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THE USE OF A REGULARIZED HEAT-KERNEL EXPANSION ON
SUPERSYMMETRIC AND/OR GAUGE INVARIANT FIELD THEORIES

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ABSTRACT

We regularize (superfield) Green's functions using a cut-off in the proper time integration of the heat-kernel expansion. Counterterms are easily computed using the background field method, up to two-loop order. Supersymmetry is explicitly maintained through use of supergraphs. The supermultiplet structure of the superconformal anomaly is easily obtained.

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Regularization and renormalization of supersymmetric (gauge) theories is a complex problem. Momentum space subtraction schemes as BPHZ¹⁾, although supersymmetric are not gauge invariant²⁾. Dimensional regularization³⁾ on the other hand, does not maintain the balance between fermionic and bosonic degrees of freedom, unless one uses the method of dimensional reduction⁴⁾, which contains ambiguities in the procedure⁵⁾.

Introduction of the heat-kernel⁶⁾ provides a new parameter, namely the proper-time τ , which can be used to regularize the (otherwise divergent) product of Green's functions (and eventually their derivatives) at the same point. Using the background field method⁷⁾, it is possible to compute the effective action in terms of the background fields, and read the counterterms needed to render it finite, order by order.

In order to define the method, we introduce the (unregularized) heat-kernel expansion for the Green's function:

$$G(x, y) = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\tau}{(4\pi\tau)^2} e^{-\frac{(x-y)^2}{4\tau} - \Lambda\tau} \tau^n a_n(x, y) \quad (1)$$

We rewrite the above expression as⁸⁾

$$G(x, y) = \sum_{n=0}^a G_n(x-y) a_n(x, y) + \tilde{G}(x, y) \quad (2)$$

where

$$G_n(x) = \int_0^{\infty} \frac{d\tau}{(4\pi\tau)^2} \tau^n e^{-\frac{x^2}{4\tau} - \Lambda\tau}$$

and "a" is a finite number (a = 2 for a four-dimensional renormalizable model).

Products of (different) functions $G_n(x)$ and their derivatives at the same point are divergent. Introducing a regularization we obtain well-defined products. For the case of dimensional regularization in non-supersymmetric theories this procedure was completely developed⁸⁾. With the supersymmetric case as a final goal, we introduce a cut-off in the heat-kernel integration parameter τ . Moreover, we enforce the regularized theory to obey the relation

$$G_n(x, y; \tau) = \int d^4 x_1 \int \dots \int d^4 x_{n-1} G_0(x, x_1; \tau_1) \dots G_0(x_{n-1}, y; \tau_n) \quad (3)$$

where $\tau = \tau_1 + \dots + \tau_n$. Then, in the regularized theory, we have

$$G_n(x) = \int_{(n+1)\epsilon}^{\infty} \frac{d\tau}{(4\pi\tau)^2} \tau^n \left(1 - (n+1) \frac{\epsilon}{\tau}\right)^n e^{-\frac{x^2}{4\tau} - \Lambda\tau} \quad (4)$$

It is now possible to compute the previously divergent products. For theories without derivative couplings, it is enough to know $G_0(x)^2$, $G_0(x)^3$ and $G_0(x)^2 G_1(x)$. In order to do that, we Fourier transform them, and expand in powers of momentum. For the first term we have

$$\int d^4 x G_0(x)^2 e^{ikx} = \sum \frac{i k_{\mu_1} \dots i k_{\mu_n}}{n!} \int d^4 x_{\mu_1} \dots x_{\mu_n} G_0(x)^2 \quad (5)$$

In this case, only $n = 0$ is divergent. In momentum space renormalization language¹⁾, this means that we need only subtract once (a constant). Using (4)

$$G_0(x)^2 = \frac{-1}{(4\pi)^2} \ln e^{\delta+1} \Lambda^2 \epsilon \delta^4(x) \quad (6)$$

By the same procedure we get

$$G_0(x)^3 = c \delta^4(x) - \frac{1}{(4\pi)^4} \ln e^{\delta+1} \Lambda^2 \epsilon \partial^2 \delta^4(x) \quad (7)$$

where c is related to a mass renormalization, and

$$G_0(x)^2 G_1(x) = \frac{-1}{2(4\pi)^4} \left(\ln e^{\delta} \Lambda^2 \epsilon \right)^2 \delta^4(x) \quad (8)$$

As an example, we consider the two-loop renormalization of the ϕ^4 theory, and compute the β -function. The Lagrangian is (we will set $m = 0$):

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{4!} \phi^4 \quad (9)$$

The background quantum splitting can be obtained by $\phi \rightarrow \phi + \xi$, and the Seeley coefficients are

$$a_0(x, y) = 1 \quad (10a)$$

$$a_1(x, x) = -\frac{g}{2} \phi^2(x) \quad (10b)$$

$$a_2(x, x) = \frac{g^2}{8} \phi^4(x) - \frac{g}{12} \partial^2 \phi^2(x) \quad (10c)$$

The one-loop counterterm is chosen to cancel the divergent contribution of the determinant of the quadratic operator $\square + g/2 \phi^2$, or equivalently, the contribution of Fig. 1. From it we compute the one-loop coupling constant renormalization

$$\delta g_1 = -\frac{3g^2}{32\pi^2} \ln e^{\sigma+1} m^2 2\epsilon \quad (11)$$

The two-loop contribution comes from Fig. 2. Graph 2.1 is the effect of δg_1 and is multiplied by $a_1(x, x)$. Graph 2.2 is $G_1(x, x)^2$. Those only contribute to the coupling constant renormalization (we are not computing mass counterterms). Finally Graph 2.3 contributes to the coupling constant renormalization [through $G_0(x)^2 G_1(x)$], and to the wave function renormalization [through $G_0(x)^3$]. From those figures we have the usual value for the β -function:

$$\beta(g) = -\frac{2}{2 \ln \epsilon} g_0 = \frac{3g^2}{(4\pi)^2} - \frac{17}{3} \frac{g^3}{(4\pi)^4} \quad (12)$$

This whole procedure can be extended to the supersymmetric case. The (unrenormalized) heat-kernel expansion has already been considered by several authors⁹⁾, and can be written as

$$G((x, \theta_x), (y, \theta_y)) = \int_0^\infty \frac{d\tau}{(4\pi\tau)^2} e^{-\frac{(x-y)^2}{4\tau}} \sum_{n=0}^\infty \tau^n a_n((x, \theta_x), (y, \theta_y)) \delta(\theta_x - \theta_y) \quad (13)$$

where $\delta(\theta)$ is $\delta^4(\theta)$ for general superfields and $\delta^2(\theta)(\delta^2(\bar{\theta}))$ for chiral (anti-chiral) superfields. Moreover a_n can contain derivatives with respect to θ or $\bar{\theta}$.

As an example, we consider the Wess-Zumino model

$$\mathcal{L} = \int d^4\theta \phi \bar{\phi} + \frac{\lambda}{3!} \int d^2\theta \phi^3 + \frac{\lambda}{3!} \int d^2\bar{\theta} \bar{\phi}^3 \quad (14)$$

With the usual background-quantum splitting $\phi \rightarrow \phi + \xi$ (respectively $\bar{\phi} \rightarrow \bar{\phi} + \bar{\xi}$) we have

$$\begin{aligned} \mathcal{L} &= \int d^4\theta \xi \bar{\xi} + \lambda \int d^2\theta \left(\frac{1}{3!} \xi^3 + \frac{1}{2} \xi^2 \phi \right) \\ &+ \lambda \int d^2\bar{\theta} \left(\frac{1}{3!} \bar{\xi}^3 + \frac{1}{2} \bar{\xi}^2 \bar{\phi} \right) = \\ &= \int d^4\theta \left(\xi \bar{\xi} \right) \Theta \left(\frac{\xi}{\bar{\xi}} \right) + \frac{\lambda}{3!} \left[\int d^2\theta \xi^3 + \int d^2\bar{\theta} \bar{\xi}^3 \right] \end{aligned} \quad (15)$$

where

$$\Theta = \begin{bmatrix} -\frac{\mathcal{D}^2}{\square} \frac{\lambda}{2} \phi & 1 \\ 1 & -\frac{\bar{\mathcal{D}}^2}{\square} \frac{\lambda}{2} \bar{\phi} \end{bmatrix}$$

The Seeley coefficients for the Green's function corresponding to $\square\theta$ are easily calculable

$$a_1((x, \theta), (x, \theta)) = -\lambda \begin{bmatrix} \bar{D}^2 \bar{\phi} & 0 \\ 0 & D^2 \phi \end{bmatrix} \quad (16a)$$

$$a_2((x, \theta), (x, \theta)) = \frac{\lambda^2}{2} \begin{bmatrix} 0 & \bar{D}^2 D^2 \bar{\phi} \phi \\ D^2 \bar{D}^2 \bar{\phi} \phi & 0 \end{bmatrix} \quad (16b)$$

For the one-loop computation we consider the graph in Fig. 3, which corresponds to a counterterm

$$2 \int d^4x d^4y \left(\frac{\lambda}{2}\right)^2 \phi(x) \bar{\phi}(y) g_0^2(x-y) = \frac{\lambda^2 \ln(e^{\gamma} m^2 2\varepsilon)}{2(4\pi)^2} \int d^4x \bar{\phi} \phi \quad (17)$$

The graph contributing to the two-loop β -function is shown in Fig. 4. The contribution of this diagram is given by the a_2 coefficient:

$$\begin{aligned} \int d^4x d^4y g_0^2(x-y) \square g_2(x-y) a_2(x, y) = \\ = \frac{\lambda^4}{4(4\pi)^4} \left(\ln(e^{\gamma} m^2 2\varepsilon) \right)^2 \int \bar{\phi}(x) \phi(x) d^4x \end{aligned} \quad (18)$$

From (17) and (18) we have

$$\beta(\lambda) = \frac{3\lambda^3}{32\pi^2} - \frac{3\lambda^5}{512\pi^4} \quad (19)$$

which is the correct result¹⁰⁾.

With this regularization procedure, we can study the supermultiplet structure of the anomaly equation for the breaking of superconformal symmetry in $N = 1$ super Yang-Mills.

It is well known¹¹⁾ that in a supersymmetric field theory, the axial

current, the supersymmetry current and the energy momentum tensor lie in the same supermultiplet. They are expected to be components of a supervector $J_{\alpha\dot{\alpha}}$ [see (5) for conventions]. In a classical superconformally invariant field theory, this vector is conserved, obeying the equation

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = 0 \quad (20)$$

Quantum mechanically superconformal symmetry is broken by the regularization procedure, and we have anomalies which are expected to form a supermultiplet. In fact, the Adler anomaly, the γ -trace of the supersymmetry current and the trace of the energy-momentum tensor fit into a superfield, the supertrace J . In terms of component fields, this was verified recently in one-loop order¹²⁾. We shall now derive the anomaly equation in terms of superfields using the present regularization. We derive the equation

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = -\frac{1}{3} \frac{\beta(g)}{g} D_{\alpha} J \quad (21)$$

at one-loop order.

We consider the $N = 1$ super Yang-Mills Lagrangian

$$\mathcal{L} = \int d^2\theta \, W^2 + \int d^2\bar{\theta} \, \bar{W}^2 + \text{gauge fixing} + \text{ghosts}$$

The background field method consists of a small set of rules derived in (5) and (13). The action, appropriate to one-loop computation is

$$\mathcal{L} = -\frac{1}{2g} \int d^4\theta \, \text{tr} V \square_B V + \text{ghosts} \quad (22)$$

where the following rules apply

$$(i) \quad D_{\alpha} (\bar{D}_{\dot{\alpha}}) \longrightarrow \mathcal{D}_{\alpha} (\bar{\mathcal{D}}_{\dot{\alpha}})$$

where $\mathcal{D}_{\alpha} (\bar{\mathcal{D}}_{\dot{\alpha}})$ are covariantized derivatives with respect to the background field

$$(ii) \quad \square_B = \mathcal{D}^a \mathcal{D}_a + W^{\alpha} \mathcal{D}_{\alpha} - \bar{W}^{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}}$$

where $W_{\alpha} (\bar{W}_{\dot{\alpha}})$ is the background field strength.

(iii) Chiral (antichiral) matter fields are now covariantly chiral (anti-chiral), obeying

$$\mathcal{D}_\alpha \bar{\phi} = 0, \text{ resp. } \bar{\mathcal{D}}_{\dot{\alpha}} \phi = 0$$

The heat-kernel series is obtained expanding

$$\exp -\tau (\mathcal{D}^2 \mathcal{D}_\alpha - w^\alpha \mathcal{D}_\alpha - \bar{w}^{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}})$$

in powers of $w^\alpha \mathcal{D}_\alpha + \bar{w}^{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}}$ (an analogous expansion must be done for chiral/anti-chiral fields, using $\square_B = \mathcal{D}^a \mathcal{D}_a - w^\alpha \mathcal{D}_\alpha - \frac{1}{2} \mathcal{D}^\alpha w_\alpha$). We obtain:

$$\begin{aligned} G(x, y) = & G_{0B}(x, y) + \\ & + \int d\tau_1 d\tau_2 \int d^4 z G_{0B}(x, z; \tau_1) (w^\alpha \mathcal{D}_\alpha + \bar{w}^{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}}) G_{0B}(z, y; \tau_2) \\ & + \frac{1}{2} \int d\tau_1 d\tau_2 d\tau_3 \int d^4 z d^4 w G_{0B}(x, z; \tau_1) (w^\alpha \mathcal{D}_\alpha + \bar{w}^{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}}) G_{0B}(z, w; \tau_2) \\ & \times (w^\beta \mathcal{D}_\beta + \bar{w}^{\dot{\beta}} \bar{\mathcal{D}}_{\dot{\beta}}) G_{0B}(w, y; \tau_3) + \tilde{G}(x, y) \end{aligned} \quad (23)$$

where G_{0B} contains yet the background field through $\mathcal{D}^a$, but its expansion is well known. We now consider the (point-split) left-hand side of the anomaly equation,

$$\begin{aligned} \left[\bar{\mathcal{D}}^{\dot{\alpha}} (W_\alpha(x')) \bar{W}_{\dot{\alpha}}(x) \right]_{x=x'} &= \mathcal{C}_2(G) \bar{\mathcal{D}}^{\dot{\alpha}} \tau \bar{\mathcal{D}}^2 \mathcal{D}_\alpha V(x') \mathcal{D}^2 \bar{\mathcal{D}}_{\dot{\alpha}} V(x) \Big|_{x=x'} \\ &= \mathcal{C}_2(G) \tau \bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{D}'_\alpha \bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{D}^2 \bar{\mathcal{D}}_{\dot{\alpha}} G((x', \theta'), (x, \theta)) \Big|_{\substack{x=x' \\ \theta=\theta'}} \end{aligned} \quad (24)$$

The Laplacian operator appearing in (24), after the D algebra, selects out the a_2 coefficient. We have, as a result¹⁴⁾

$$\bar{D}^\alpha J_{\alpha\dot{\alpha}} = -\frac{1}{3} \frac{\beta(g)}{g} D_\alpha w^2 \quad (25)$$

where $\beta(g)$ can be computed from the one-loop effective action (note that only chiral loops contribute):

$$S^{(1)} = 3 \int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \text{tr} \, g^{ch}_{((x,\theta),(x,\theta);\tau)} \quad (26)$$

with the well-known result [the factor 3 comes⁵⁾ from the Faddeev-Popov plus the Nielsen-Kallosh¹⁵⁾ ghosts]:

$$\beta^{(1)} = -3 C_2(G) \frac{g^3}{(4\pi)^2} \quad (27)$$

The two-loop β function comes from the diagrams in Fig. 5, where in the first diagram we have only gauge field propagators, whereas in the second one there are two ghost lines. Using now the expansion (2.3) (actually the first three terms), and also a Seeley expansion for G_{0B} (which has the background curvature F_{ab}^2 as the trace of the a_2 coefficient), it is possible to compute those two-loop diagrams with a gauge invariant result, the computation being very similar to Ref. 16). Note that using (Fig. 5), one can easily see the non-Abelian structure and the square of the Casimir $C_2(G)^2$ shows up.

In order to understand the relation of (25) with the Adler-Bardeen theorem, it must be computed at order two-loops. This is presently under study.

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Fig. 1

Fig. 2.1

Fig. 2.2

Fig. 2.3

Fig. 3

Fig. 4

Fig. 5.1

Fig. 5.2