformed,

$$-i(\omega + p^2 + i\epsilon) A_\lambda^a(p, \omega) = I_\lambda^a(p, \omega) + J_\lambda^a(p, \omega) + \eta_\lambda^a(p, \omega) \quad (5.3)$$

Here I^a denotes the usual interaction terms and J^a is a new interaction derived from (5.1) and (5.2):

$$J_\lambda^a(p, \omega) = \frac{g f^{abc}}{2(2\pi)^{n+1}} \int d^n q \, d^n r \, d\omega_1 \, d\omega_2 \, \delta^{(n)}(p+q+r) \quad (5.4)$$

$$\delta(\omega_1 + \omega_2 - \omega) A^{b\mu}(-q, \omega_1) A^{c\nu}(-r, -\omega_2)$$

$$v_{\lambda\mu\nu}^J(p, q, r)$$

with

$$v_{\lambda\mu\nu}^J(p, q, r) = g_{\lambda\mu} r_\nu - g_{\lambda\nu} q_\mu \quad (5.5)$$

This new interaction is a three-gluon interaction: see Fig. 5. Unlike the usual three-gluon interaction contained in I^a , for which

$$v_{\lambda\mu\nu}^I(p, q, r) = g_{\lambda\mu} (p-q)_\nu + \text{cyclic perms.} \quad (5.6)$$

it is a directed vertex because it is symmetric under the interchange of two lines only. We have indicated this by an arrow in the figure: in our stochastic diagrams the arrow will always point into the vertex from the left.

The directedness complicates the analysis of this new interaction and makes the relation between stochastic diagrams and Feynman diagrams non-trivial. This complication is presumably because the new interaction is not derivable from a Lagrangian^{10),15)}.

Consider first the contribution from the new vertex to V_3 in lowest order. It takes a different form in each of the three stochastic diagrams (Fig. 3 and two similar diagrams). The three terms sum to

$$-i(2\pi)^n \delta^{(n)}(p_1 + p_2 + p_3) g(p_1^2 + i\epsilon)^{-1} (p_2^2 + i\epsilon)^{-1} (p_3^2 + i\epsilon)^{-1} \\ f^{abc} v_{\lambda\mu\nu}^K(p_1, p_2, p_3)$$

with

$$v_{\lambda\mu\nu}^K(p_1, p_2, p_3) = (p_1^2 + p_2^2 + p_3^2)^{-1} \left[(p_1^2 - p_2^2) g_{\lambda\mu} p_{3\nu} \right. \\ \left. + \text{cyclic perms.} \right] \quad (5.7)$$

This is an effective, undirected vertex, which we shall denote in diagrams by a black spot.

Stochastic diagrams of a given topology that contain only vertices of the ordinary type v^I sum to the corresponding Feynman diagram; this follows directly from the work of Section 2. Because of our choice (5.2) for χ , this Feynman diagram is calculated in Feynman gauge. However, for stochastic diagrams that contain one or more vertices of the new type v^J the situation is much more complicated.

If we sum all such diagrams, and use them to calculate a gauge-invariant quantity, we expect¹⁰⁾ that they should just reproduce the usual ghost contributions. The methods of Section 2 may still be used to express the sum of all the stochastic diagrams of a given topology in terms of residual Feynman-like diagrams having one less vertex. This is much simpler than calculating all the stochastic diagrams individually. However, because the residual diagrams are not gauge-invariant, even when summed, we are unable to set up a simple induction procedure like that of Section 2 which will prove that finally the ghost contributions, and only the ghost contributions, will emerge.

As an example, consider the order- g^2 stochastic diagrams for the two-point function V_2 . The details are in the Appendix. The stochastic diagrams that do not contain a vertex of the new type indeed sum to the ordinary Feynman graphs. With the help of the recursive methods of Section 2, we find that the stochastic diagrams having either or both of their vertices of the new type v^J sum to the Feynman-like diagrams of Fig.6. If we write down the integrals corresponding to

these diagrams, but do not perform the integration over the loop momentum, we recognize a term which is just what is normally associated with a ghost loop. However, the remaining contribution is unrecognizable. We expect that this unrecognizable term should cancel if we use V_2 to calculate some gauge-invariant quantity.

We have checked this in two examples. The first is quark-quark scattering in order g^4 , and the second is

$$\langle 0 | F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) | 0 \rangle \quad (5.8)$$

in order g^2 . In the first example, we suppose that we are using dimensional regularization, so that the quark self-energy and vertex corrections are gauge-invariant. Then the remaining diagrams must together be gauge invariant: they are shown in Fig. 7. In order to keep the calculation simple, we do not apply stochastic quantization to the external quark lines: we merely use them to project out a gauge-invariant quantity from the gluon fields:

$$\begin{aligned} & u_{ri}(p_1) u_{r'i'}(p_2) \bar{u}_{sj}(p_3) \bar{u}_{s'j'}(p_4) \\ & \{ \langle 0 | A_\lambda^a(p) A_{\lambda'}^{a'}(p') | 0 \rangle \gamma_{rs}^\lambda \gamma_{r's'}^{\lambda'} t_{ij}^a t_{i'j'}^{a'} - \\ & \frac{g}{(2\pi)^n} \int d^n q \langle 0 | A_\mu^a(q) A_\nu^b(r) A_\lambda^c(p') | 0 \rangle \\ & \frac{(\gamma^\mu \gamma \cdot (p_1 - q) \gamma^\nu)_{rs} \gamma_{r's'}^{\lambda'} t_{il}^a t_{lj}^b t_{i'j'}^c + \dots }{q^2 - 2p_1 \cdot q} \} \end{aligned} \quad (5.9)$$

where $p = p_1 - p_3$, $p' = p_2 - p_4$ and $p' + q + r = 0$. According to what we have said, the contributions from the ordinary vertices v^I are the normal ones. The anomalous contributions to the terms of Fig. 7 are those in the gluon propagator (Fig. 6) in the first term and a new vertex contribution to the second and third terms. Without doing any loop integration, we find that the unrecognizable integrands just cancel, leaving the wanted ghost term behind.

In the case of $\langle FF \rangle$ in (5.8) the result is similar, but the details are different. The unwanted integrands do not cancel, but it is easy to see that they integrate to zero in any regularization scheme. [We find that the terms disagree in detail with those of Ref. 6), though the conclusion is the same.]

It is tempting to modify the Langevin equation after applying stochastic gauge fixing by including an additional kernel operator K . With the choice

$$K = \frac{i}{p^2 + i\epsilon} \quad (5.10)$$

things apparently become very much simpler, in particular the effective vertex v^K of (5.7) then vanishes and so only the first and fourth terms of Fig. 6 survive. However, we cannot use the proof of the last section to show that the sum of stochastic diagrams is independent of K . If we repeat the calculation of $\langle FF \rangle$, we do find that the answer is the same. But this is not true for quark-quark scattering: the unrecognizable terms no longer cancel, even after integration. We conclude that it is not generally valid to introduce a kernel after the Langevin equation has been modified by stochastic gauge fixing.

Finally in this section, we mention that there is a choice of χ which seems to yield the axial gauge. Having added (5.1) to the right-hand side of the Parisi-Wu form (3.5) of the Langevin equation, we choose χ by imposing

$$n \cdot A(p, t) = n \cdot \dot{A}(p, t) = 0 \quad (5.11)$$

This gives

$$\chi = i p \cdot A - n \cdot (I + \eta) / n \cdot p$$

The result is a Langevin equation rather like (3.9), except that there are additional interaction terms which make things very much more complicated. Presumably, they cancel when gauge-invariant quantities are calculated.

Our main conclusion in this section is that the relationship between the stochastic approach to non-Abelian gauge theory and the ordinary perturbation expansion is not at all simple. That, of course, is its interest.

6. - THE PARISI-WU APPROACH TO GAUGE THEORY

We have emphasized the role of Feynman's ϵ in our analysis. Although the coefficient of A^{av} in (3.5) is not invertible when $\epsilon \rightarrow 0$, there is no problem before this limit is taken. Then

$$\left[D_{\mu\nu}^{ab}(p) \right]^{-1} = \delta^{ab} \left[\frac{g^{\mu\nu} + p^\mu p^\nu / i\epsilon}{p^2 + i\epsilon} \right] \quad (6.1)$$

and the vertices of the theory are solely those of the conventional type. The work of Section 2 is then valid, so that the sum of the stochastic diagrams gives ordinary Feynman diagrams with the propagator (6.1). We have repeated the calculations of $\langle FF \rangle$ and quark-quark scattering in this formalism. When $\epsilon \rightarrow 0$, the negative powers of ϵ cancel in these gauge-invariant quantities and leave behind a term which is just what is normally associated with a ghost loop. However, it is too small by a factor of two.

This discovery is not new¹⁶⁾. By first setting $\epsilon \neq 0$, we have effectively given the gauge field a (complex) mass, and it is known that the zero-mass limit of a massive "Yang-Mills" theory has just this disease.

In the present context, the problem arises because we have taken the equilibrium limit of the stochastic process before $\epsilon \rightarrow 0^+$. If we take these limits in the reverse order, the result is different and apparently then correct.

We explore this. Introduce the transverse and longitudinal projection operators

$$\begin{aligned} T^{\mu\nu}(p) &= g^{\mu\nu} - p^\mu p^\nu / p^2 \\ L^{\mu\nu}(p) &= p^\mu p^\nu / p^2 \end{aligned} \quad (6.2)$$

With D as in (6.1), the stochastic Green's function (2.6) is

$$G_{\mu\nu}^{ab}(p, \omega) = \delta^{ab} \left[\frac{T_{\mu\nu}(p)}{\omega + p^2 + i\epsilon} + \frac{L_{\mu\nu}(p)}{\omega + i\epsilon} \right] \quad (6.3)$$

and its Fourier transform is

$$G_{\mu\nu}^{ab}(p, t) = -i \theta(t) \delta^{ab} \left[T_{\mu\nu}(p) e^{i(p^2 + i\epsilon)t} + L_{\mu\nu}(p) e^{-\epsilon t} \right] \quad (6.4)$$

To zeroth order this gives

$$\begin{aligned}
 A_{\mu}^a(p, t) &= \frac{i}{2\pi} \int d\omega e^{-i\omega t} G_{\mu\nu}^{ab}(p, \omega) \eta^{\nu}(p, \omega) \\
 &= i \int dt' G_{\mu\nu}^{ab}(p, t - t') \eta^{\nu}(p, t')
 \end{aligned} \tag{6.5}$$

The θ -function in (6.4) gives an effective upper limit $t' = t$ in this last integration, but there is no lower limit. This is because, as we explained in Section 2, the Fourier transform method of solving the Langevin equation gives a particular solution, which vanishes as $t \rightarrow \infty$. Consequently our stochastic correlation functions at finite t are already the equilibrium Green's functions. If we let $\varepsilon \rightarrow 0^+$ in our correlation functions, this is after we have gone to the equilibrium limit, which gives the wrong result when it is applied to the Parisi-Wu form of the Langevin equation.

If, as instructed by Parisi and Wu¹⁾, we impose the boundary condition that the stochastic field vanish at some finite t , which we may as well take to be $t = 0$, when $\varepsilon = 0$ the part of the solution which depends on the initial conditions is no longer transient: we cannot neglect it when t now becomes large. This complication may be traced back to the occurrence of the pole at $\omega = 0$ in the longitudinal part of G : see (6.3). Such a pole arises because D is not invertible when $\varepsilon = 0$.

We impose our initial condition on $A(p, t)$ by including an additional factor $\theta(t')$ under the second integral in (6.5). So effectively $\eta(p, t')$ is multiplied by $\theta(t')$, which means that its Fourier transform $\eta(p, \omega)$ must be replaced with

$$\frac{i}{2\pi} \int d\Omega \frac{1}{\Omega + i0} \eta(p, \omega - \Omega) \tag{6.6}$$

The effect on stochastic diagrams is that the sum of the ω 's on the two lines attached to a cross no longer vanishes. This means that we no longer have the overall δ -function (2.9), and the stochastic average (2.10) now varies with t . It is convenient to modify the stochastic diagrams by including an additional line attached to each cross, emerging from it to the right. This line carries the sum Ω of the two ω 's entering the cross from the left and has a "propagator"

$$\frac{i}{2\pi} \frac{1}{\Omega + i0} \quad (6.7)$$

As the simplest possible example, consider the modified stochastic diagram of Fig. 8, corresponding to the two-point function V_2 to zeroth order in the interaction. Its contribution to the stochastic average

$$\langle A_\mu^a(p_1, \omega_1) A_\nu^b(p_2, \omega_2) \rangle \quad (6.8)$$

is

$$2 (2\pi)^{n+1} \delta^{(n)}(p_1 + p_2) G_{\mu e}^{ac}(p_1, \omega_1) G_\nu^{bce}(p_2, \omega_2) \frac{i}{2\pi} \frac{1}{\omega_1 + \omega_2 + i0} \quad (6.9)$$

We go over to t -space by applying the Fourier integration (2.11) (with $N = 2$). Because we are interested in $t > 0$, we complete the contours of the ω_1 and ω_2 integrations with infinite semicircles in the lower half planes, and then evaluate the integrals by taking the residues at the pole of the integrand. This gives

$$\langle A_\mu^a(p_1, t) A_\nu^b(p_2, t) \rangle = -(2\pi)^n \delta^{(n)}(p_1 + p_2) \delta^{ab} i \left[\frac{T_{\mu\nu}(p_1)}{p_1^2 + i\epsilon} (1 - e^{2it(p_1^2 + i\epsilon)}) + \frac{L_{\mu\nu}(p_1)}{i\epsilon} (1 - e^{-2\epsilon t}) \right] \quad (6.10)$$

If, incorrectly, we now take the limit $t \rightarrow \infty$, the exponentials disappear. They are the part of the solution which depends on our initial condition. If, however, we let $\epsilon \rightarrow 0^+$ before letting t become large, although the first exponential again becomes suppressed because it oscillates infinitely rapidly, the coefficient of $L_{\mu\nu}$ in the square bracket becomes $-2it$. According to Parisi and Wu¹⁾, when one calculates a gauge-invariant expression such powers of t cancel, and leave behind terms that in the usual quantization procedure are associated with ghosts.

It is fairly easy to check this in simple examples, such as we described in Section 3. In calculating stochastic diagrams more complicated than Fig. 8, it is helpful again to use an identity like (2.21). However, because we no longer have the $\delta(\omega_1 + \omega_2 + \dots + \omega_N)$, the right-hand side has the additional term $-i \partial/\partial t V_N^{(\alpha)}$. The resulting differential equation is easy to solve, but we have not been able to derive from it any general results that may be stated in a concise form.

We have to conclude that, in the Parisi-Wu approach, the connection between stochastic diagrams and Feynman diagrams is at least as complicated as in the Zwanziger approach. In this sense, both quantization procedures are very different from the canonical one.

ACKNOWLEDGEMENTS

We are grateful to several members of TH Division at CERN for patient discussions, in particular D. Amati and P. Damgaard.

APPENDIX

In this Appendix we show how the new interaction (5.4) contributes, in gauge-invariant quantities, terms normally associated with a ghost field.

If we use the recursive method of Section 2 to sum the appropriate stochastic diagrams, we find that in $O(g^2)$ the contributions to

$$\langle 0 | T A_\lambda^a(p) A_\lambda^b(p') | 0 \rangle \quad (A.1)$$

from terms involving the new vertex are those shown in Fig. 6. There the vertex with the arrow is the new vertex v^J of (5.5), and the black spot denotes the effective vertex v^K of (5.7). Four of the diagrams include also an ordinary vertex (5.6). The diagrams are Feynman-like: they imply the usual integration over the internal loop momentum. An elementary calculation of these diagrams gives

$$-\delta^{(n)}(p+p') g^2 f^{acd} f^{bcd} \int d^n q d^n r \delta^{(n)}(p+q+r) \frac{1}{p^2 q^2 r^2} \\ \left[\frac{q_\lambda r_{\lambda'}}{p^2} - g_{\lambda\lambda'} \frac{2p^2 - q^2 - r^2}{p^2 + q^2 + r^2} \right] + \dots \quad (A.2)$$

where the usual $i\epsilon$ prescription is to be applied to the denominators. The terms which we have not written explicitly are proportional to $p_\lambda p_{\lambda'}$; these will not contribute to the gauge-invariant quantities which we calculate below. The first term in the square bracket corresponds just to the usual ghost term. We show that the other term does not contribute to the two gauge-invariant quantities discussed in Section 5.

In the case of quark-quark scattering, we must insert (A.2) into the first term of (5.9). We must also use the effective vertex v^K to calculate the second term of (5.9), together with another term in which the roles of the two quarks are reversed. With some elementary Dirac and colour algebra, we find that these two terms yield an integral which exactly cancels the contribution from the unwanted term in (A.2). (The integrands cancel; we do not have to do the loop integration.)

In the case of the gauge-invariant expression $\langle FF \rangle$ of (5.8), things work out differently. We interpret (5.8) as

$$\lim_{y \rightarrow x} \langle 0 | T F_{\mu\nu}^a(x) F_{\rho\sigma}^{a'}(y) | 0 \rangle \quad (A.3)$$

The contribution of the two-point function to this is

$$\begin{aligned} & - \frac{1}{(2\pi)^{2n}} \int d^n p \, d^n p' \, e^{-i(p+p') \cdot x} \\ & \{ p_\mu p'_\nu \langle 0 | T A_\nu^a(p) A_\sigma^{a'}(p') | 0 \rangle \\ & - (\mu \leftrightarrow \nu) - (e \leftrightarrow \sigma) + (\mu \leftrightarrow \nu, e \leftrightarrow \sigma) \} \end{aligned} \quad (A.4)$$

If we insert (A.2) into this expression, the unwanted term in the square bracket of (A.2) contributes

$$\begin{aligned} & \frac{-1}{(2\pi)^{2n}} g^2 f^{abc} f^{abc} \int d^n p \, d^n q \, d^n r \, \delta^{(n)}(p+q+r) \\ & \left\{ g_{\nu\sigma} p_\mu p'_e \frac{q^2 + r^2 - 2p^2}{p^2 q^2 r^2 (p^2 + q^2 + r^2)} + \dots \right\} \end{aligned} \quad (A.5)$$

where the unwritten terms in the curly bracket denote the same permutations as in (A.4).

We obtain (A.4) by using the terms linear in A in both $F_{\mu\nu}^a$ and $F_{\rho\sigma}^{a'}$. If instead we use the quadratic term in one of these two operators, we obtain in order g^2 terms such as depicted in Fig. 9. When the vertex here is the effective vertex v^K , we obtain an integral like (A.5), but with the curly bracket replaced with

$$\begin{aligned} & - p_\mu v_{\nu\rho\sigma}^K(p, q, r) + p_\nu v_{\mu\rho\sigma}^K(p, q, r) \\ & + p_e v_{\sigma\mu\nu}^K(-p, -q, -r) - p_\sigma v_{e\mu\nu}^K(-p, -q, -r) \end{aligned} \quad (A.6)$$

In order to see that these terms, when integrated, just cancel (A.5) one needs two identities:

$$\int d^n q \, d^n r \, \delta^{(n)}(p+q+r) \frac{q^2 - r^2}{q^2 r^2 (p^2 + q^2 + r^2)} = 0$$

$$\int d^n p \, d^n q \, d^n r \, \delta^{(n)}(p+q+r) \tag{A.7}$$

$$p_\lambda q_\lambda \frac{p^2 - q^2}{p^2 q^2 r^2 (p^2 + q^2 + r^2)} = 0$$

Each of these identities follows trivially from the symmetry properties of the integrand, and each would be valid also if we were using some regularization scheme different from dimensional regularization.

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FIGURE CAPTIONS

- Fig. 1: Diagrams for the perturbation expansion of the stochastic field.
- Fig. 2: Diagrams for the perturbation expansion of a stochastic average (2.8).
- Fig. 3: One of the stochastic diagrams in the calculation of the three-point function in lowest order.
- Fig. 4: Two different ways in which external lines couple into stochastic diagrams.
- Fig. 5: The new vertex v^J .
- Fig. 6: Feynman-like diagrams encountered in the calculation of the two-point function in order g^2 from the Langevin equation (5.3). The black spots denote the effective vertex v^K .
- Fig. 7: Contributions to quark-quark scattering in order g^4 .
- Fig. 8: A generalized stochastic diagram, which takes account of the initial condition on the stochastic field.
- Fig. 9: A contribution to $\langle FF \rangle$ of (A.3).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9